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Three Symbol Partially Balanced Arrays of Strength (2m +1)

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SUMMARY

Using image method of Dey *et al.* (1972) on a tactical configuration $(\alpha - \beta - k - v)$ converted into design parameters by standard relationship, a three symbol PB array of strength $(2m + 1)$ has been constructed. In view of this, an example with PB array in three symbols of strength 5 has been given. A catalogue of two new designs that can be obtained through the PB array has also been included. Of two, one is useful for obtaining new design for practical situations, an actual example of intercropping experiments with 9 intercrops has been added.

Key words: Tactical configuration, Partially Balanced (PB) arrays, Balanced incomplete block (BIB) design, Doubly balanced incomplete block (DBIB) design, Strength.

1. INTRODUCTION

A new class of arrays called partially balanced arrays, was first introduced and studied by Chakravarti (1956). He obtained some two symbol (2 level) PB arrays by omitting suitably certain assemblies from an orthogonal array. Chakravarti (1961) subsequently gave a further method of construction of these arrays involving six symbols. Bose and Srivastava (1964) have shown certain important principal submatrices of the 'information matrix' corresponding to a balanced fractional factorial design (i.e. a PB array) belong to the linear associative algebra generated by certain well known partially balanced association schemes. These algebra have been proved very helpful in certain statistical studies given by Srivastava and Chopra (1971a). Rafter and Seiden (1974) have found the bounds on the maximum possible number of rows and with the problem of constructing PB arrays for given sets of parameters. Rafter (1971) and Srivastava (1972) have rightly pointed out that the PB arrays give a mathematically challenging field of research which unites various branches of the combinatorial theory of design of experiments. Further, Sinha and Nigam (1983) and Nigam (1985) constructed a series of $(n + 1)$ symbol PB arrays of strength two from regular group divisible designs.

Dey *et al.* (1972) have constructed PB arrays of strength two and three with three symbols using balanced incomplete block (BIB) and doubly balanced incomplete block (DBIB) designs. A tactical configuration, introduced by Sprott (1955) is a generalised structure of a balanced incomplete block design. Sharma and Chandak (1999) obtained a tactical configuration of order $(2m + 1)$ from a tactical configuration of order $2m$. An attempt has been made to construct a three symbol PB array of strength $(2m + 1)$ using the method of Dey *et al.* (1972) on a tactical configuration converted into design parameters by standard relationship.

2. DEFINITIONS AND NOTATIONS

Partially Balanced (PB) Arrays

Let **A** be an v x b matrix, with elements 0, 1, 2, ..., $s - 1$. Consider the $\mathbf{s}^t$ ordered t-plet $(x_1, x_2, \ldots, x_t)$ that can be formed from a t-rowed submatrix of **A** and let there be associated a positive integer $\mu(x_1, x_2, \ldots, x_t)$ that is invariant under permutations of $x_1, x_2, \ldots, x_t$. If for every t-rowed submatrix of **A** the $\mathbf{s}^t$ ordered t-plets $(x_1, x_2, \ldots, x_t)$ occur $\mu(x_1, x_2, \ldots, x_t)$ times, the matrix **A** is called a partially balanced array of strength t in b assemblies with v constraints, s symbols and the specified $\mu(x_1, x_2, \ldots, x_t)$ parameters.

The set of all permutations of $x_1, x_2, \ldots, x_t$ of an array of strength t in s symbols will be called the index set of the array and will be denoted by $\Lambda_{s,t}$. The array of **A** will be represented as the PB array (v, b, s, t) with index set $\Lambda_{s,t}$.

Tactical Configuration

Given a set Ω of v elements, and given positive integers $k, \beta\ (\beta \le k \le v)$ and α, we denote by a tactical configuration $c(\alpha - \beta - k - v)$ a system of blocks (subsets of Ω), having k elements each and such that every subset of Ω having β elements is included in exactly α blocks. If $\alpha = 1$, then the configuration is called the Steiner system i.e., it is a complete $(1 - \beta - k - v)$ configuration of v elements arranged in blocks of k so that each set of β elements occurs exactly once (see also Carmichael (1956)).

The symbol λ_t denotes the frequency of the number of blocks in which any t treatments a, b, c,..., occur together.

