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# Construction of Orthogonal and Balanced Arrays in Two and Three Symbols of Strength (2m+1)

**HL Sharma, Sanjeeta Biswas and Sweta Yadav**
*Jawaharlal Nehru Krishi Vishwavidyalaya, Jabalpur*

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## SUMMARY

Orthogonal arrays (OAs) and balanced arrays (BAs) in two and three symbols of strength (2m + 1) have been constructed by considering a tactical configuration (α -β - k - v) converted into design parameters by standard relationship. In view of this, two example with OA in two symbols and one example of BA in three symbols of strength five have been added. In the last, example of intercropping experiment with two main crops and eight intercrops has been provided.

*Keywords:* Tactical configuration, Orthogonal arrays (OA), Balanced incomplete block (BIB) design, Doubly balanced incomplete block (DBIB) design, Strength.

## 1. INTRODUCTION

Orthogonal arrays are objects that are most often generated through algebraic arguments. They have a number of applications in statistics, and have often been studied by algebraic mathematicians as objects of interest in their own right. Our treatment will reflect their use as representations of statistical experimental designs. Orthogonal arrays, first introduced by Rao [1946, 1947] have been used extensively in factorial designs. Specifically, it is of size N, k constraints, s levels and strength t, denoted by OA (N, k, s, t); a k × N matrix X of s symbols, such that all the ordered t-tuples of the symbols occur equally often as column vectors of any t × N sub matrix of X. It is clear that N must be of the form $\lambda s^t$, where λ is usually called the index of the orthogonal array. In applications to factorial designs, each row corresponds to a factor, the symbols are factor levels and each column represents a combination of the factor levels. Thus every OA (N, k, s, t) defines an N-run factorial design for k factors, each having s levels . When λ = 1 we refer to such arrays as "orthogonal arrays of index unity". Most of the techniques for the construction of 2-symbol orthogonal arrays are special cases of techniques for s-symbol arrays. In this study, a structure of primary interest to us is that of a Hadamard matrix. A Hadamard matrix of order n is an n × n matrix $H_n$ with entries 1 or -1, such that $H_n H_n' = nI_n$ where $I_n$ denotes the nxn identity matrix.

Paley [1933] was interested in orthogonal arrays with t = 2, s = 2 because of their applications to the theory of polytopes. Plotkin [1972] obtained the very strong conjecture that every Hadamard matrix of order 8n can be obtained by specializing some orthogonal design of order 8n. He showed that the existence of a Hadamard matrix of order n implies the existence of three types of orthogonal design. If we write n = 4λ in the Hadamard matrix, it is easily seen that $A_{(nxn-1)}$ is an OA(4λ, 4λ − 1, 2, 2), based on the symbols 1 and -1.

A comprehensive reference on the use of orthogonal arrays (OAs) as factorial design in diverse problems of statistical parameter optimization is provided by Wu and Hamada [2000]. Stufken and Tang [2008] provided a complete solution to enumerating non-isomorphic two-level OAs of strength t with t + 2 constraints

*Corresponding author:* H.L. Sharma
*E-mail address:* hlsharma123_jnkvv@rediffmail.com

for any t and any run size $N = \lambda\ 2^t$. More recently, Bulutoglu and Ryan [2018] and Bulutoglu and Margot [2008] formulated an integer linear programming (ILP) method for classifying OAs of strength 3 and 4 with run size at most 162. A few specific construction methods of OAs have been proposed in Brouwer *et al*., [2006] and Nguyen [2008]. Mixed-level OAs of strength 3 with run-size at most 100 are available online at *http://elearning.cse.hcmut.edu.vn/samgroup/OA.jap* given by V.M.M. Nguyen and strength at least 2 at *http://www2.research.att.com/njas/oadir/index.html/* given by N.J.A. Sloane.

A new class of arrays called balanced arrays (BAs) was first introduced and studied by Chakravarti (1956). He obtained some two symbol (2 level) balanced arrays by omitting suitably certain assemblies from an orthogonal array. . A tactical configuration, introduced by Sprott (1955) is a generalized structure of a balanced incomplete block design. Sharma and Chandak (1999) obtained a tactical configuration of order (2m + 1) from a tactical configuration of order 2m for all positive integral values of m. An attempt has been made to construct a two symbol OA arrays of strength (2m + 1) and three symbol BAs arrays on a tactical configuration converted into design parameters by standard relationship.

