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Richards Stochastic Differential Equation Growth Model and its Application

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SUMMARY

Richards four-parameter nonlinear growth model, which is a generalization of the well-known logistic and Gompertz models, is a very versatile model for describing many growth processes. However, one limitation of the corresponding Richards nonlinear statistical model is that it is applicable only when the data are available at equidistant epochs, which is not always possible. The other limitation is that it is not able to describe the underlying fluctuations of the system satisfactorily particularly for longitudinal data, as merely an error term is added to the deterministic model to obtain it. Accordingly, in this article, the general approach of 'Stochastic differential equations' is considered. Specifically, the methodology for Richards growth model in random environment is developed. The optimal predictor of untransformed data along with prediction error variance is also derived. Relevant computer programs for its application are written and the same are included as an Appendix. Finally, as an illustration, pig growth data are considered and superiority of our proposed model is shown over the Richards nonlinear statistical model for given data.

Keywords: Richards nonlinear growth model, Stochastic differential equation, Interval estimation, Out-of-sample forecasting.

1. INTRODUCTION

It is well recognized that any type of statistical inquiry in which principles from some body of knowledge enter seriously into the analysis is likely to lead to a 'Nonlinear model' (Seber and Wild 2003). Such models play a very important role in understanding the complex inter-relationships among variables. Nonlinear growth models describe the development of variable of interest and are applicable in almost all disciplines related to plants, animals, fisheries, etc. (See, e.g. Gupta and Iannuzzi 1998). Richards four-parameter nonlinear growth model, which is a generalization of the well-known logistic and Gompertz models, is generally able to describe many growth processes. A large number of research papers in research dealing with various aspects of this model, if nothing else, is a testimony to the important role played by it (See e.g. Iquebal *et al.* 2009, Matis *et al.* 2011, Ghosh *et al.* 2011, Wang *et al.* 2012, Lv *et al.* 2015, and Roman-Roman and Torres-Ruiz 2015). Therefore, in this article, we shall confine our attention to only the Richards growth model.

Richards four-parameter nonlinear growth model arises as a result of making assumptions about the type of growth and expressing them in terms of a differential equation. A heartening feature of this model is that the underlying nonlinear differential equation is soluble by means of a transformation. In order to fit the same to data, usual practice is to assume an additive error term on the right hand side of the functional form of the model. The errors are generally assumed to be independently and identically distributed (iid). Nonlinear estimation procedures, are then employed to estimate the underlying parameters. Further, they can also be estimated efficiently, if the errors are autocorrelated. A good description of Richards growth model and various procedures for estimation of its parameters is available in Seber and Wild (2003). It may be emphasized that, unlike Gompertz and logistic models, it is extremely difficult to fit

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Richards model due to its high nonlinearity. Further, Nonlinear estimation procedures are applicable only when the data are available at equidistant epochs. However, collection of growth data over time involves constraints of time, personnel, and budgets, etc. that do not always satisfy this requirement. The data that do exist in studies with missing data or data at unequal time-intervals are potentially informative, and precluding such data from analysis could affect conclusions adversely (Dennis and Ponciano 2014). The other limitation is that, by simply adding an error term, a nonlinear statistical model is not capable of describing the underlying fluctuations of the system satisfactorily, particularly for longitudinal data.

Both the above issues can, however, be handled by employing the more general approach of 'Stochastic differential equations'. These are generally obtained by adding a stochastic term to the differential equation form of deterministic model. In a physical situation, random environmental fluctuations due to variations in parameters, such as Birth and death rates generally occur with great rapidity as compared to the time-scale of population growth. Therefore, the stochastic term is generally taken to be a Gaussian white noise stochastic process. The heartening aspect of this prescription is that the resultant process becomes Markovian. However, the price to be paid is that the sample paths are very irregular and do not admit of derivatives in the conventional sense. To handle this situation, two types of stochastic calculi due respectively to Stratonovich and Itô have been developed in the literature. In the former, usual rules of calculus continue to apply whereas in the latter, these are suitably modified. However, for the present article, both these calculi yield identical results as we are concerned only with the case of additive noise which is independent of state variable. Standard Itô formula is applied to solve these stochastic differential equation (SDE) models through their equivalent Itô stochastic integral representations. The present research paper is organized as follows. In Section 2, deterministic and statistical versions of Richards growth model are discussed along with stabilization of variances using appropriate transformation. Linearized Richards SDE (LRSDE) model is proposed for transformed data and the same is described. The optimal predictor of untransformed data along with prediction error variance is also derived. In Section 3, the methodology is applied to real data and it is shown that, for given data, the proposed LRSDE model performs better compared to Transformed Richards nonlinear statistical (TRNS) model for both modelling and forecasting purposes. The entire data analysis is carried out in SAS software package, Ver. 9.4 and salient codes are written in C++ and included as an Appendix.

2. METHODOLOGY

2.1 Deterministic Model

Richards growth model is expressed by the nonlinear differential equation

$$d\mu_t/dt = r\mu_t(K^m - \mu_t^m)/(mK^m), \qquad (1)$$

where μ_t denotes the variable of interest at time t, r is intrinsic growth rate, K is carrying capacity, and m is a parameter. To solve eq.(1), let $\lambda_t = \mu_t^{-m}$; then

$$d\lambda_t/dt + r\lambda_t = rK^{-m},$$

which is a linear differential equation. Solving it, we get

$$\mu_t = \frac{K\mu_0}{\left[\mu_0^m + \left(K^m - \mu_0^m\right)exp(-rt)\right]^{1/m}}, \qquad (2)$$

where μ_0 indicates the initial value of μ_t at time $t = 0$. The graph of μ_t versus t is generally sigmoid, i.e. elongated S-shaped. There are four parameters in this model, viz. r, K, μ_0 and m. The ranges of the first three parameters are positive, while that for the last parameter is from -∞ to +∞. Further, eq. (2) reduces respectively to monomolecular, Gompertz, and logistic models when $m = -1, 0, 1$. The point of inflexion of Richards model is at $\mu_t = K/(m+1)^{1/m}, m \neq -1$, and occurs at

$$t = \frac{1}{r}\log_e\left(\frac{K^m - \mu_0^m}{m\mu_0^m}\right), m \neq -1.$$

Thus, Richards model is very flexible as the point of inflexion is not fixed but can occur at any fraction of the carrying capacity K depending on the value of m.

