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Optimal Nested Row-Column Designs

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SUMMARY

The universal optimality of non-proper block designs with nested rows and columns is studied under the usual homoscedastic model. Some general methods of construction of universally optimal non-proper block designs with nested rows and columns are given. A catalogue of universally optimal proper/non-proper block designs with nested rows and columns is included. Two methods of construction of most balanced group divisible designs with nested rows and columns (MBGDN-RC designs) are given along with a catalogue of such designs.

Key Words: Block designs with nested rows and columns, Universal optimality, Variance balance, Most balanced group divisible designs.

1. Introduction

Many a time the experimenters come across situations in which the experimental material cannot or need not be divided into blocks with equal number of experimental units but the elimination of heterogeneity in two directions is desirable within each block and is achieved by forming rows and columns within each block. For example, in agricultural field experiments, particularly experimenting in hilly areas, often it is found that the blocks formed are physically separate fields (say different farmers fields, some blocks in the plains and some in the terraces in the hilly tracks) and two (crossed) sources of variation are included in the analysis of data to account for heterogeneity in two directions within each field. However, it is indeed possible that the fields have unequal number of plots within them and, therefore, the fields cannot or need not be divided into equal number of rows and equal number of columns. In hilly areas when some fields are on the plains and some are on the hilly tracks it may happen that the number of plots within the fields may vary widely. For instance, the number of experimental units in the fields in the plain may be quite high while the number of plots possible on the fields that are on the terraces in the hills may be very small. To obtain efficient designs for these and similar situations is the problem addressed in this paper.

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It is well known that block designs with nested rows and columns are useful in the experimental situations just described. A design $d \in \mathbf{D}(v, b, p_1, p_2, \cdots, p_b, q_1, q_2, \cdots, q_b)$, a class of connected designs in which v treatments denoted by 1, 2, ..., v are to be applied to a set of experimental units arranged in b blocks of sizes $k_1 = p_1q_1$, $k_2 = p_2q_2, \cdots$, $k_b = p_bq_b$, is said to be a block design with nested rows and columns with unequal block sizes or simply a nested row-column design with unequal block sizes.

Earliest known nested row-column designs with equal block sizes are the lattice square designs. Several methods of construction of nested row-column designs with equal block sizes can be found in Srivastava [18], Singh and Dey [15], Agarwal and Prasad [1, 2], Street [19], Ipinyomi and John [11], Cheng [8], Sreenath [16, 17], Uddin [20,21] and Uddin and Morgan [23]. Optimality studies of nested row-column designs with equal block sizes are recent and have been made by Chang and Notz [5, 6, 7], Bagchi *et al.* [3] and Morgan and Uddin [13]. These authors studied the optimality aspects in the class of connected designs D(v, b, p, q), with v treatments arranged in b blocks of common size k = pq. For the non-proper setting, the optimality aspects were studied by Uddin *et al.* [22] who gave several methods of construction of equireplicate balanced nested row-column designs with at most two block sizes and gave a catalogue of designs with $v \le 10, r \le 10, p_1 \le q_1 \le v, p_2 \le q_2 \le v$. Some methods of construction of non-proper variance balanced nested row-column designs are also given by Chakraborty [4].

This paper studies the universal optimality of block designs with nested rows and columns in a wider class of designs $\mathcal{D} = \mathcal{D}(v, b, b^*, n)$ under a linear, additive, fixed effects, homoscedastic model. Here $\mathcal{D}$ is the class of connected block designs with nested rows and columns having v treatments arranged in b blocks, $b^*(= \sum_{j=1}^{b} q_{dj})$ columns and n $(= \sum_{j=1}^{b} p_{dj}q_{dj})$ experimental units, where p_{dj} and q_{dj} are the numbers of rows and columns respectively in the j^{th} block of a design $d \in \mathcal{D}$. In proving the universal optimality, use is made of a sufficient condition of Kiefer [12, Proposition 1] and results of Gupta *et al.* [10] on universally optimal non-proper block designs. Further, some methods of construction of universally optimal non-proper nested row-column designs are given. A catalogue of proper and non-proper balanced block designs with nested rows and columns is given in Table 1. Bagchi *et al.* [3] defined a most balanced group divisible design with nested rows and columns (MBGDN-RC design) and showed that an MBGDN-RC design, whenever it exists, is optimal with respect to all generalized criteria of type 1 and gave a method of construction of MBGDN-RC designs. In this paper two more methods of construction of MBGDN-RC designs are given and a catalogue is provided in Table 2.