## 2. DEFINITIONS AND NOTATIONS

### 2.1 Orthogonal (OA) arrays

An OA is generally presented as a two-dimensional array, table, or matrix of N rows and k columns. Each entry in the array is one element of a set of s “symbols”, often taken to be {0, 1, 2,…, s − 1} or {1, 2, 3, ..., s}.. The final basic quantity required for defining the array is its strength, a positive integer $t \le k$. The single requirement that an Nxk array of s symbols must meet to be an OA of strength t is that every subset of t columns (from among the k columns), when considered alone, must contain each of the possible $s^t$ ordered rows the same number of times. A standard notation often used to reference an OA of N rows, k columns, and s symbols, of strength t is OA (N, k, s, t). The number of times each unique row of a t-column subset appears is called the index of the array, often designated by the symbol λ used in OAs as a class of factorial experimental designs.

### 2.2 Balanced (PB) Arrays

Let A be an m×N matrix, with elements 0, 1, 2…, or s-1. Consider the $s^t$ (1×t) vectors, $X' = (x_1,x_2,\ldots, x_t)$, which can be formed from t-rowed sub-matrix of A and associate with each (t×1) vector X a positive integer $\mu\ (x_1,x_2,\ldots, x_t)$, which is invariant under permutations of $(x_1,x_2,\ldots, x_t)$, where $x_i = 0,1,2,\ldots,s-1;\ i=1,2,\ldots, t$. If for every t-rowed sub-matrix of A the $s^t$ distinct (t×1) vectors X occur as columns $\mu\ (x_1,x_2,\ldots, x_t)$ times, then the matrix A is called a balanced (PB) array of strength t in N assemblies with m constraints, s symbols and the specified $\mu\ (x_1,x_2,\ldots, x_t)$, parameters. In view of the fact that $\mu\ (x_1,x_2,\ldots, x_t)$ is invariant under permutation of $(x_1,x_2,\ldots, x_t)$ one can denote by $i_1,i_2,\ldots,$ ir μ ′s $x_1,x_2,\ldots, x_r$ The number of repetitions of a fixed column of any t×N sub array of A, where the column contains i1x1’s, i2x2’s,…and irxr’s, (xj = 0, 1,2, …, s-1), ij = t , r = min ({s,t}). The set of all permutations $i_1,i_2,\ldots, i_r\ \mu\ x_1,x_2,\ldots, x_r$ of an array of strength t in s symbols will be called the index set of the array and will be denoted by Λs,t. The array A will be represented as the PB arrays ( m, N, s, t) with index set $\Lambda s^t$,

### 2.3 Tactical Configuration

Given a set Ω of v elements, and given positive integers k, β ($\beta \le k \le v$) and α, we designate by a tactical configuration (α -β-k-v), a system of blocks (subset of Ω), having k elements each such that every subset of Ω having β elements is included in exactly α blocks. If α =1, then the configuration is called the Steiner system i.e., it is a complete (1-β-k-v) configuration of v elements arranged in blocks of k so that each set of β elements occurs exactly once. The symbol $\lambda_t$ denotes the frequency of the number of blocks in which any t treatments a, b, c,…, occur together. It is very obvious that t=1,2,…,β (β may be odd or even) and $\lambda_1 = r$ (number of replication), $\lambda_0 = b$ = number of blocks.

## 3. THEOREM

**Theorem 3.1**

Using the BIB design

$$( 2k+1,b, r, k, \lambda=\lambda_2) \tag{3.1}$$

($\lambda=\lambda_2$ is taken for pair of two treatments in the sense that a set of j elements appears $\lambda_j$ times in tactical configuration).

with additional property that each set of j elements occurs $\lambda_j$ times where j = 3,4,5,…,β. The configuration is possible when β =2m.

If series (3.1) is used, then an OA [N=(2b+B), k=(v+1), s=2, t=(2m+1)] is constructed, where $B = B_{(2m+1)} + (2m+1)c_{2m}\, B_{2m} + (2m+1)c_{2m-1}\, B_{2m-1} + (2m+1)c_{2m-2}\, B_{2m-2} + \ldots\ (2m+1)c_1 B_1 + B_0$. blocks, where $B_{(2m+1)}$, $B_{2m}$, $B_{2m-1}$,……. $B_1$, and $B_0$ are the number of blocks of strength (2m+1), 2m, (2m-1),…..1 and 0 respectively.

**Proof**

Applying the method given by Sprott (1955) the resulting complete configuration consists of blocks Bi of (3.1) together with blocks $Bi^c$ (which are the compliments of the blocks Bi in addition to the element ∞. It is obvious that the resulting complete configuration has v=2k elements and all blocks contain k distinct elements. It is the generalized modification of Sprott (1955) for tactical configuration done in this paper.