2.2 Nonlinear Statistical Model

In order to apply eq. (2) to data (y_t), the usual practice is to assume an additive error term on its right hand side. Thus, the corresponding Richards nonlinear statistical model is given by

$$y_t = \mu_t + \varepsilon_t = \frac{K\mu_0}{\left[\mu_0^m + \left(K^m - \mu_0^m\right)exp(-rt)\right]^{1/m}} + \varepsilon_t \qquad (3)$$

where the error term ε_t is assumed to be independently and identically distributed with mean zero and variance σ^2. The parameters of the model given by eq. (3) are then estimated through Nonlinear estimation procedures, such as Levenberg-Marquardt procedure. Most of the standard statistical software packages, such as SPSS and SAS contain computer programs for fitting nonlinear statistical models. It may be highlighted that the assumption of homoscedastic errors is quite often violated in practice due to the presence of increase in variation of growth curve as time increases. However, the square-root transformation of μ_t is quite often able to achieve homoscedasticity. Accordingly, in our further discussion, we shall confine our attention to the case $m = -1/2$. Thus, from eq. (3), the model to be considered reduces to

$$y_t = \mu_t + \varepsilon_t = \lambda_t^2 + \varepsilon_t$$
$$= \left[K^{1/2} + \left(\mu_0^{1/2} - K^{1/2}\right)exp(-rt)\right]^2 + \varepsilon_t, \quad (4)$$

where ε_t are heteroscedastic. To tackle the realistic situation of a data set having heteroscedastic errors in the model given by eq. (4), there is a need to consider the following transformation

$$y_t = \lambda_t^2(1 + \eta_t/\lambda_t)^2 \quad (5)$$

where the error terms η_t are homoscedastic. From eq. (5), note that the error term ε_t in eq. (4) is $\eta_t^2 + 2\eta_t\lambda_t$, whose variance depends on t, and the mean-equation of eq. (4) is retained. Thus, the square-root transformation of eq. (5) leads to Transformed Richards nonlinear statistical (TRNS) model given by

$$Z_t = y_t^{1/2} = K^{1/2} + \left(y_0^{1/2} - K^{1/2}\right)exp(-rt) + \eta_t. \quad (6)$$

The confidence-intervals for parameters K and r may be obtained as follows. Consider a fixed-regressor nonlinear statistical model with known functional relationship $y_i = f(x_i, \boldsymbol{\theta}) + \varepsilon_i, i = 1,2,\ldots,n$, where $E[\varepsilon_i] = 0$, and the true value $\boldsymbol{\theta}^*$ of $\boldsymbol{\theta}$ is known to belong to Θ, which is a subset of $\Re^p$. Let $\hat{\theta}$ be the iterative least-squares estimator of $\boldsymbol{\theta}$ based on Gauss-Newton method of minimizing $S(\boldsymbol{\theta}) = \{\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{\theta})\}'\{\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{\theta})\}$. Under appropriate regularity conditions, $\hat{\theta} - \theta^* \sim N\left(0, \sigma^2\left(F_\bullet{}'F_\bullet\right)^{-1}\right)$, where $F_\bullet = \partial \boldsymbol{f}(\boldsymbol{\theta})/\partial\boldsymbol{\theta}'$ is a $n \times p$ matrix of first-order derivatives of regression function $\boldsymbol{f}(\boldsymbol{\theta})$. For TRNS model, $F_\bullet$ is estimated at $\hat{\theta} = \left(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3\right)'$, where $\hat{\theta}_1 = K^{1/2}$, $\hat{\theta}_2 = y_0^{1/2} - K^{1/2}$, $\hat{\theta}_3 = r$. Assuming that the process $\{\varepsilon_t^*\}$ is Gaussian white noise, the 100 $(1-\gamma)\%$ confidence-interval for the i^{th} element of $\boldsymbol{\theta}$ is $\hat{\theta}_i \mp t_{n-3, \frac{\gamma}{2}} s\sqrt{\hat{c}_{ii}}$, where $\left(F_\bullet{}'F_\bullet\right)^{-1} = \left(\hat{c}_{ij}\right)$ and $s^2 = \| \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{\theta}) \|^2/(n-p)$. Note that the t^{th} row of $F_\bullet = \left(1, e^{-rt}, -t\left(y_0^{1/2} - K^{1/2}\right)e^{-rt}\right)$. Finally, interval estimation for carrying capacity K is obtained by squaring the lower and upper limits of interval estimator of θ_1.