2. Preliminaries

In the usual setting of block designs with nested rows and columns, suppose that v treatments are to be compared using a design d in which n experimental units are arranged in b blocks with j^{th} block of size $k_{dj} = p_{dj}q_{dj}$, $\forall$ j = 1(1)b. Let $\mathbf{N}_d$ be the v × b incidence matrix of treatments versus blocks; $\mathbf{N}_{d1}$ the $v \times \sum_{j=1}^{b} p_{dj}$ incidence matrix of treatments versus rows and $\mathbf{N}_{d2}$ the $v\times\sum_{j=1}^{b} q_{dj}$ incidence matrix of treatments versus columns. $\mathbf{Q}_d$ and $\mathbf{P}_d$ denote respectively the $\sum_{j=1}^{b} p_{dj}\times\sum_{j=1}^{b} p_{dj}$ and $\sum_{j=1}^{b} q_{dj}\times\sum_{j=1}^{b} q_{dj}$ diagonal matrices of row sizes and column sizes given by $Q_d = \sum_{j=1}^{b}{}^{+} q_{dj}\mathbf{I}_{p_{dj}}, P_d = \sum_{j=1}^{b}{}^{+} p_{dj}\mathbf{I}_{q_{dj}}$ and $\mathbf{K}_d = \text{Diag}(p_{d1}q_{d1}, \ldots, p_{db}q_{db})$, the b × b diagonal matrix of block sizes. Here $\sum^{+}$ denotes the direct sum of matrices. We also have $\mathbf{R}_d = \text{Diag}(r_{d1},\ldots,r_{dv})$, where r_{di}, i=1(1)v is the replication number of the i^{th} treatment. Under the usual homoscedastic, fixed effects, additive, linear model, the coefficient matrix of reduced normal equations for estimating linear functions of treatment effects using a block design with nested rows and columns is

$$\begin{aligned} \mathbf{C}_d &= \mathbf{R}_d - \mathbf{N}_{d1}\mathbf{Q}_d^{-1}\mathbf{N}'_{d1} - \mathbf{N}_{d2}\mathbf{P}_d^{-1}\mathbf{N}'_{d2} + \mathbf{N}_d\mathbf{K}_d^{-1}\mathbf{N}'_d \\ &= \mathbf{R}_d - \mathbf{N}_{d2}\mathbf{P}_d^{-1}\mathbf{N}'_{d2} - \mathbf{L}_d \end{aligned} \tag{2.1}$$

where $\mathbf{L}_d = \mathbf{N}_{d1}\mathbf{Q}_d^{-1}\mathbf{N}'_{d1} - \mathbf{N}_d\mathbf{K}_d^{-1}\mathbf{N}'_d$. It may be seen easily that $\mathbf{L}_d$ is non-negative definite matrix. The matrix $\mathbf{C}_d$ is symmetric, non-negative definite with row and column sums zero, and for a connected design Rank ($\mathbf{C}_d$) = v - 1. Henceforth, we consider only connected designs. We may allow $p_j > v$ for some or all j=1(1)b.

Let $\mathcal{B} = \mathcal{B}$ (v, b, n) denote the class of all connected block designs with v treatments, b blocks and n experimental units and $\mathbf{B} = \mathbf{B}$ (v, b, $k_1,\ldots,k_b$) denote the class of all connected block designs with v treatments, b blocks and the j^{th} block size as k_j, j = 1(1)b. For a block design $d \in \mathcal{B}$ or $\mathbf{B}$, $\mathbf{N}_d = ((n_{dij}))$ denotes the v × b treatments versus blocks incidence matrix, where n_{dij} denotes the number of times the i^{th} treatment is applied in the j^{th} block, i = 1(1)v , j = 1(1)b.