Let Bi be the blocks of general BIB design ( v, b, r, k, $\lambda=\lambda_2$) with additional property that each set of j elements occurs $\lambda_j$ times where j= 3,4,5,………..β. Then $Bi^c$ are the blocks of a BIB design (v, b, b-r, v-k,b-2r+$\lambda_2$) with additional parameters $\lambda_3$=b-3r+3$\lambda_2$-$\lambda_3$; $\lambda_4$ = b–4r+6$\lambda_2$–4$\lambda_3$+$\lambda_4$,…, $\lambda_\beta = b + \sum_{k=1}^{k=\beta+1} (-1)^k \binom{\beta+1}{k} \lambda_k$

The configuration is possible when β=2m. Suppose that a specified set $A_{(\beta+1)}$ of strength (β+1) occurs in exactly $\lambda_{(\beta+1)}$ blocks of Bi, then exactly the set of strength β of the specified set $A_{(\beta+1)}$ occurs together in $\lambda_\beta - \lambda_{(\beta+1)}$ blocks of Bi; and exactly set of strength (β-1) of the same set occurs in $\lambda_\beta - 2\lambda_\beta + \lambda_{(\beta+1)}$ blocks of Bi.

Similarly, a specified set of strength [β-(p-1)] of $A_{(\beta+1)}$ occurs in

$$\lambda_{\beta-(p-1)} - \binom{p}{1}\lambda_{\beta-(p-2)} + \binom{p}{2}\lambda_{\beta-(p-3)} + \ldots (-1)^p \binom{p}{p}\lambda_{\beta+1}$$

blocks of Bi where p= 0, 1, 2,…, β+1.

Sharma and Chandak (1999) have rightly pointed out that the configuration is possible when β=2m for all positive integral values of m and it would be located at the middle point of (m+1). The expression of various values of p can be given as follows:

When p=0, $\quad_2 \quad_1$ (3.2)

When p=1, $\lambda_{2m} - \lambda_{2m+1}$ (3.3)

When p=2, $\lambda_{2m-1} - \binom{2}{1}\lambda_{2m} + \lambda_{2m+1}$ (3.4)

When p=3, $\lambda_{2m-2} - \binom{3}{1}\lambda_{2m-1} + \binom{3}{2}\lambda_{2m} + \lambda_{2m+1}$ (3.5)

When p=4,

$$\lambda_{2m-3} - \binom{4}{1}\lambda_{2m-2} + \binom{4}{2}\lambda_{2m-1} + \binom{4}{3}\lambda_{2m} + \lambda_{2m+1} \quad (3.6)$$

When p = m-1

$$\lambda_{m+2} - \binom{m-1}{1}\lambda_{m+3} + \binom{m-1}{2}\lambda_{m+4} + (-1)^{m-1}\binom{m-1}{m-1}\lambda_{2m+1} = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k}\lambda_{m+k+2} \quad (3.7)$$

When p=m,

$$\lambda_{m+1} - \binom{m}{1}\lambda_{m+2} + \binom{m}{2}\lambda_{m+3} + \ldots (-1)^m \binom{m}{m}\lambda_{2m+1} = \sum_{k=0}^{m} (-1)^k \binom{m}{k}\lambda_{m+k+1} \quad (3.8)$$

When p=m+1

$$\lambda_m - \binom{m+1}{1}\lambda_{m+1} + \binom{m+1}{2}\lambda_{m+2} + \ldots (-1)^{m+1}\binom{m+1}{m+1}\lambda_{2m+1} = \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k}\lambda_{m+k} \quad (3.9)$$

When p=m+2

$$\lambda_{m-1} - \binom{m+2}{1}\lambda_m + \binom{m+2}{2}\lambda_{m+1} + \ldots (-1)^{m+2}\binom{m+2}{m+2}\lambda_{2m+1} = \sum_{k=0}^{m+2} (-1)^k \binom{m+2}{k}\lambda_{m+k-1} \quad (3.10)$$

When p=2m,

$$\lambda_1 - \binom{2m}{1}\lambda_2 + \binom{2m}{2}\lambda_3 + \ldots (-1)^{2m}\binom{2m}{2m}\lambda_{2m+1} = \sum_{k=0}^{2m} (-1)^k \binom{2m}{k}\lambda_{k+1} \quad (3.11)$$

When p=2m+1

$$\lambda_0 - \binom{2m+1}{1}\lambda_1 + \binom{2m+1}{2}\lambda_2 + \ldots (-1)^{2m+1}\binom{2m+1}{2m+1}\lambda_{2m+1} = \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k}\lambda_k \quad (3.12)$$