Further, to test the suitability of TRNS model, comparative study of "pure error" vis-a-vis "lack of fit" may be carried out. To this end, the design framework of nonlinear regression model is given by

$$y_{ij} = \mu_i + \varepsilon_{ij}, i = 1,2,\ldots,n; j = 1,2,\ldots,J \quad (7)$$

where $\mu_i = f(x_i, \boldsymbol{\theta})$. Using partition of error sum of squares into "pure error" and "lack of fit" sum of squares, least squares estimators of $\boldsymbol{\theta}$ are obtained by minimizing the latter. The normal equations for $\widetilde{\boldsymbol{\theta}}$ are

$$-2\sum_{i=1}^{n} J\left(\bar{y}_i - f(x_i, \boldsymbol{\theta})\right)\frac{\partial f(x_i, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}_l}\Big|_{\widetilde{\boldsymbol{\theta}}_l} = 0, l = 1,2,\ldots,p.$$

It may be noted that $f(t, \boldsymbol{\theta}) = A + Be^{-Ct}$. Replacing μ_i by $\hat{\mu}_t = f\left(t, \widetilde{\boldsymbol{\theta}}\right)$, residual sum of squares Q_H is split into a "pure" sum of squares Q and a "lack of fit" sum of squares $Q_H - Q$. Using asymptotic linear approximation for a nonlinear function, following test statistic may be used to test the null hypothesis of nonlinear regression function under independent error

$$F = \frac{\{\sum_{i=1}^{n} J(\bar{y}_i - \hat{\mu}_i)^2\}(N-n)}{\sum_{i=1}^{n}\sum_{j=1}^{J}\left(y_{ij} - \bar{y}_i\right)^2(n-p)} \quad (8)$$

which follows $F_{n-p,N-n}$ distribution.

2.3 Stochastic Differential Equation Model

It may be pointed out that errors in relative growth rate are closer to have constant variance than those in growth rates. Accordingly, models of the following form have been considered in the literature (Seber and Wild 2003)

$$(1/y_t)(dy_t/dt) = h(t, y_t; \theta) + \varepsilon_t \quad (9)$$

Even when the errors in eq. (9) are independent, the process $\{y_t\}$ is dependent with changing variance.

However, it suffers from the drawback that it may not be able to allow conditional variance of y_t to depend on past growth data. To this end, corresponding to eq. (1) with $m=-1/2$ under random environment, the SDE version of Richards growth model is given by

$$dy_t = \left\{-2ry_t\left(k^{-1/2} - y_t^{-1/2}\right)k^{-1/2}\right\}\left\{dt + \left(\sigma dW_t / r(k^{1/2} - y_t^{1/2})\right)\right\} \tag{10}$$

Using Markov property of the solution y_t, note that expected value of the error term $\varepsilon_t = -2\sigma dW_t k^{1/2}(k^{-1/2} - y_t^{-1/2})/(k^{1/2} - y_t^{1/2}) = 2\sigma dW_t\, y_t^{-1/2}$ is zero. Further, it may be noted that conditional variance of $y_{t+\delta t}$ depends on past growth data due to the fact that ε_t is a function of y_t. Therefore, using the variance stabilization transformation $Z_t = g(y_t) = y_t^{1/2}$ and using chain rule of differentiation, linear SDE in transformed variable Z_t may be obtained. Hence, advantage of the nonlinear SDE model given in eq. (10) is that it is capable of yielding closed form solution by getting solution of SDE in transformed variable. Thus, after necessary simplification, eq. (10) is reduced to Linearized Richards SDE (LRSDE) model, given by

$$dZ_t = r(\alpha - Z_t)dt + \sigma dW_t, \tag{11}$$

where $\alpha = K^{1/2}$ and W_t is the Brownian or Wiener process with variance parameter unity. Given $\mathcal{F}_{t_k} = \{Z_t : t \le t_k\}$, solution of the LRSDE model, obtained by using Ito calculus, is given by (Filipe *et al.* 2013)

$$Z_t = K^{1/2} + \left(Z_{t_k} - K^{1/2}\right)e^{-r(t-t_k)} + \sigma exp(-rt)\int_{t_k}^{t} exp(rs)\, dW_s \tag{12}$$

Note that solution of the above LRSDE model is Markovian and follows Gaussian process with conditional mean and variance given by $\mu_{Z:t|t_k} = E\{Z_t|F_{t_k}\} = \alpha + \left(Z_{t_k} - \alpha\right)e^{-r(t-t_k)}$ and $\sigma^2_{Z:t|t_k} = V\{y_t|F_{t_k}\} = \sigma^2\left(1 - e^{-2r(t-t_k)}\right)/(2r)$ respectively. The mean-value function of $\{Z_t\}$, i.e. $E[Z_t] = \alpha + \left(Z_{t_0} - \alpha\right)e^{-r(t-t_k)}$ is a sigmoid curve, whereas the transition probability is homogeneous and homoscedasic. It may be noted that variance-function of $\{Z_t\}$ depends on time, which allows the variance of $\{y_t\}$ to change over time. The process is also asymptotically stationary with mean and variance given by α and $\sigma^2/(2r)$. Further, $\varepsilon_t = 2\sigma dW_t Z_t^{-1}$. As in TRNS model, note that $E\{\varepsilon_t\} = E\{2\sigma dW_t Z_t^{-1}\} = 2E[Z_t^{-1}E(\sigma dW_t\,|\,F_t)] = 2E[Z_t^{-1}E\{\sigma dW_t\,|\,F_t\}] = 0$, due to the fact that Z_t is function of $\{W_s : s \le t\}$. Using same argument, variance of error term ε_t is obtained as $V\{\varepsilon_t\} = 4\,E[Z_t^{-2}E\{(\sigma dW_t)^2\,|\,F_t\}] \approx 4\sigma^2\{\sigma^2/(2r) + \alpha^2\}^{-1}$ which is constant. Therefore, modelling of error structures as given in eqs. (5) and (10) is capable to fit heteroscedastic growth data to TRNS model and its stochastic analogue, viz. LRSDE model given respectively by eqs. (6) and (11).