It is obvious that $t = 1, 2, \ldots \beta$, $\lambda_\beta = \alpha$, and $\lambda_1 = r$ (replication)

Sharma and Chandak (1999) have shown that a configuration of order $(2m + 1)$ can always be constructed for all positive integral values of m.

Let μ_{ijk}^{fgh} denote the frequency of the t-plet in the $t \times b\ (t \le v)$ sub-array of the $b \times v$ array in three symbols i, j, k with frequencies f, g, and h respectively such that $f + g + h = t$.

For completeness, the image method of Dey *et al.* (1972) is reproduced below:

Consider a balanced incomplete block (BIB) design with usual parameters v, b, r, k, and λ.

Let $\mathbf{N}\ (= n_{ij})$ be the incidence matrix of this BIB design, where

n_{ij} = 1, if the j^{th} treatment occurs in the i^{th} block

= 0, otherwise

Evidently, **N** is a $b \times v$ array of symbols (0 1). Let any assembly of this array be denoted by a row vector $\mathbf{z} = (z_1, z_2, \ldots, z_v)$, $z_i = 0$ or 1.

Then they defined the "images" of **z** as **z***, given by $\mathbf{z}^* = (z_1^*, z_2^*, \ldots, z_v^*)$, $z_i + z_i^* \equiv 2 \pmod 3$ for all $i = 1, 2, \ldots, v$. Now, let **M** be a $b \times v$ array of "images" of each of the assemblies of **N**.

3. THEOREM

The columns of $\mathbf{A}'$ when treated as assemblies give rise to a PB arrays with three symbols, $2b$ assemblies and strength $(2m + 1)$ where $\mathbf{A}'$ is given by

$$\mathbf{A}' = [\mathbf{N}'\ \mathbf{M}']$$

and $\mathbf{A}'$ denotes the transpose of **A**.

Proof : The frequency of the ordered t-plet (1, 1, 1, ..., (2m + 1)) *i.e.*

$$\mu_{0\ \ 1\ \ \ \ \ 2}^{0\ \ 2m+1\ \ *}$$

in any t-columned sub-array of **N** is obviously the number of blocks in which any $(2m + 1)$ treatments a, b, c, ..., occur together and is therefore equal to λ_{2m+1} (Sharma and Chandak (1999)). The frequency of the other t-plet (0, 1, 1, ..., 2m) *i.e.*

$$\mu_{0\ \ 1\ \ 2}^{1\ \ 2m\ \ *}$$

in any t-columned sub array of **N** is the number of blocks in which all treatments occur with only one treatment absent. Clearly, the number of such blocks is $\lambda_{2m} - \lambda_{2m+1}$ and similarly the frequency of the blocks of ordered $t-$ plet

$$\mu_{0\ \ \ 1\ \ \ \ 2}^{2\ \ 2m-1\ \ *} \text{ is } \lambda_{2m-1} - 2\lambda_{2m} + \lambda_{2m+1}$$

Proceeding like this

$$\mu_{0\ \ \ 1\ \ \ \ 2}^{3\ \ 2m-2\ \ *}$$

$$= \lambda_{2m-2} - 3C_1\lambda_{2m-1} + 3C_2\lambda_{2m} - \lambda_{2m+1}$$

In the same fashion

$$\mu_{0\ \ \ \ \ 1\ \ \ \ \ \ 2}^{p\ \ 2m-(p-1)\ \ *}$$

$$= \lambda_{2m-(p-1)} - {}^{p}C_1\lambda_{2m-(p-2)} + {}^{p}C_2\lambda_{2m-(p-3)} \cdots$$

$$\ldots (-1)^{p\,p}C_p\, \lambda_{2m+1} \text{ where } p = 0, 1, 2 \ldots, 2m$$

Therefore, the total number of assemblies containing the part or whole of the blocks of the strength $(2m + 1)$ is

$$\sum_{k=1}^{2m+1} (-1)^k\, {}^{2m+1}C_k \lambda_k$$

(see, Sharma and Chandak (1999)) and hence the frequency of the blocks of ordered t-plet not containing a single treatment *i.e.*

$$\mu\begin{matrix}2m+1 & 0 & * \\ 0 & 1 & 2\end{matrix} = b + \sum_{k=1}^{2m+1} (-1)^k\, {}^{2m+1}C_k \lambda_k$$