Therefore, in order to construct OA, the additional blocks would be added with 2b blocks· The difference of the equation (3.9) and (3.2) would provide the number of blocks $B_{(2m+1)}$ of strength (2m+1). Similarly, The difference of the equation (3.9) and (3.3) would provide the number of blocks $B_{2m}$ of strength 2m of the set $A_{2m+1}$ and multiplied by (2m+1) $(2m+1)c_{2m}$. The difference of the equation (3.9) and (3.4) would provide the number of blocks $B_{(2m-1)}$ of strength 2m-1 of the set $A_{2m+1}$ and multiplied by (2m+1) $(2m+1)c_{2m-1}$. In the same way ,the difference of the equation (3.9) and (3.11) would provide the number of blocks $B_1$ of the strength 1 of the same set and it will also be multiplied by (2m+1) $(2m+1)c_1$. In the last, the difference of the equation (3.9) and (3.12) would provide the number of blocks $B_0$ of strength zero i.e. no treatment would appear in this block. Thus, the total number of blocks becomes:

$$B=B_{2m+1}+(2m+1)c_{2m}\,B_{2m}+(2m+1)c_{2m-1}\,B_{2m-1}+(2m+1)c_{2m-2}\,B_{2m-2}+\ldots,(2m+1)c_1\,B_1+\quad B_0 \quad \text{blocks},$$

where $B_{(2m+1)}$, $B_{2m}$, $B_{2m-1}$,…….. $B_1$, and $B_0$ are the number of blocks of strength (2m+1),2m,(2m-1),(2m-2)…..1 and 0 respectively.

Thus, it provides the construction of OA's with parameters [N= (2b+B), k=(v+1), s=2, t=(2m+1)].

Hence the theorem.

### 3.1.1 Illustrative Examples

**Example 1**

Let us consider a BIB design with parameters (5, 10, 4, 2, 1) with b=3r-2λ in addition to its compliment with parameters (5, 10, 6, 3, 3). Thus, we have doubly balanced incomplete block design with parameters (6, 20, 10, 3, 4, 1). Then, addition of four blocks would provide OA (24, 6, 2, 3 ) which is given below. In order to have another DBIBD , BIB design with parameters (5, 10, 6, 3, 3) and (5, 5, 4, 4, 3) will be taken together so that we have DBIBD with parameters (6, 15, 10, 4, 6, 3). Then, two DBIBD with parameters (6, 20, 10, 3, 4, 1) and (6, 15, 10, 4, 6, 3) will be taken at a time resulting into a tactical configuration of strength four with parameters (7, 35, 20, 4, 10, 4, 1). In order to get a tactical configuration of strength five, compliment of (7, 35, 20, 4, 10, 4, 1) will be added together and then we have (8, 70, 35, 4, 15, 5, 1, 0). Applying Theorem 3.1, we have OA's with parameters [N=96, k=8, s=2, t=5] with index 3. Two examples of OA are obtained using Theorem 3.1 as follows:

(i) OA (24,6,2,3 ) of index 3

| | |
|---|---|
| 111000 | 110001 |
| 110100 | 101001 |
| 110010 | 100101 |
| 001110 | 100011 |
| 010110 | 011001 |
| 011010 | 010101 |
| 011100 | 010011 |
| 100110 | 001101 |
| 101010 | 001011 |
| 101100 | 000111 |
| 111111 | 000000 |
| 111111 | 000000 |

(ii) OA (96,8,2,5 ) of index 3

| | | | |
|---|---|---|---|
| 11100010 | 01111000 | 01110001 | 00000000 |
| 11010010 | 11100100 | 10011001 | 11111111 |
| 11001010 | 11010100 | 10101001 | 11111111 |
| 00111010 | 11001100 | 10110001 | 11111111 |
| 01011010 | 00111100 | 11001001 | 11110000 |
| 01101010 | 01011100 | 11100001 | 01111000 |
| 01110010 | 01101100 | 11010001 | 00111100 |
| 10011010 | 01110100 | 01000111 | 00011110 |
| 10101010 | 10011100 | 00100111 | 00001111 |
| 10110010 | 10101100 | 00010111 | 11110000 |
| 11000110 | 10110100 | 00001111 | 01111000 |
| 10100110 | 00011101 | 10000111 | 00111100 |
| 10010110 | 00101101 | 00011011 | 00011110 |
| 10001110 | 00110101 | 00101011 | 00001111 |
| 01100110 | 11000101 | 00110011 | 00010000 |
| 01010110 | 10100101 | 11000011 | 00001000 |
| 01001110 | 10010101 | 10100011 | 00000100 |
| 00110110 | 10001101 | 10010011 | 00000010 |
| 00011110 | 01100101 | 10001011 | 00000001 |
| 00101110 | 01010101 | 01100011 | 00010000 |
| 10111000 | 01001101 | 01010011 | 00001000 |
| 11011000 | 00111001 | 01001011 | 00000100 |
| 11101000 | 01011001 | 00000000 | 00000010 |
| 11110000 | 01101001 | 00000000 | 00000001 |