Definition 2.1: (Gupta *et al.* [10]). A design $d \in \mathbf{B}$ is called a Generalized Binary Balanced Block (GBBB) design if

(i) $n_{dij} = \text{int}(k_j/v)$ or $\text{int}(k_j/v) + 1, \quad \forall\ j = 1(1)b$

(ii) $\sum_{j=1}^{b} n_{dij} n_{di'j}/k_j = \lambda_1$, a constant, $\forall\ i \neq i' = 1(1)v$ (2.2)

Definition 2.2: (Gupta *et al.* [10]). A design $d \in \mathcal{B}$ is called a Binary Balanced Block (BBB) design if

(i) $n_{dij} = 0$ or 1

(ii) $\sum_{j=1}^{b} n_{dij} n_{di'j}/k_{dj} = \lambda_2$, a constant, $\forall\ i \neq i' = 1(1)v$ (2.3)

where $k_{d1}, \ldots, k_{db}$ denote the block sizes of the design $d \in \mathcal{B}$, with $k_{d1} + \ldots + k_{db} = n$. With these definitions, we now introduce generalized binary balanced block designs with nested rows and columns (GBBBN-RC design) and binary balanced block designs with nested rows and columns (BBBN-RC design).

Definition 2.3: A design $d \in \mathbf{D}$ is said to be a GBBBN-RC design if

(i) $\mathbf{L}_d = \mathbf{N}_{d1}\mathbf{Q}^{-1}\mathbf{N}'_{d1} - \mathbf{N}_d\mathbf{K}^{-1}\mathbf{N}'_d = \mathbf{0}$

(ii) $\mathbf{N}_{d2}$ is the incidence matrix of a GBBB design.

Definition 2.4: A design $d \in \mathcal{D}$ is called a BBBN-RC design if

(i) $\mathbf{L}_d = \mathbf{N}_{d1}\mathbf{Q}_d^{-1}\mathbf{N}'_{d1} - \mathbf{N}_d\mathbf{K}_d^{-1}\mathbf{N}'_d = \mathbf{0}$

(ii) $\mathbf{N}_{d2}$ is the incidence matrix of a BBB design.

3. Universally Optimal Designs

In this section we prove the universal optimality of GBBBN-RC designs and BBBN-RC designs over $\mathbf{D}$ and $\mathcal{D}$. We first state, as Theorem 3.1 and Corollary 3.1, the results obtained by Uddin *et al.* [22] on universal optimality of non-proper block designs with nested rows and columns.

Theorem 3.1: Consider a design $d^* \in \mathbf{D}$ satisfying

(i) $\mathbf{L}_{d^*} = \mathbf{N}_{d^*1}\mathbf{Q}^{-1}\mathbf{N}'_{d^*1} - \mathbf{N}_{d^*}\mathbf{K}^{-1}\mathbf{N}'_{d^*} = \mathbf{0}$

(ii) $\mathbf{N}_{d^*2}$ is the incidence matrix of a block design which is universally optimal over $\mathbf{B}(v, \sum_{j=1}^{b} q_j, p_1 1'_{q_1}, \ldots, p_b 1'_{q_b})$.

The design d^*, whenever it exists, is universally optimal over $\mathbf{D}$.

Corollary 3.1: A GBBBN-RC design $d^* \in \mathbf{D}$, whenever existent, is universally optimal over $\mathbf{D}(v, b, p_1, \ldots, p_b, q_1, \ldots, q_b)$.

If the column component design is binary, then using definitions 2.3 and 2.4, the universal optimality can be established in a wider class of designs.

Theorem 3.2: Consider a design $d^* \in \mathcal{D}$ satisfying

(i) $\mathbf{L}_{d^*} = \mathbf{N}_{d^*1}\mathbf{Q}_{d^*}^{-1}\mathbf{N}'_{d^*1} - \mathbf{N}_{d^*}\mathbf{K}_{d^*}^{-1}\mathbf{N}'_{d^*} = \mathbf{0}$

(ii) $\mathbf{N}_{d^*2}$ is the incidence matrix of a block design which is universally optimal over $\mathcal{B}(v, b^*, n)$.

The design d*, whenever it exists, is universally optimal over $\mathcal{D}$.