Since the assemblies of **M** are "images" of those of **N**, it follows that in any t-columned sub-array of **M**, the frequency of the ordered t-plets will be corresponding to **N** *i.e.*, the frequency of the ordered t-plets viz., no factor absent, one factor absent, two factor absent and so on in **N** are:

$$\mu\begin{matrix}0 & 2m+1 & * \\ 0 & 1 & 2\end{matrix},\quad \mu\begin{matrix}1 & 2m & * \\ 0 & 1 & 2\end{matrix},\quad \mu\begin{matrix}2 & 2m-1 & * \\ 0 & 1 & 2\end{matrix},$$

$\ldots,\quad \mu\begin{matrix}p & 2m-(p-1) & * \\ 0 & 1 & 2\end{matrix}$ will give rise in **M**

$$\mu\begin{matrix}* & 2m+1 & 0 \\ 0 & 1 & 2\end{matrix},\quad \mu\begin{matrix}* & 2m & 1 \\ 0 & 1 & 2\end{matrix},\quad \mu\begin{matrix}* & 2m-1 & 2 \\ 0 & 1 & 2\end{matrix},$$

$\ldots,\quad \mu\begin{matrix}* & 2m-(p-1) & p \\ 0 & 1 & 2\end{matrix}$, respectively

Clearly the frequencies

$$\mu\begin{matrix}* & 2m+1 & 0 \\ 0 & 1 & 2\end{matrix} = \lambda_{2m+1}$$

$$\mu\begin{matrix}* & 0 & 2m+1 \\ 0 & 1 & 2\end{matrix} = b + \sum_{k=1}^{2m+1} (-1)^k\, {}^{2m+1}C_k \lambda_k$$

$$\mu\begin{matrix}* & 2m & 1 \\ 0 & 1 & 2\end{matrix} = \lambda_{2m} - \lambda_{2m+1}$$

$$\mu\begin{matrix}* & 2m-1 & 2 \\ 0 & 1 & 2\end{matrix} = \lambda_{2m-1} - 2\lambda_{2m} + \lambda_{2m+1}$$

$$\mu\begin{matrix}* & 2m-(p-1) & p \\ 0 & 1 & 2\end{matrix}$$

$$= \lambda_{2m-(p-1)} - {}^pC_1\lambda_{2m-(p-2)} + {}^pC_2\lambda_{2m-(p-3)} \cdots$$

$\ldots (-1)^p\, {}^pC_p \lambda_{2m+1}$ where $p = 0, 1, 2 \ldots 2m$

Therefore, in the whole array **A**, the frequency of all ordered t-plets are given by

$$\mu\begin{matrix}0 & 2m+1 & 0 \\ 0 & 1 & 2\end{matrix} = \lambda_{2m+1} + \lambda_{2m+1} = 2\lambda_{2m+1}$$

$$\mu\begin{matrix}1 & 2m & 0 \\ 0 & 1 & 2\end{matrix} = \lambda_{2m} - \lambda_{2m+1} = \mu\begin{matrix}0 & 2m & 1 \\ 0 & 1 & 2\end{matrix}$$

$$\mu\begin{matrix}2 & 2m-1 & 0 \\ 0 & 1 & 2\end{matrix} = \lambda_{2m-1} - 2\lambda_{2m} + \lambda_{2m+1}$$

$$= \mu\begin{matrix}0 & 2m-1 & 2 \\ 0 & 1 & 2\end{matrix}$$

$$\mu\begin{matrix}p & 2m-(p-1) & 0 \\ 0 & 1 & 2\end{matrix}$$

$$= \lambda_{2m-(p-1)} - {}^pC_1\lambda_{2m-(p-2)}$$

$$+ {}^pC_2\lambda_{2m-(p-3)} \ldots\ldots (-1)^p\, {}^pC_p\, \lambda_{2m+1}$$

$$= \mu\begin{matrix}0 & 2m-(p-1) & p \\ 0 & 1 & 2\end{matrix}$$

where p = 0, 1, 2 ..., 2m

and

$$\mu\begin{matrix}2m+1 & 0 & 0 \\ 0 & 1 & 2\end{matrix} = b + \sum_{k=1}^{2m+1} (-1)^k\, {}^{2m+1}C_k \lambda_k$$

$$= \mu\begin{matrix}0 & 0 & 2m+1 \\ 0 & 1 & 2\end{matrix}$$

Thus, **A** is a three symbol PB arrays of strength $(2m + 1)$ for all positive integral values of m. The frequency of all other t-plets combinations are zero.