**Example 2**

Let us consider a BIB design with parameters (9, 18, 8, 4,3) with b=3r-2λ in addition to its compliment with parameters (9, 18, 10, 5, 5). Thus, we have doubly balanced incomplete block design with parameters (10, 36,18, 5, 8, 3). Thus, we have OA (40, 10, 2 ,3) of index 5 after applying Theorem 3.1. In order to have another DBIBD BIB design with parameters (9, 18, 8, 4, 3) and (9, 12, 4, 3, 1) will be taken together so that we have DBIBD with parameters (10, 30, 12, 4, 4, 1). Then, two DBIBD with parameters(10, 36, 18, 5, 8, 3) and (10, 30, 12, 4, 4, 1) will be taken at a time resulting into a tactical configuration of strength four with parameters (11, 66, 30, 5, 12, 4, 1). In order to get a tactical configuration of strength five, compliment of (11, 66, 30, 5, 12, 4, 1) as (11, 66, 36, 6, 18, 8, 3) will be added together and then we have (12, 132, 66, 6, 30, 12, 4, 1), as tactical configuration (1-5-6-12). Applying Theorem 3.1, we have OA's with parameters (N=160, k=12, s=2, t=5) with index 5.

**Theorem 3.1.2**

The columns of *A'* when treated as assemblies give rise to a BAs arrays with three symbols, [2(b+B)] assemblies and strength (2m + 1) where *A'* is given by

*A'* = *[N'*| M'1

and *A'* denotes the transpose of A.

**Proof:**

Let $\mu_{ijk}^{fgh}$ denote the frequency of the t - plet in the t×b (t≤v) sub array of the b×v array in three symbols i, j, k with frequencies f, g and h respectively, such that f+g+h=t.

For completeness, the image method of Dey *et al.*, (1972) is reproduced below which is the additional modification of Dey *et al.*, (1972) for generalized balanced arrays (BAs).

Consider a BIB design with usual parameters v, b, r, k, and λ.

Let N (=$n_{ij}$) be the incidence matrix of this BIB design, where

$N_{ij}$=1,if the jth treatment appears in the ith block = 0, otherwise.

Evidently, N is a bxv array of symbols (0, 1). Let any assembly of this array be denoted by a row vector

z=($z_1$,$z_2$,…,$z_v$), z=0 or 1, z being the vector and the points within, factor binary points in incidence matrix of BIB design.

Then, they defined the 'image 'of z as z* given by z*=($z_1$*,$z_2$*,….$z_v$*), zi+zi*=2(mod3) for all i=1,2,…v. Now, let M be a bxv array of 'images' of each of the assemblies of N.

The frequency of the ordered t-plet (1, 1, 1,…, (*2m* + 1) *i.e.*

$$\mu_0^0 \quad {2m+1 \atop 1} \quad {* \atop 2}$$

in any t-columned sub-array of N is obviously the number of blocks in which any (2*m* + 1) treatments *a, b,* c, ..., occur together and is therefore equal to $\lambda_{2m+1}$ (Sharma and Chandak (1999). The frequency of the other t-plet (0, 1, 1, …, 2m) *i.e.*

$$\mu_0^1 \quad {2m \atop 1} \quad {* \atop 2}$$

In any t-columned sub array of N is the number of blocks in which all treatments occur with only one treatment absent. Clearly, the number of such blocks is $\lambda_{2m} - \lambda_{2m+1}$ and similarly the frequency of the blocks of ordered t -plet

$$\mu_0^2 \quad {2m-1 \atop 1} \quad {* \atop 2} \quad \text{is}$$

$$\lambda_{2m-1} - 2\lambda_{2m} + \lambda_{2m+1}$$

Proceeding like this

$$\mu_0^3 \quad {2m-2 \atop 1} \quad {* \atop 2}$$

$$=\lambda_{2m-2} - 3C_1\lambda_{2m-1} + 3C_2\lambda_{2m} - \lambda_{2m+1}$$

In the same fashion

$$\mu_0^p \quad {2m-(p-1) \atop 1} \quad {* \atop 2}$$

$$=\lambda_{2m-(p-l)} - PC_1\lambda_{2m-(p-2)} + PC_2\lambda_{2m-(p-3)} - \ldots$$