Corollary 3.2: A BBBN-RC design $d^* \in \mathcal{D}$, whenever existent, is universally optimal over $\mathcal{D}(v, b, b^*, n)$.

Proof: The proof follows from definition 2.3 and Theorem 3.3 of Gupta *et al.* [10].

A design d^* of Theorem 3.1 is also universally optimal over $\mathcal{D}(v, b, b^*, n)$, where $n = \sum_{j=1}^{b} p_j q_j$ provided $p_j \le v$ and $q_j = q_{dj}$, $\forall\, j = 1(1)b$. Similarly, a design d^* of Theorem 3.2 is also universally optimal in $\mathbf{D}(v, b, p_1, \ldots, p_b, q_1, \ldots, q_b)$ if $p_{d^*j} = p_j$ and $q_{d^*j} = q_j$, $\forall\, j = (1)b$. As a consequence of Theorem 3.2 and Corollary 3.2, all the designs hitherto known in the literature as universally optimal over D (v, b, p, q) and $\mathbf{D}$ (v, b, $p_1, \ldots, p_b, q_1, \ldots, q_b$) and binary with respect to columns are also universally optimal over $\mathcal{D}$ (v, b, b^*, n). Therefore, the UNRC designs [GBBBN-RC or BBBN-RC designs] given in Uddin *et al.* [22] are also optimal over $\mathcal{D}$ (v, b, b^*, n). In Table 1 of Uddin *et al.* [22] designs at serial numbers 3 and 5 with respective parameters $v = 4, b_1 = 2, p_1 = 2, q_1 = 2, b_2 = 2, p_2 = 2, q_2 = 4$, $n = 24$ and $v = 4, b_1 = 4$, $p_1 = 2$, $q_1 = 3$, $n = 24$ are optimal over $\mathcal{D}$ (4, 4, 12, 24). Similarly designs at serial numbers 10, 15 and 18 are universally optimal over $\mathcal{D}$ (5, 7, 20, 40). Other designs in Table 1 of Uddin *et al.* [22] can also be checked similarly.

4. *Methods of Construction*

This section gives some methods of constructing universally optimal BBBN-RC (GBBBN-RC) designs. It is easy to verify that for these designs the matrix $\mathbf{L}_d = \mathbf{0}$, and therefore, ignoring block and row classifications and considering columns as blocks, we get a BBB (GBBB) design of

Gupta *et al.* [10]. Hence, using Theorem 3.2 (Theorem 3.1), these designs are universally optimal over $\mathcal{D}$ (**D**).

Method 4.1: Consider a pairwise balanced binary block (PBBB) design with parameters v, $b_1, b_2, \cdots, b_s$, $k_1\mathbf{1}'_{b_1}$, $k_2\mathbf{1}'_{b_2}$,..., $k_s\mathbf{1}'_{b_s}$, λ. For h = 1,2,...,s suppose that there exist row regular Generalized Youden Designs (GYD's) with parameters k_h, p_h, q_h, λ^*_h, where λ^*_h denotes the common off diagonal elements of $\mathbf{NN}'$ and $\mathbf{N}$ is the incidence matrix of the column component block design and $k_h|q_h$, where $x|y$ means x divides y. Arrange the k_h treatments belonging to each of the b_h blocks of size k_h of the PBBB design to form a row regular GYD (k_h, p_h, q_h, λ^*_h). Take the copies of the blocks so obtained of sizes respectively p_1q_1, p_2q_2, ..., p_sq_s in the ratio $\phi_1 : \phi_2 : ... : \phi_s$, where $\phi_h = \theta_h / c$, θ_h = ((L.C.M. of $\lambda_1, \lambda_2, ..., \lambda_s$) p_h / λ_h) and c = HCF of θ_1, θ_2, ..., θ_s. Take the set of all blocks so obtained. The resulting design is a GBBBN-RC design with parameters v, $b'_1 = \phi_1 b_1$, $b'_2 = \phi_2 b_2$, ..., $b'_s = \phi_s b_s$, $p_1\mathbf{1}'_{\phi_1 b_1}$, $p_2\mathbf{1}'_{\phi_2 b_2}$,..., $p_s\mathbf{1}'_{\phi_s b_s}$, $q_1 1'_{\phi_1 b_1}$, $q_2 1'_{\phi_2 b_2}$, ..., $q_s\mathbf{1}'_{\phi_s b_s}$ and is universally optimal over **D**.