Hence the theorem.

The results of Dey *et al.* (1972) become a particular case when $m = 1$ in this theorem.

4. ILLUSTRATIVE EXAMPLES

Example 4.1

Consider the incidence matrix of the tactical configuration (1-5-6-12) having $v = 12$, $b = 132$, $r = 66$, $k = 6$, $\lambda_2 = 30$, $\lambda_3 = 12$, $\lambda_4 = 4$, $\lambda_5 = 1$ and applying the construction method given in Section 3 of this paper, we get **X** is a PB array ($v = 12, b = 264, s = 3, t = 5$), with index set $\Lambda_{3,5}$.

$$\mu_{012}^{050} = \lambda_5 + \lambda_5 = 2$$

$$\mu_{012}^{140} = \lambda_4 - \lambda_5 = 3, \quad \mu_{012}^{041} = \lambda_4 - \lambda_5 = 3$$

$$\mu_{012}^{230} = \lambda_3 - 2\lambda_4 + \lambda_5 = 5$$

$$\mu_{012}^{032} = \lambda_3 - 2\lambda_4 + \lambda_5 = 5$$

$$\mu_{012}^{320} = \lambda_2 - 3\lambda_3 + 3\lambda_4 - \lambda_5 = 5$$

$$\mu_{012}^{023} = \lambda_2 - 3\lambda_3 + 3\lambda_4 - \lambda_5 = 5$$

$$\mu_{012}^{410} = r - 4\lambda_2 + 6\lambda_3 - 4\lambda_4 + \lambda_5 = 3$$

$$\mu_{012}^{014} = r - 4\lambda_2 + 6\lambda_3 - 4\lambda_4 + \lambda_5 = 3$$

$$\mu_{012}^{500} = b - 5r + 10\lambda_2 - 10\lambda_3 + 5\lambda_4 - \lambda_5 = 1$$

$$\mu_{012}^{005} = b - 5r + 10\lambda_2 - 10\lambda_3 + 5\lambda_4 - \lambda_5 = 1$$

The frequency of other treatment combinations of strength 5 is zero *i.e.*

$$\mu_{012}^{131} = 0 \qquad \mu_{021}^{221} = 0 \qquad \mu_{012}^{311} = 0$$

$$\mu_{012}^{113} = 0 \qquad \mu_{012}^{212} = 0 \qquad \mu_{012}^{122} = 0$$

$$\mu_{012}^{104} = 0 \qquad \mu_{012}^{401} = 0 \qquad \mu_{012}^{203} = 0$$

and $\mu_{012}^{302} = 0$

0	0	0	1	1	1	0	0	1	1	0	1	2	2	2	1	1	1	2	2	1	1	2	1
0	0	1	0	1	0	1	0	1	1	0	1	2	2	1	2	1	2	1	2	1	1	2	1
1	0	0	1	0	1	1	0	0	1	0	1	1	2	2	1	2	1	1	2	2	1	2	1
1	0	0	0	1	0	1	1	0	1	0	1	1	2	2	2	1	2	1	1	2	1	2	1
0	1	1	1	0	0	1	1	0	0	0	1	2	1	1	1	2	2	1	1	2	2	2	1
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0	1	0	0	0	1	0	1	1	1	0	1	2	1	2	2	2	1	2	1	1	1	2	1
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1	0	1	1	1	0	1	0	0	0	0	1	1	2	1	1	1	2	1	2	2	2	2	1
0	1	0	1	1	1	1	0	0	0	0	1	2	1	2	1	1	1	1	2	2	2	2	1

$$X = \begin{array}{cccccccccccc|cccccccccccc}
0&0&0&0&1&1&1&1&1&0&0&1&2&2&2&2&1&1&1&1&1&2&2&1\\
0&1&1&1&0&1&0&0&1&0&0&1&2&1&1&1&2&1&2&2&1&2&2&1\\
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1&0&0&1&1&0&0&1&1&0&0&1&1&2&2&1&1&2&2&1&1&2&2&1\\
1&0&1&0&0&1&1&0&1&0&0&1&1&2&1&2&2&1&1&2&1&2&2&1\\
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1&0&0&1&0&0&1&0&0&1&1&1&1&2&2&1&2&2&1&2&2&1&1&1\\
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0&0&0&1&1&1&0&0&1&0&1&1&2&2&2&1&1&1&2&2&1&2&1&1\\
0&0&1&0&1&0&1&0&1&0&1&1&2&2&1&2&1&2&1&2&1&2&1&1\\
1&0&0&1&0&1&1&0&0&0&1&1&1&2&2&1&2&1&1&2&2&2&1&1\\
1&0&0&0&1&0&1&1&0&0&1&1&1&2&2&2&1&2&1&1&2&2&1&1
\end{array}$$