$(-I)^p PC_p \lambda_{2m+1}$ where $p$ =0, 1, 2..., 2$m$

Therefore, the total number of assemblies containing the part or whole of the blocks of the strength*(2m*+1)is

$$\sum_{k=1}^{2m+1} (-1)^k \binom{2m+1}{k} \lambda_k$$

(see, Sharma and Chandak (1999) and hence the frequency of the blocks of ordered t-plet not containing a single treatment *i.e.*

$$\mu_{0\,1\,2}^{2m+1\,0\,*} = b + \sum_{k=1}^{2m+1} (-1)^k \binom{2m+1}{k} \lambda_k$$

Since the assemblies of M are "images" of those of N, it follows that in any t-columned sub-array of M, the frequency of the ordered t-plets will be corresponding to N *i.e.,* the frequency of the ordered t-plets viz., no factor absent, one factor absent, two factors absent and so on in N are:

$\mu_{0\,1\,2}^{0\,2m+1\,*}$, $\mu_{0\,1\,2}^{1\,2m\,*}$, $\mu_{0\,1\,2}^{2\,2m-1\,*}$ ……..
$\mu_{0\,1\,2}^{p\,2m-(p-1)\,*}$

will give rise in M $\mu_{0\,1\,2}^{*\,2m+1\,0}$, $\mu_{0\,1\,2}^{*\,2m\,0}$,
$\mu_{0\,1\,2}^{*\,2m-1\,2}$, $\mu_{0\,1\,2}^{*\,2m-(p-1)\,p}$.

Clearly the frequencies

$$\mu_{0\,1\,2}^{*\,2m+1\,0} = \lambda_{2m+1}$$

$$\mu_{0\,1\,2}^{*\,2m\,1} = \lambda_{2m} - \lambda_{2m+1}$$

$$\mu_{0\,1\,2}^{*\,2m-1\,2} = \lambda_{2m-1} - 2\lambda_{2m} + \lambda_{2m+1}$$

$$\mu_{0\,1\,2}^{*\,0\,2m+1} = b + \sum_{k=1}^{2m+1} (-1)^k \binom{2m+1}{k} \lambda_k$$

$$\mu_{0\,1\,2}^{*\,2m-(p-1)\,p} = \lambda_{2m-(p-1)} - pC_1 \lambda_{2m-(p-2)} + pC_2 \lambda_{2m-(p-3)} - \ldots$$

$(-1)^p pC_p \lambda_{2m+1}$ where $p$ =0, 1, 2...,$2m$. Therefore, in the whole array, the frequencies of all ordered t-plets are given by

$$\mu_{0\,1\,2}^{0\,2m+1\,0} = \mu_{0\,1\,2}^{0\,2m+1\,0} = \lambda_{2m+1}$$

$$\mu_{0\,1\,2}^{1\,2m\,0} = \mu_{0\,1\,2}^{0\,2m\,1} = \lambda_{2m} - \lambda_{2m+1}$$

$$\mu_{0\,1\,2}^{2\,2m-1\,0} = \mu_{0\,1\,2}^{0\,2m-1\,2} = \lambda_{2m-1} - 2\lambda_{2m} + \lambda_{2m+1}$$

$$\mu_{0\,1\,2}^{p\,0\,2m-(p-1)} = \mu_{0\,1\,2}^{2m-(p-1)\,0\,p}$$

$$= \lambda_{2m-(p-1)} - pC_1 \lambda_{2m-(p-2)} + \mu_{0\,1\,2}^{2m+1\,0\,0} = b + \sum_{k=1}^{2m+1} (-1)^k \binom{2m+1}{k} \lambda_k \; pC_2 \lambda_{2m-(p-3)} - \ldots (-1)^p pC_p \lambda_{2m+1}$$

where $p$ =0, 1, 2...,$2m$, *and*

$$\mu_{0\,1\,2}^{2m+1\,0\,0} = \mu_{0\,1\,2}^{0\,0\,2m+1} = b + \sum_{k=1}^{2m+1} (-1)^k \binom{2m+1}{k} \lambda_k$$

Thus, A is a three symbol BAs of strength $(2m + 1)$ for all positive integral values of m. The frequencies of all other t-plets combinations are zero.

Hence the theorem.

The results of Dey *et al.* (1972) become a particular case when $m = 1$ in this theorem.

**Example 3**

Let us consider the incidence matrix of the tactical configuration (1-5-6-12) having v = 12, $b$ = 132, $r$ = 66, $k$ =6,$\lambda_2$=30,$\lambda_3$=12,$\lambda_4$=4, $\lambda_5$=1, and applying the construction method given in Section 3 of this paper.