If the row regular GYD's are binary with respect to columns then we get a BBBN-RC design with the above parameters that is universally optimal over $\mathcal{D}$ $(v, b = \sum_{h=1}^{s}\phi_h b_h, \sum_{h=1}^{s}\phi_h b_h q_h, \sum_{h=1}^{s}\phi_h b_h p_h q_h)$. Latin Square designs (LSD's) and Youden Square designs (YSD's) are row regular GYD's and, therefore, can also be used either separately or in combination in place of row regular GYD's. In the above procedure using YSD's and LSD's in combination is useful in the situations when there exists a Youden square design (in number of treatments equal to one of the block sizes of a PBBB design) whose number of rows equals the other block size of the PBBB design. In fact, the designs TE1 and TE8 in Uddin *et al.* [22] obtained through trial and error can be obtained using the above procedure. To be clearer, consider the following example.

Example 4.1.1: Consider a PBBB design with parameters $v = 6, b_1 = 2$, $b_2 = 9, k_1 = 3, k_2 = 2$ with block contents as (1, 2, 3), (4, 5, 6), (1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6). Arrange the contents of the blocks of size 3 in a YSD (3, 2, 1) and the treatments from the blocks of size 2 in a Latin square of side 2. Then taking the set of blocks obtained by taking copies of the blocks of sizes 6 and 4 in the ratio 2:1 we get a BBBN-RC design with parameters $v = 6, b_1 = 4, p_1 = 2, q_1 = 3, b_2 = 9, p_2 = 2, q_2 = 2$. This is infact the design TE8 in Uddin *et al.* [22].

Remark 4.1.1: Consider a PBBB design with parameters v, b, $k_1, \ldots, k_b, \lambda$. For j = 1(1) b, arrange the contents of the j^{th} block as a Latin square of order k_j. The resulting design is a BBBN-RC design with parameters v, b, $p_j = q_j = k_j$, $\forall$ j =1(1) b and is universally optimal over $\mathcal{D}$ $(v, b, \sum_{j=1}^{b} k_j, \sum_{j=1}^{b} k_j^2)$.

Several methods of construction of PBBB designs are available in the literature. For a review of methods of construction of PBBB designs, one may refer to Parsad *et al.* [14]. A PBBB design can always be obtained using the following procedure:

For given v, let $B_1, \ldots, B_m$ be a partition of the set V = {1, ... , v} of v treatment labels such that the l^{th} partition B_l is of cardinality t_l (≥ 2), $\forall$ l=1(1)m, $\sum_{l=1}^{m} t_l = v$ and $B_l \cap B_{l'} = \phi$. For each pair of the sets B_l and $B_{l'}$ (l<l'=1(1) m) form all possible pairs of treatments such that one treatment is from B_l and the other from $B_{l'}$. This procedure gives a PBBB design with parameters $v, b_l, k_l = t_l, \forall l = 1(1)m$, $b_{m+1} = \sum_{l=1}^{m} \sum_{l'>l=1}^{m} t_l t_{l'}$, $k_{m+1} = 2$. Now, following the procedure of remark 4.1.1, we get a BBBN-RC design with parameters v, $b_l = 1$, $p_l = t_l$, $q_l = t_l$, l = 1(1)m, $b_{m+1} = \sum_{l=1}^{m} \sum_{l'>l=1}^{m} t_l t_{l'}$, $p_{m+1} = 2$, $q_{m+1} = 2$, $n = \sum_{s=1}^{m+1} b_s p_s q_s$.