Since the transformation $g(.)$ is monotonically non-decreasing, therefore, approximate mean-value function of process $\{y_t\}$ viz. $E\left(y_t|\mathcal{F}_{y:t_k}\right) = g^{-1}[\{E\left(Z_t|\mathcal{F}_{t_k}\right)\}] = E^2\left(Z_t|\mathcal{F}_{t_k}\right)$ is also monotonically non-decreasing and tends to α^2 as $t \to \infty$. However, an attempt is made here to obtain exact conditional expectation of $\{y_t\}$ following LRSDE model. Note that, given $\mathcal{F}_{Z:t_k}$, $U_t = Z_t/\sigma_{z:t|t_k}$ is normally distributed with variance unity and means $\mu_{1:Z}$ and $\mu_{2:Z}$. Write $E\left(y_t\,|\mathcal{F}_{y:t_k}\right) = E\left\{E\left(y_t\,|\mathcal{F}_{Z:t_k}\right)|\mathcal{F}_{y:t_k}\right\}$, where inner conditional expectations are taken with respect to information sets $\mathcal{F}^{(i)}_{Z:t_k}, i = 1,2$. The expressions for $E\left(y_t\,|\mathcal{F}_{Z:t_k}\right)$ are evaluated as $\sigma^2_{Z:t|t_k}\mu_1$ and $\sigma^2_{Z:t|t_k}\mu_2$ when information about the processes $\{Z_t\}$ are $\mathcal{F}^{(1)}_{Z:t_k}$ with $Z_{t_k} = \sqrt{y_{t_k}}$ and $\mathcal{F}^{(2)}_{Z:t_k}$ with $Z_{t_k} = -\sqrt{y_{t_k}}$ respectively. Note that

$$\sigma^2_{Z:t|t_k}\mu_1 = \sigma^2_{Z:t|t_k}(1 + \mu^2_{1:Z}),$$
$$\sigma^2_{Z:t|t_k}\mu_2 = \sigma^2_{Z:t|t_k}(1 + \mu^2_{2:Z}) \tag{13}$$

$$\mu_{1:Z} = \left\{\alpha + \left(\sqrt{y_{t_k}} - \alpha\right)e^{-\beta(t-t_k)}\right\}/\sigma_{z:t|t_k},$$
$$\mu_{2:Z} = \left\{\alpha + \left(-\sqrt{x_{t_k}} - \alpha\right)e^{-\beta(t-t_k)}\right\}/\sigma_{z:t|t_k}. \tag{14}$$

The outer conditional expectation may be carried out based on density at $\mp\sqrt{y_{t_k}}$ of the stationary distribution of Z_t, which is Gaussian with mean $\mu_0 = \alpha$ and variance $\sigma_0^2 = \sigma^2/(2r)$. Finally, conditional expectation of y_t given $\mathcal{F}_{y:t_k}$, i.e. $\mu_{y:t|t_k}$ is obtained by weighted means of two conditional expectations and is given by

$$E\left(y_t \mid \mathcal{F}_{y:t_k}\right) = \frac{\sigma^2_{Z:t|t_k}\left\{exp\left(-.5\sigma_0^{-2}\left(\sqrt{y_{t_k}}-\alpha\right)^2\right)\mu_1 + exp\left(-.5\sigma_0^{-2}\left(-\sqrt{y_{t_k}}-\alpha\right)^2\right)\mu_2\right\}}{exp\left\{-.5\sigma_0^{-2}\left(\sqrt{y_{t_k}}-\alpha\right)^2\right\}+exp\left\{-.5\sigma_0^{-2}\left(-\sqrt{y_{t_k}}-\alpha\right)^2\right\}}. \quad (15)$$

To evaluate conditional variance using conditioning principle, note that

$$V\left(y_t \mid \mathcal{F}_{y:t_k}\right) = E\left\{V\left(y_t \mid \mathcal{F}_{Z:t_k}\right) \mid \mathcal{F}_{y:t_k}\right\} + V\left\{E\left(y_t \mid \mathcal{F}_{Z:t_k}\right) \mid \mathcal{F}_{y:t_k}\right\}.$$

Along similar lines as above definition of information sets at time t_k, the inner conditional variance expression in the first expression of $V\left(y_t \mid \mathcal{F}_{y:t_k}\right)$ may be evaluated at two realized values of Z_t, viz. $\mp\sqrt{y_{t_k}}$, and writing

$$V\left(y_t \mid \mathcal{F}^{(i)}_{Z:t_k}\right) = V\left(Z_t^2 \mid \mathcal{F}^{(i)}_{Z:t_k}\right) = \sigma^4_{z:t|t_k} V\left\{\left(Z_t/\sigma_{z:t|t_k}\right)^2 \mid \mathcal{F}^{(i)}_{Z:t_k}\right\} = \sigma^4_{z:t|t_k}\left[E(U_t^4) - E^2(U_t^2)\right],$$

higher moments of U_t are expressed in terms of central moments. Thus, expected conditional variance of y_t is given by

$$E\left\{V\left(y_t \mid \mathcal{F}_{Z:t_k}\right) \mid \mathcal{F}_{y:t_k}\right\} = \frac{\sigma^4_{Z:t|t_k}\left[exp\left\{-.5\sigma_0^{-2}\left(\sqrt{y_{t_k}}-\alpha\right)^2\right\}\left(3+6\mu^2_{1:Z}+\mu^4_{1:Z}-\mu_1^2\right)+exp\left\{-.5\sigma_0^{-2}\left(-\sqrt{y_{t_k}}-\alpha\right)^2\right\}\left(3+6\mu^2_{2:Z}+\mu^4_{2:Z}-\mu_2^2\right)\right]}{exp\left\{-.5\sigma_0^{-2}\left(\sqrt{y_{t_k}}-\alpha\right)^2\right\}+exp\left\{-.5\sigma_0^{-2}\left(-\sqrt{y_{t_k}}-\alpha\right)^2\right\}}. \quad (16)$$