0	1	0	0	0	1	0	1	1	0	1	1	2	1	2	2	2	1	2	1	1	2	1	1
1	0	1	0	0	0	0	1	1	0	1	1	1	2	1	2	2	2	2	1	1	2	1	1
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1	0	1	0	0	1	0	1	0	0	1	1	1	2	1	2	2	1	2	1	2	2	1	1
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0	0	1	0	1	0	1	0	1	1	1	0	2	2	1	2	1	2	1	2	1	1	1	2
1	1	0	0	1	0	1	0	0	1	1	0	1	1	2	2	1	2	1	2	2	1	1	2
0	1	1	0	1	1	0	0	0	1	1	0	2	1	1	2	1	1	2	2	2	1	1	2
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1	1	0	1	0	0	0	0	1	1	1	0	1	1	2	1	2	2	2	2	1	1	1	2
1	0	0	0	1	1	0	0	1	1	1	0	1	2	2	2	1	1	2	2	1	1	1	2
1	0	1	0	0	1	0	1	0	1	1	0	1	2	1	2	2	1	2	1	2	1	1	2
0	0	0	1	0	1	1	1	0	1	1	0	2	2	2	1	2	1	1	1	2	1	1	2
0	0	1	1	1	0	0	1	0	1	1	0	2	2	1	1	1	2	2	1	2	1	1	2
0	1	0	0	0	0	1	1	1	1	1	0	2	1	2	2	2	2	1	1	1	1	1	2
1	1	0	1	1	0	1	1	0	0	0	0	1	1	2	1	1	2	1	1	2	2	2	2
0	1	1	0	1	1	0	1	1	0	0	0	2	1	1	2	1	1	2	1	1	2	2	2
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1	0	1	1	1	1	0	0	0	0	1	0	1	2	1	1	1	1	2	2	2	2	1	2
0	1	1	1	0	0	1	1	0	0	1	0	2	1	1	1	2	2	1	1	2	2	1	2
0	1	0	1	1	0	1	0	1	0	1	0	2	1	2	1	1	2	1	2	1	2	1	2
0	0	1	0	1	1	1	1	0	0	1	0	2	2	1	2	1	1	1	1	2	2	1	2
1	0	0	1	0	0	1	1	1	0	1	0	1	2	2	1	2	2	1	1	1	2	1	2
0	0	1	1	0	1	0	1	1	0	1	0	2	2	1	1	2	1	2	1	1	2	1	2
1	1	0	0	0	1	1	0	1	0	1	0	1	1	2	2	2	1	1	2	1	2	1	2
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1	1	1	1	0	0	0	0	0	1	1	0	1	1	1	1	2	2	2	2	2	1	1	2
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0	1	1	0	0	1	1	0	0	1	1	0	2	1	1	2	2	1	1	2	2	1	1	2
1	0	1	0	0	1	1	0	1	1	0	0	1	2	1	2	2	1	1	2	1	1	2	2
0	0	0	0	1	1	1	1	1	1	0	0	2	2	2	2	1	1	1	1	1	1	2	2
1	1	0	0	1	1	0	1	0	1	0	0	1	1	2	2	1	1	2	1	2	1	2	2

0	1	1	1	0	0	1	1	0	1	0	0
0	0	1	0	1	1	1	1	0	1	0	0
0	0	1	1	0	1	0	1	1	1	0	0
1	0	1	1	1	0	0	1	1	0	0	0
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0	1	0	1	1	0	1	0	1	1	0	0
1	0	0	1	0	0	1	1	1	1	0	0
1	1	0	0	0	1	1	0	1	1	0	0