This design is universally optimal over $\mathcal{D}$ $(v,\ b = \sum_{s=1}^{m+1} b_s,\ b^* = \sum_{s=1}^{m+1} b_s q_s,\ n = \sum_{s=1}^{m+1} b_s p_s q_s)$. If some of the t_l, say t of them, are equal to one, i.e. $t_{m-t+1} = \ldots = t_m = 1$, then $b_l = 1$, $p_l = t_l$, $q_l = t_l$, l = 1(1)m-t and $b_{m-t+1} = \sum_{l=1}^{m} \sum_{l'>l=1}^{m} t_l t_{l'}$, $p_{m-t+1} = 2$, $q_{m-t+1} = 2$, $n = \sum_{s=1}^{m-t+1} b_s p_s q_s$ and the design is universally optimal over $\mathcal{D}$ $(v,\ b = \sum_{s=1}^{m-t+1} b_s,\ b^* = \sum_{s=1}^{m-t+1} b_s q_s,\ n = \sum_{s=1}^{m-t+1} b_s p_s q_s)$.

Example 4.1.2: There always exists a BBBN-RC design with parameters v = 4, $b_1 = 1$, $p_1 = 3$, $q_1 = 3$, $b_2 = 3$, $p_2 = 2$, $q_2 = 2$. For v = 4 and $t_1 = 3$, $t_2 = 1$ the design is

1 2 3	1 4	2 4	3 4
2 3 1	4 1	4 2	4 3
3 1 2			

which is universally optimal over $\mathcal{D}$ (4, 4, 9, 21). It can easily be seen that fewer experimental units are required than for the designs catalogued in Uddin *et al.* [22] for the same variance of the estimated elementary contrasts.

Method 4.2: Consider a $(k_1,k_2,...,k_s)$ resolvable binary balanced block (BBB) design with parameters $v,b_1,b_2,...,b_s,r_1,r_2,...,r_s,k_1,k_2,...,k_s$. Arrange the contents of the blocks of same set pertaining to blocks of sizes k_h in the form of a $k_h \times v$ array, where v is the number of blocks in one set of blocks of the k_h resolvable portion of the BBB design. The resulting design is a BBBN-RC design with parameters $v, b_1^* = b_1/v, p_1 = k_1, q_1 = v$, ..., $b_s^* = b_s/v$, $p_s = k_s$, $q_s = v$.

Remark 4.2: The block design with nested rows and columns obtained from a $(k_1,k_2,...,k_s)$ resolvable block design subjected to the procedure of method 4.3, retains the same characterization properties as those of the original block design, *e.g.* variance balance, partially balance, efficiency balance, etc.

Example 4.2: Consider the (2,3) resolvable BBB design with parameters $v = 6$, $b_1 = 18$, $b_2 = 6$, $r_1 = 6$, $r_2 = 3$, $k_1 = 2$, $k_2 = 3$ given below with columns as blocks:

1 3 5 6 2 4	1 3 5 2 4 6	1 3 5 4 6 2	1 2 3 4 5 6
2 4 6 1 3 5	4 6 2 1 3 5	6 2 4 1 3 5	3 4 5 6 1 2
			5 6 1 2 3 4

Following the procedure of the above method, we get a BBBN-RC design with parameters $v = 6, b_1 = 3, p_1 = 2, q_1 = 6$, $b_2 = 1$, $p_2 = 3, q_2 = 6$ that is universally optimal over $\boldsymbol{D}$ (6, 4, 24, 54).

Method 4.3: Consider a GBBBN-RC (BBBN-RC) design **D** (v, b_1, ..., b_h, ...,b_s, p_1, ..., p_h, ..., p_s, q_1, ..., q_h, ..., q_s). The row-wise union of any subset or all b_h blocks (h=1(1)s) yields a GBBBN-RC (BBBN-RC) design.

This method is an extension of Theorem 3.2.5 of Bagchi *et al.* [3] and Theorem 9 of Morgan and Uddin [13].

Example 4.3: Consider the BBBN-RC design with parameters $v = 5$, $b_1 = 1$, $b_2 = 7$, $p_1 = 3$, $p_2 = 2$, $q_1 = 3$ and $q_2 = 2$ constructed using remark 4.1.1. The design is given as

1 2 3	1 4	1 5	2 4	2 5	3 4	3 5	4 5
2 3 1	4 1	5 1	4 2	5 2	4 3	5 3	5 4
3 1 2							

which is universally optimal over $\boldsymbol{D}$ (5, 8, 17, 37). Now in block number 2 to 7 with two rows, we take the union of two blocks each and retain the contents of the block number 8. The resulting design is a BBBN-RC design

1 2 3	1 4 1 5	2 4 2 5	3 4 3 5	4 5
2 3 1	4 1 5 1	4 2 5 2	4 3 5 3	5 4
3 1 2				

which is universally optimal over $\mathcal{D}$ (5, 5, 17, 37).