In N, we have

$$\mu_{012}^{05*} = \lambda_5 + \lambda_3 - 2\lambda_4$$

$$\mu_{012}^{14*} = 5c_4[\lambda_4 - \lambda_5] + 5c_4[\lambda_3 - 3\lambda_4 + 2\lambda_5]$$

$$\mu_{012}^{23*} = 5c_3[\lambda_3 - 2\lambda_4 + \lambda_5]$$

$$\mu_{012}^{32*} = 5c_2[\lambda_2 - 3\lambda_3 - 3\lambda_4 + \lambda_5]$$

$$\mu_{012}^{41*} = 5c_1[\lambda_1 - 4c_1\lambda_2 - 4c_2\lambda_3 + 4c_3\lambda_4 - \lambda_5] + 5c_1[\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_1 + 4c_1\lambda_2 - 4c_2\lambda_3 + 4c_3\lambda_4 - \lambda_5]$$

$$\mu_{012}^{50*} = [\lambda_0 - 5c_1\lambda_1 + 5c_2\lambda_2 - 5c_3\lambda_3 + 5c_4\lambda_4 - \lambda_5] + [\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_0 + 5c_1\lambda_1 - 5c_2\lambda_2 + 5c_3\lambda_3 - 5c_4\lambda_4 + \lambda_5]$$

Similarly in M, we have

$$\mu_{012}^{*50} = \lambda_5 + \lambda_3 - 2\lambda_4$$

$$\mu_{012}^{*41} = 5c_4[\lambda_4 - \lambda_5] + 5c_4[\lambda_3 - 3\lambda_4 + 2\lambda_5]$$

$$\mu_{012}^{*32} = 5c_3[\lambda_3 - 2\lambda_4 + \lambda_5]$$

$$\mu_{012}^{*23} = 5c_2[\lambda_2 - 3\lambda_3 - 3\lambda_4 + \lambda_5]$$

$$\mu_{012}^{*14} = 5c_1[\lambda_1 - 4c_1\lambda_2 - 4c_2\lambda_3 + 4c_3\lambda_4 - \lambda_5] + 5c_1[\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_1 + 4c_1\lambda_2 - 4c_2\lambda_3 + 4c_3\lambda_4 - \lambda_5]$$

$$\mu_{012}^{*05} = [\lambda_0 - 5c_1\lambda_1 + 5c_2\lambda_2 - 5c_3\lambda_3 + 5c_4\lambda_4 - \lambda_5] + [\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_0 + 5c_1\lambda_1 - 5c_2\lambda_2 + 5c_3\lambda_3 - 5c_4\lambda_4 + \lambda_5]$$

In overall, we get X is a PB arrays (v=12, b=320, s=3, t=5 with index set $\Lambda_{3,5.}$

$$\mu_{012}^{050} = \lambda_5 + \lambda_3 - 2\lambda_4 = 10,$$

$$\mu_{012}^{140} = \mu_{012}^{041} = 5c_4[\lambda_4 - \lambda_5] + 5c_4[\lambda_3 - 3\lambda_4 + 2\lambda_5] = 25$$

$$\mu_{012}^{230} = \mu_{012}^{032} = 5c_3[\lambda_3 - 2\lambda_4 + \lambda_5] = 50$$

$$\mu_{012}^{320} = \mu_{012}^{023} = 5c_2[\lambda_2 - 3\lambda_3 - 3\lambda_4 + \lambda_5] = 50$$

$$\mu_{012}^{410} = \mu_{012}^{014} = 5c_1[\lambda_1 - 4c_1\lambda_2 - 4c_2\lambda_3 + 4c_3\lambda_4 - \lambda_5] + 5c_1[\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_1 + 4c_1\lambda_2 - 4c_2\lambda_3 + 4c_3\lambda_4 - \lambda_5]$$

=25

$$\mu_{012}^{500} = \mu_{012}^{005} = [\lambda_0 - 5c_1\lambda_1 + 5c_2\lambda_2 - 5c_3\lambda_3 + 5c_4\lambda_4 - \lambda_5] + [\lambda_3 - 2\lambda_4 + \lambda_5 - \lambda_0 + 5c_1\lambda_1 - 5c_2\lambda_2 + 5c_3\lambda_3 - 5c_4\lambda_4 + \lambda_5]$$

=5

The frequency of other treatment combinations of strength five is zero.