Finally, obtaining variance of conditional expectation, i.e. $E\left(y_t \mid \mathcal{F}^{(i)}_{Z:t_k}\right)$ from eq. (15), and evaluating conditional variance of y_t given $\mathcal{F}_{y:t_k}$, we get

$$V\left(y_t \mid \mathcal{F}_{y:t_k}\right) = E\left\{V\left(y_t \mid \mathcal{F}_{Z:t_k}\right) \mid \mathcal{F}_{y:t_k}\right\} - E^2\left(y_t \mid F_{y:t_k}\right). + \frac{\sigma^4_{Z:t|t_k}\left[exp\left\{-.5\sigma_0^{-2}\left(\sqrt{y_{t_k}}-\alpha\right)^2\right\}\mu_1^2+exp\left\{-.5\sigma_0^{-2}\left(-\sqrt{y_{t_k}}-\alpha\right)^2\right\}\mu_2^2\right]}{exp\left\{-.5\sigma_0^{-2}\left(\sqrt{y_{t_k}}-\alpha\right)^2\right\}+exp\left\{-.5\sigma_0^{-2}\left(-\sqrt{y_{t_k}}-\alpha\right)^2\right\}}. \quad (17)$$

It may be noted that, $V\left(y_t \mid \mathcal{F}_{y:t_k}\right)$ depends on past growth data thus conforming to LRSDE model given in eq. (10).

3. AN ILLUSTRATION

As an illustration, pig growth data, reported in Das (2015) and collected at the piggery farm of ICAR-Indian Veterinary Research Institute, Izatnagar, Bareilly, India, are considered for data analysis. An attempt is made to fit various types of Richards growth models to this data and to study their relative performances through computations of goodness-of-fit and forecast accuracy criteria. To this end, variance stabilization of the error terms in TRNS model is carried out by square-root transformation of raw data for weights of 210 pigs. The weights of each pig are observed at ages 0,1,2,…,8 months and thereafter at ages of 12, 16, 20, and 24 months. As discussed in Section 2, under the condition that error in y_t is proportional to λ_t, TRNS model is fitted to the data. Since these are equispaced for the age ranging from month zero up to eighth, variances of the error terms in the above model are estimated by computing observed variances, viz. $\hat{\sigma}_t^2$ of estimated error series $\{\hat{\varepsilon}_{t,j}: 1 \le j \le 210\}$, $t = 0,1,\ldots,8$, and the same are reported in Table 1. From this table, it is noticed that the error variances remain more or less constant in case of TRNS model. Therefore, from the viewpoint of inference about interval estimates of parameters of TRNS model, an attempt is made to estimate Kernel densities of estimated error distributions from Richards growth model at various time-epochs ranging from zero to eight and some of them are exhibited in Figs. 1 and 2. It is observed that error distribution resembles Gaussian distribution only for TRNS model. Further, from eq. (5), parameters of Richards nonlinear growth model are estimated by using NLIN procedure available in SAS package, Ver. 9.4. Subsequently, non-parametric Run test is applied to estimated residual series and it is found that all the calculated values are coming out to be less than 1.96, implying thereby that the assumption of independence of error terms is not violated at 5% level of significance. It may be highlighted that this assumption is violated at 5% level of significance for fitted untransformed Richards nonlinear statistical model given by eq. (4). Further, using the methodology discussed in Section

2, interval estimates of parameters, viz. carrying capacity K and growth rate r are obtained for all the 210 pigs and the same, to save space, are reported in Table 2 for randomly selected 07 pigs only. From Table 2, it is observed that the interval estimates of carrying capacity are not capable to describe observed maximum weights of pigs under consideration.

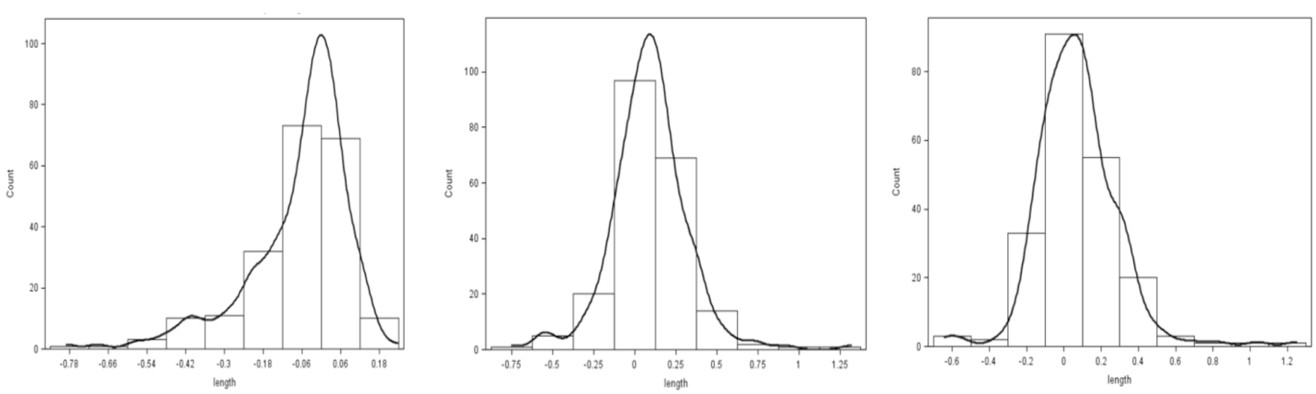
Table 1. Estimated variances of TRNS model