Remark 4.3.1: If there exist s BN-RC designs (Bagchi *et al.* [3]), $\mathbf{D}_h(v, b_h, p_h, q_h)$, $\forall$ h =1(1)s, then $\mathbf{D} = \bigcup_{h=1}^{s} \mathbf{D}_h$ is a GBBBN-RC design which is universally optimal over $\mathbf{D}(v, b = \sum_{h=1}^{s} b_h, p_1 \mathbf{1}'_{b_1}, ..., p_s \mathbf{1}'_{b_s}, q_1 \mathbf{1}'_{b_1}, ..., q_s \mathbf{1}'_{b_s})$. However, when a BN-RC design is binary in columns, then the resulting design is a BBBN-RC design that is universally optimal over $\mathcal{D}(v, b = \sum_{h=1}^{s} b_h, b^* = \sum_{h=1}^{s} b_h q_h, n)$.

Note: Theorem 3.1 of Uddin *et al.* [22] can easily be extended to construct a BBBN-RC design with s distinct block sizes by making s-groups of the blocks of the BIB design (v, b, r, k, λ) and then taking s BNRC designs with parameters of the hth BNRC design as (k, b_h, p_h, q_h) and $\lambda_h = b_h p_h q_h (p_h - 1)/(k(k-1))$ such that $\lambda_1 / p_1 = \lambda_2 / p_2 = \cdots = \lambda_s / p_s$. However, example 1 in that paper seems to be incorrect, as it does not satisfy the condition $\lambda' / p' = \lambda'' / p''$.

A catalogue of BBBN-RC designs for $v \le 10, \bar{r} \le 10$, $p_1 \le q_1 \le 10, p_2 \le q_2 \le 10$ and $p_j q_j \le 20, j = 1,2$, where $\bar{r}$ denotes the average replication number, have been given in Table 1. Some proper block designs with nested rows and columns have also been included.

5. *Methods of Construction of MBGDN-RC Designs*

Due to combinatorial problems, it may not always be possible to get a GBBBN-RC design or a BBBN-RC design or such a design may require a large number of experimental units. Therefore, one has to use partially balanced designs. Bagchi *et al.* [3] showed that a most balanced group divisible design with nested rows and columns (MBGDN-RC design), whenever existent, is optimal according to Type 1 optimality criteria. A MBGDN-RC design is defined as below:

Definition 5.1: A design $d \in D$ is said to be a MBGDN-RC design if

(i) $\mathbf{L}_d = \mathbf{N}_{d*1} \mathbf{Q}^{-1} \mathbf{N}'_{d*1} - \mathbf{N}_{d*} \mathbf{K}^{-1} \mathbf{N}'_{d*} = \mathbf{0}$

(ii) $\mathbf{N}_{d2}$ is the incidence matrix of a most balanced group divisible design.

In this section, we give some methods of construction of MBGDN-RC designs.

Method 5.1: Consider a most balanced group divisible design with parameters $v, b, r, k, \lambda_1, \lambda_2 = \lambda_1 + 1$ and a YSD $(k, p, \lambda = 1)$. Rearrange the contents of the j^{th} block of the most balanced group divisible design as a Youden Square design in p rows and k columns. Repeat this process for all $j = 1,2,\ldots,b$. The resulting design is a MBGDN-RC design with parameters $v^* = v, b^* = b, r^* = rp, p^* = p, q^* = k, \lambda_1, \lambda_2 = \lambda_1 + 1$.