**Example 4**

Hedayat and Wallis (1978) have given a theorem stating that the existence of Hadamard matrix of order 4t implies the existence of BIB designs with parameters:

v= 2t , b=4t-2, r=2t-1,k= t and λ=t-1

On the basis of t=2, let us consider BIB design v = 4, b = 6, r =3, k = 2, $\lambda_2$= 1, so that N' of Example 3, can be obtained. Taking the images of N' as M' using zi + zi* = 2 (mod 3) for all i = 1, 2,…, *v* treatments. The blocks are given below:

$$A' = \begin{bmatrix} 100011 \\ 010101 \\ 001110 \\ 111000 \end{bmatrix} \begin{bmatrix} 122211 \\ 212121 \\ 221112 \\ 111222 \end{bmatrix}$$ where A' is the transpose of A

The combinatorial arrangements, in particular, orthogonal and partially balanced arrays of specified strength t are used in the construction of balanced symmetrical and asymmetrical confounded factorial experiments, multi factorial designs (fractional replications) and so on (Rao ,1947; 1949 and Nair and Rao (1948)). Balanced arrays satisfy the same properties as orthogonal arrays when used as fractional replicated factorial designs in terms of estimability of main effects and interactions, but the estimates, of main effects and interactions may have different precisions besides being correlated. The construction and use of such designs have been indicated in Chakravarti (1956), (1961), (1963) and extensively investigated by Srivastava (1972), Srivastava and Anderson (1970) and Srivastava and Chopra (1971a), (1971b), (1971c), (1973) in the special case s =2, i.e., S has two symbols 0 and 1.

A catalogue of two new designs that can be obtained through the BAs has been given below:

*OA (24, 6, 2, 3) and OA (96, 8, 2, 5) of index 3 are given.

** The N' and its images M' are BAs of strength (2*m* + 1) with three symbols (0, 1, 2). In particular, Example 4.4 is a BAs of strength 5 with 3 symbols with index set $A_{3,5}$ constructed by author in the present paper.

***The constructed PB array in the present paper can be used for conducting intercropping experiments when the intercrops are sub-divided into various groups based on agronomic practices including main crop assuming that some of the interaction of intercrops are negligible. We construct design for experiments where each plot consists of main crop *p* and *q* intercrops, such that each of these intercrops is selected from a group of *r* intercrops following Rao and Rao (2001).

Now, let us consider an intercropping experiment using two main crops and 8 intercrops where the intercrops are partitioned into four groups $Q_1$, $Q_2$, $Q_3$ and $Q_4$ with 2 in each group viz., $Q_1$ = [1, 2 ], $Q_2$ = [3,4] , $Q_3$ = [5, 6 ] and $Q_4$=[7,8]. Let us designate the symbols 0,2 of first row of PB array with intercrops 1, 2 of $Q_1$, second row with intercrops 3, 4 of $Q_2$ , third row with intercrops 5, 6 of $Q_3$ and fourth row with intercrops 7,8 of $Q_4$. Considering the column of the array as the plots of the intercrop experiment in addition to the two main crops $m_1$ and $m_2$ in each plot. The resulting intercropping experiment will consist of the following 12 plots:

($m_1$, $m_2$, 3, 5); ($m_1$, $m_2$, 1, 5); ($m_1$, $m_2$, 1, 3); ($m_1$, $m_2$, 1, 7); ($m_1$, $m_2$, 3, 7); ($m_1$, $m_2$, 5, 7)

($m_1$, $m_2$, 4, 6); ($m_1$, $m_2$, 2, 6); ($m_1$, $m_2$, 2, 4); ($m_1$, $m_2$, 2, 8), ($m_1$, $m_2$, 4, 8), ($m_1$, $m_2$, 6, 8).

It is to be noted that this method provides intercropping design with two main crops and eight intercrops divided into four groups of two intercrops each. It is claimed that this design for intercropping experiment has lesser number of blocks as compared to Rao and Rao (2001)

In the context of an actual example of intercropping experiment, Pandey *et al.* (2003) have studied the effect of maize *(Zea mays L.)* based intercropping systems on maize yield as main crop and six intercrops viz., pigeon pea, sesamum, groundnut, blackgram, turmeric and forage meth by conducting an experiment during the rainy seasons of 1998 and 1999 at the research farm of Rajendra Agricultural University, Pusa, Samastipur (Bihar). The experiment consisting of 6 intercrops with one main crop was conducted in randomized complete block design with 4 replications. Maize was sown at 75 cm row spacing in sole as well as in intercropping on 26 and 22 June, respectively, in the first and second year of experimentation. One row of pigeon pea at distance of75 cm and 2 rows of other intercrops at 30 cm distance were accommodated between 2 rows of maize. The intra row spacing of 30, 30, 10, 15, 10 and 15 cm were maintained by thinning for 6 intercrops.

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