2	1	1	1	2	2	1	1	2	1	2	2
2	2	1	2	1	1	1	1	2	1	2	2
2	2	1	1	2	1	2	1	1	1	2	2
1	2	1	1	1	2	2	1	1	2	2	2
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1	2	1	1	1	1	2	2	2	1	2	2
2	1	2	1	1	2	1	2	1	1	2	2
1	2	2	1	2	2	1	1	1	1	2	2
1	1	2	2	2	1	1	2	1	1	2	2

Example 4.2

Let us consider BIB design $v = b = 3$, $r = k = 1$, $\lambda_2 = 0$, so that N′ of Example 4.1, can be made. Taking the images of N′ as M′ using $z_i + z_i^* \equiv 2 \pmod 3$ for all $i = 1, 2, \ldots, v$ treatments. The blocks are given below:

$$A' = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right)$$

The combinatorial arrangements, in particular, orthogonal and partially balanced arrays of specified strength t are used in the construction of balanced symmetrical and asymmetrical confounded factorial experiments, multifactorial designs (fractional replications) and so on (Rao ((1947), (1949)) and Nair and Rao (1948)). Partially balanced arrays satisfy the same properties as orthogonal arrays when used as fractional replicated factorial designs in terms of estimability of main effects and interactions, but the estimates, of main effects and interactions may have different precisions besides being correlated. The construction and use of such designs have been indicated in Chakravarti ((1956), (1961), (1963)) and extensively investigated by Srivastava (1972), Srivastava and Anderson (1970) and Srivastava and Chopra ((1971a), (1971b), (1971c), (1973)) in the special case $s = 2$, *i.e., s* has two symbols 0 and 1.

A catalogue of two new designs that can be obtained through the PB arrays has been given below:

1. *The **N′** and its images **M′** are PB arrays of strength $(2m + 1)$ with three symbols (0, 1, 2). In particular, Example 4.1 is a PB array of strength 5 with 3 symbols with index set $\Lambda_{3,5}$ constructed by author in the present paper.
2. **The constructed PB array in the present paper can be used for conducting intercroping experiments when the intercrops are sub-divided into various groups based on agronomic practices including main crop assuming that

some of the interaction of intercrops are negligible. We construct design for experiments where each plot consists of main crop p and q intercrops, such that each of these intercrops is selected from a group of r intercrops following Rao and Rao (2001).

Now, let us consider an intercropping experiment using a main crop p and 9 intercrops where the intercrops are partitioned into three groups Q_1, Q_2 and Q_3 with 3 in each group viz., Q_1= [1, 2, 3], Q_2= [4, 5, 6] and Q_3 = [7, 8, 9]. Let us designate the symbols 0,1,2, of first row of PB array with intercrops 1, 2, 3 of Q_1, second row with intercrops 4, 5, 6 of Q_2 and third row with intercrops 7, 8, 9 of Q_3. Considering the column of the array as the plots of the intercrop experiment in addition to the main crop 'p' in each plot. The resulting intercropping experiment will consist of the following 6 plots:

(p, 2, 4, 7), (p, 1, 5, 7), (p, 1, 4, 8)

(p, 2, 6, 9), (p, 3, 5, 9), (p, 3, 6, 8)

It is to be noted that this method provides intercropping design with one main crop and nine intercrops divided into three groups of three intercrops each.

In the context of an actual example of intercropping experiment, Pandey *et al.* (2003) have studied the effect of maize (*Zea mays L.*) based intercropping systems on maize yield as main crop and six intercrops viz., pigeonpea, sesamum, groundnut, blackgram, turmeric and forage meth by conducting an experiment during the rainy seasons of 1998 and 1999 at the research farm of Rajendra Agricultural University, Pusa, Samastipur (Bihar). The experiement consisting of 6 intercrops with one main crop was conducted in randomized complete block design with 4 replications. Maize was sown at 75 cm row spacing in sole as well as in intercropping on 26 and 22 June, respectively, in the first and second year of experimentation. One row of pigeonpea at distance of 75 cm and 2 rows of other intercrops at 30 cm distance were accommodated between 2 rows of maize. The intra-row spacing of 30, 30, 10, 15, 10 and 15 cm were maintained by thinning for 6 intercrops.

* New arrays

** New designs for conducting intercropping experiments

The PB array mentioned in Example 4.2 can be used for intercropping experiment for research purposes including three more intercrops viz., greengram, pearlmillet and soybean in addition to the above intercrops.

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