Age	Untransformed	Transformed
0	0.03	0.002
1	0.02	0.004
2	0.06	0.007
3	0.05	0.005
4	0.07	0.006
5	0.06	0.005
6	0.11	0.008
7	0.10	0.003
8	0.05	0.005

Here, $N = 1890$, $n = 9$, $p = 3$, and calculated value of F, using eq. (8), is 55.57, which is significant at 5% level of significance, implying thereby that there is a need to employ SDE approach. Rejection of null hypothesis is plausibly caused by an unpredictably varying environment due to period of bad nutrition, bout of sickness, and so on, which may affect growth patterns for some time into the future; these may not just be perturbations of the current measurements. Therefore, using eq. (9), parameters of LRSDE model are estimated by maximizing likelihood function, which is the product of conditional distributions of Z_t given Z_{t-1}. Subsequently, estimated conditional mean and variance of y_t given $\mathcal{F}_{y:t_k}$ are computed to study

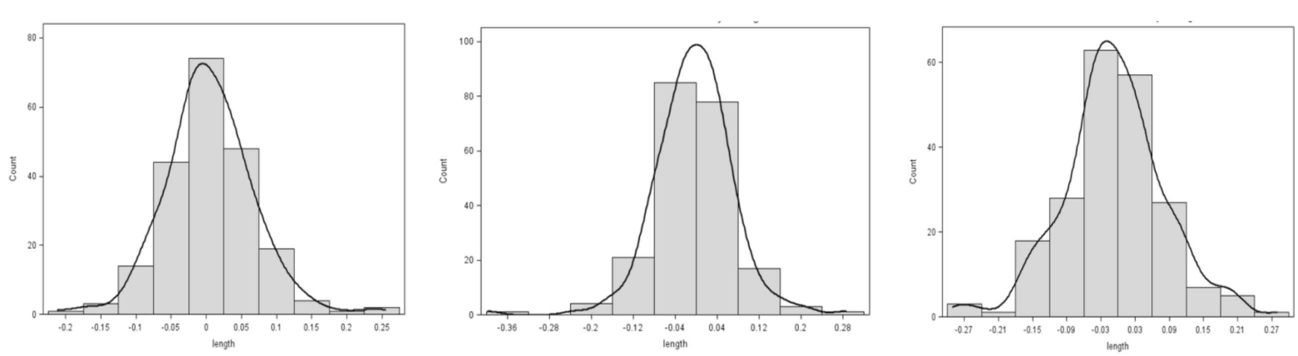
Table 2. Interval estimation for carrying capacity and growth rate parameters for randomly selected 7 pigs using TRNS model

Pig Nos.	Carrying capacity		Growth rate	
	Lower limit	Upper limit	Lower limit	Upper limit
1	11.90	23.00	0.08	0.16
2	09.53	14.70	0.13	0.24
3	07.14	23.37	0.04	0.26
4	10.80	24.34	0.07	0.20
5	17.03	29.00	0.09	0.17
6	14.40	31.41	0.08	0.19
7	10.41	16.60	0.15	0.30



(a) (b) (c)

Fig. 1. Kernel density estimates of error distribution under untransformed TRNS model at: (a) age 0 months, (b) age 3months, and (c) age 8 months



(a) (b) (c)

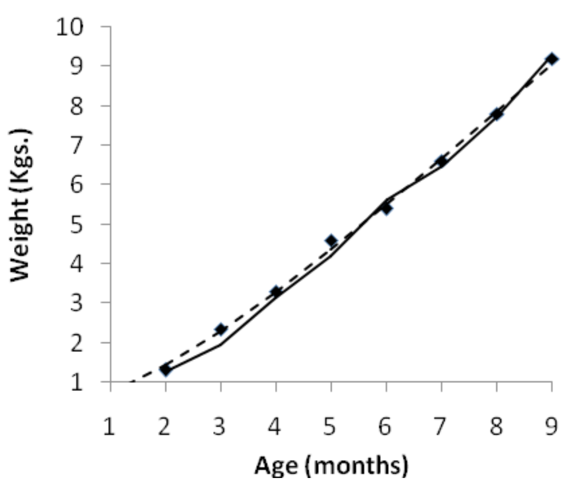
Fig. 2. Kernel density estimates of error distribution under TRNS model at: (a) age 1 month, (b) age 4 months, and (c) age 6 months

fitting and forecast performance for various models and the same are reported in Table 3. It is to be noted that, estimates of parameters r and α are used in eq. (7) to estimate relative growth rate $(1/y_t)\,dy_t/dt$ for evaluating the estimated variance of age specific error term $\varepsilon_t = 2\eta_t Z_t^{-1}$ for the model given by eq. (10). To this end, predicted values of Z_t at missing points, viz. at ages 13,14,15,17,18,19,21,22,23 are obtained and used to calculate naïve estimate of relative growth rate. The estimated variance of ε_t is reported in Table 3. It is noticed that the variance of estimated variance of ε_t becomes stable due to the fact that said variance during initial age up to 10 months is computed as 0.002, whereas it is computed only as 3.3×10^{-7} during later ages till age 24 month. Therefore, stochastic relative growth model with error term having constant variance is capable to explain the observed growth data. Also, the age-specific variance of y_t is computed, which has a non-decreasing trend, thereby explaining the need to consider LRSDE model given by eq. (12).

Table 3. Estimated variances

Age	Variance	Age	Variance
1	0.14495	13	0.00005
2	0.02930	14	0.00022
3	0.01544	15	0.00082
4	0.01268	16	0.00096
5	0.00148	17	0.00010
6	0.00424	18	0.00009
7	0.00270	19	0.00008
8	0.00769	20	0.00575
9	0.00009	21	0.00004
10	0.00199	22	0.00004
11	0.00222	23	0.00018
12	0.00113	24	0.00457

The fitted and four-step ahead forecast values for pig weights at ages 12, 16, 20, and 24 months for fitted TRNS model are obtained using first-order Taylor series approximation of $y_t = g^{-1}(Z_t) = Z_t^2$ and those for fitted LRSDE model are obtained using eq. (15). To save space, the same are reported for 2 pigs in Table 4. Further, for fitted TRNS and LRSDE models, the standard deviations of prediction errors are respectively computed using Taylor series approximation and formula of conditional variances given by eq. (17) for observed as well as 4 hold-out data points. These standard deviations are also reported within brackets ( ) in Table 4. To get a visual idea, the fitted and four-step ahead forecast values for the fitted models are exhibited in Figs. 3 and 4. Evidently, the fitted values by both the models are seen to be quite close to actual data, implying thereby that both the TRNS an LRSDE models are able to describe the given data satisfactorily. However, it may be pointed out that that fitting of TRNS model is a curve fitting approach which attempts merely to minimize the distance between observed and fitted values. On the other hand, the advantage of LRSDE model is that it is mechanistically developed by modelling the dependent error processes given by eq. (12).



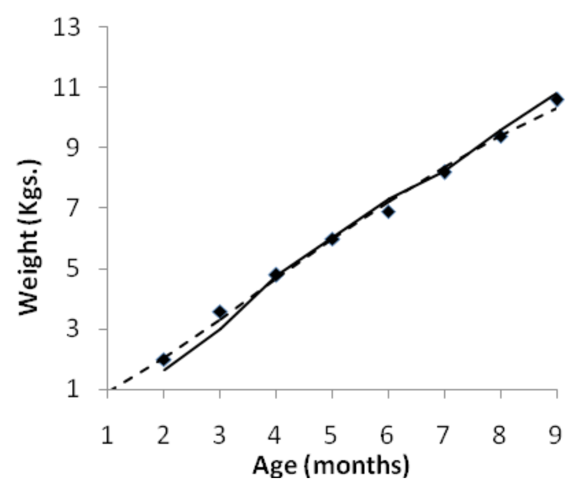


Fig. 3. Comparison of fitting performance of LRSDE model along with data (◆) for Pig 1 and Pig 2 (Solid lines indicate LRSDE model, Dotted lines indicate TRNS model).

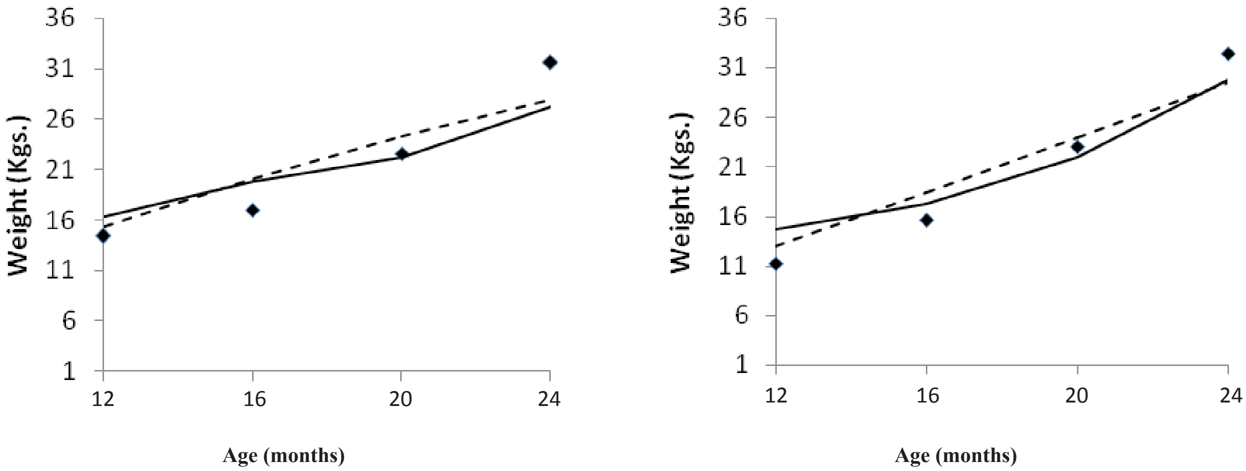


Fig. 4. Comparison of forecast performance of LRSDE model along with data (◆) for Pig 1 and Pig 2 (Solid lines indicate LRSDE model, Dotted lines indicate TRNS model).

Table 4. Fitted and forecast values of pig data for various models

Age	Pig 1	Pig 2	TRNS model				LRSDE model			
			Pig 1		Pig 2		Pig 1		Pig 2	
			Fitting	Forecast	Fitting	Forecast	Fitting	Forecast	Fitting	Forecast
0	0.9	0.8	0.93 (0.17)	-	0.71 (0.05)	-	-	-	-	-
1	2.0	1.3	2.05 (0.28)	-	1.42 (0.07)	-	1.66 (0.08)*	-	1.25 (0.05)	-
2	3.6	2.4	3.35 (0.46)	-	2.29 (0.12)	-	2.99 (0.15)	-	1.93 (0.08)	-
3	4.8	3.3	4.69 (0.64)	-	3.29 (0.17)	-	4.77 (0.24)	-	3.14 (0.13)	-
4	6.0	4.6	6.00 (0.82)	-	4.37 (0.22)	-	6.06 (0.30)	-	4.20 (0.17)	-
5	6.9	5.4	7.24 (0.98)	-	5.52 (0.28)	-	7.32 (0.37)	-	5.62 (0.24)	-
6	8.2	6.6	8.37 (1.14)	-	6.69 (0.34)	-	8.25 (0.41)	-	6.48 (0.27)	-
7	9.4	7.8	9.40 (1.28)	-	7.88 (0.40)	-	9.59 (0.48)	-	7.76 (0.33)	-
8	10.6	9.2	10.31 (1.41)	-	9.08 (0.46)	-	10.80 (0.54)	-	9.30 (0.38)	-
12	14.4	11.2	-	15.32 (409)	-	13.04 (267)	-	16.27 (2.75)	-	14.75 (2.33)
16	17.0	15.6	-	20.04 (535)	-	18.39 (377)	-	19.88 (3.36)	-	17.30 (2.69)
20	22.6	23.0	-	24.25 (647)	-	23.94 (491)	-	22.26 (3.76)	-	21.87 (3.46)
24	31.6	32.4	-	27.87 (744)	-	29.49 (605)	-	27.21 (4.60)	-	29.70 (4.70)

*Figures in parentheses indicate corresponding standard deviations of prediction errors

Undoubtedly, the main purpose of developing a model is to make reliable and accurate forecasts. To this end, the Root mean square error values for four-step ahead forecasts in respect of 7 randomly selected pigs are computed and the same are reported in Table 5. A perusal of this table shows that the fitted LRSDE model has performed better than the fitted TRNS model for the data under consideration. Finally, the interval estimates of carrying capacities and growth rates for fitted LRSDE model in respect of all the 210 pigs are computed by extensive simulation of trajectories of Z_t, but to save space, the results for 07 randomly selected pigs are reported in Table 6. Evidently, a heartening aspect of fitted LRSDE model is that the estimated intervals of carrying capacities are able to contain the actual weights of pigs at the age of 24 months. In view of all this, it may be concluded that LRSDE model is nt only best for forecasting given data but is also capable of satisfying other desirable features.

Table 5. Root mean square error for four-step ahead forecasting for various models

Pig Nos.	TRNS model	LRSDE model
1	06.68	3.37
2	05.13	4.20
3	04.70	4.54
4	10.35	5.34
5	08.01	6.56
6	03.40	3.33
7	14.22	5.44

Table 6. Interval estimation of carrying capacity and growth rate using LRSDE model

Pig Nos.	Carrying capacity		Growth rate	
	Lower limit	Upper limit	Lower limit	Upper limit
1	25.22	60.00	0.02	0.07
2	22.32	50.01	0.02	0.09
3	24.41	57.98	0.03	0.08
4	28.03	65.26	0.02	0.07
5	32.45	68.94	0.05	0.09
6	25.11	46.66	0.05	0.10
7	28.87	69.78	0.02	0.10

4. CONCLUDING REMARKS

Purpose of this article is to develop the methodology for application of Richards four-parameter model in random environment. However, a particular value of the fourth parameter, viz. *m* is considered. Work is in progress to extend this type of research work for general value of *m*, and the same shall be reported separately in due course of time. The possibility of developing the methodology when the noise term is coloured (Behera and O'Rourke 2008) also needs to be explored.

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APPENDIX

Bootstrap estimation of variances of estimates of carrying capacity and growth rate

```
proc iml;
do i=k1 to k2; /*k1= random positive value, k2= k1+size_of_data*/
seed = i;
c = j(size_of_column-1,1,seed);
b = normal(c);
b1=b1||b;
end;
print b1;
quit;
%macro data;
proc optmodel;
/*n1=last age for equi-spaced data, n2= serial number of last data point */
set l={1..n2};
set j=2..n1;
set k=n1+1..n2;
number y{l};
read data abc_&kk. into [_n_] y;
number n init n2+1;
var z{1..3}>=0;
max f=sum{i in j}log((sqrt((z[1]/(2*z[2]))*(1-exp(-2*z[2])))**(-1)))-sum{i in j}(((z[1]/z[2])*(1-
```

```
exp(-2*z[2])))**(-1))
*((y[i]-(z[3]+(y[i-1]-z[3])*exp(-
z[2])))**2)+(sum{i in k}log((sqrt((z[1]/
(2*z[2]))*(1-exp(-8*z[2])))**(-1)))
-sum{i in k}(((z[1]/z[2])*(1-exp(-8*z[2])))**(-1))
*((y[i]-(z[3]+(y[i-1]-z[3])*exp(-4*z[2])))**2));
solve;
print z;
run;
quit;
%mend;
%macro iml;
proc  iml;
x={simulated data};
%do  i=1 %to size_of_data;
y1=x[,&i];
y_&i.=(y1);
print y_&i.;
varnames={y};
create abc_&i. from y_&i.[colname=varnames];
append from y_&i.;
close  abc_&i.;
%end;
create x1 from x;
append from x;
close  x1;
%do kk=1 %to size_of_data;
%data;
%end;
proc iml;
%do kk=1 %to size_of_data;
 use abc_&kk.;
read all into y_&kk.;
use parms_&kk.;
read all into z_&kk.;
use x1;
read all into x;
zz=z_&kk.[,2];
zz1=zz1||zz;
%end;
print zz1;
%mend;
%iml;
proc iml;
varz2=variance_of_growth_rate;
varz3=variance_of_carrying_capacity;
z={estimates};
do i=1 to size_of_data;
lb=z[2][i]-sqrt(varz2);
ub=z[2][i]+sqrt(varz2);
print lb;
print ub;
end;
do i=1 to size_of_data;
lb=(z[3][i]-1.96*sqrt(varz3))**2;
ub=(z[3][i]+1.96*sqrt(varz3))**2;
print lb;
print ub;
end;
quit;
```

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