Example 5.1: Consider the group divisible design SR18 with parameters v = 6, b = 4, r = 2, k = 3, m = 3, n = 2, $\lambda_1 = 0, \lambda_2 = 1$. There also exists a Youden square design (3,2,1). Then following the procedure of method 5.1, we get a MBGDN-RC design with parameters $v = 6, b = 4, p = 2, q = 3$. The design is as follows:

1 2 3	1 5 6	2 4 6	3 4 5
2 3 1	5 6 1	4 6 2	4 5 3

Method 5.2: A MBGDN-RC design with parameters $v, b, p = k$, $q = v, \lambda_1, \lambda_2 = \lambda_1 + 1$ can easily be obtained from a k-resolvable most balanced group divisible design with parameters $v, b, r, k, \lambda_1, \lambda_2 = \lambda_1 + 1$, following the procedure of Method 4.3.

Example 5.2: Consider a group divisible design R52 (Clatworthy [9]) with parameters v = 6, b = 18, r = 9, k = 3, m = 2, n = 3, $\lambda_1 = 3, \lambda_2 = 4$. This design is one resolvable and there will be 9-sets of 2 blocks each such that each treatment is replicated once in each group. Now regrouping these sets into 3 groups such that there are 6 blocks within each group, we get a 3-reolvable group divisible design. Following the procedure of the above method, we get a MBGDN-RC design with parameters v = 6, b = 3, p = 3, q = 6, r = 9, $\lambda_1 = 3, \lambda_2 = 4$. The design is given as follows:

1 4 5 3 2 6	3 5 1 2 4 6	1 2 5 3 6 4
2 5 1 6 3 4	1 6 4 3 5 2	3 4 6 2 5 1
3 6 2 4 5 1	4 2 5 6 3 1	6 5 1 4 3 2

A catalogue of MBGDN-RC designs for $v \leq 10, r \leq 10, p \leq q \leq 10$ and $pq \leq 20$ has been given in Table 2. In the tables PBBBD denotes pairwise balanced binary block design, YSD denotes the Youden Square design, S#, SR#, R# denote respectively the singular, semi-regular and regular group divisible designs given in Clatworthy [9]. BMS# is the method from Bagchi *et al.* [3].

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Table 1. Parameters of BBBN-RC designs with $v \le 10, \bar{r} \le 10, p_j \le q_j \le 10$ and $p_j q_j \le 20, j = 1, 2$

No.	v	b_1	p_1	q_1	b_2	p_2	q_2	Source and Method	
1	3	3	2	2	-	-	-	BIBD (3, 3, 2, 2, 1)	BMS 3.2.4
2	4	1	3	3	3	2	2	PBBBD {(1, 2, 3), (1, 4), (2, 4), (3, 4)}	Method 4.1, Remark 4.1.1
3	4	2	3	4	-	-	-	3–resolvable BIBD (4, 8, 6, 3, 4)	Method 4.2
4	5	1	4	4	4	2	2	PBBBD {(1, 2, 3, 4), (1, 5) (2, 5), (3, 5), (4, 5)}	Method 4.1, Remark 4.1.1
5	5	1	3	3	7	2	2	PBBBD {(1, 2, 3), (1, 4), (1,5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}	Method 4.1, Remark 4.1.1
6	5	1	4	4	2	2	4	PBBBD {(1, 2, 3, 4), (1, 5), (2, 5), (3, 5), (4, 5)}	Method 4.1, Remark 4.1.1 and Method 4.3
7	6	1	4	4	9	2	2	PBBBD {(1, 2,3, 4), (1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)}	Method 4.1, Remark 4.1.1
8	6	1	3	3	12	2	2	PBBBD {(1, 2, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)}	Method 4.1, Remark 4.1.1
9	6	1	4	4	3	2	6	PBBBD {(1, 2, 3, 4), (1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)}	Method 4.1 and Method 4.3
10	6	1	3	3	6	2	4	PBBBD {(1, 2, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)}	Method 4.1, Remark 4.1.1 and Method 4.3
11	6	1	3	3	4	2	6	PBBBD {(1, 2, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)}	Method 4.1, Remark 4.1.1 and Method 4.3
12	6	5	2	6	-	-	-	2-Resolvable BIBD (6, 30, 10, 2, 2)	Method 4.2
13	6	3	2	6	1	3	6	2SR6, 3(2, 3)	Method 4.2
14	9	4	2	9	-	-	-	2-Resolvable BIBD (9, 36, 8, 2, 1)	Method 4.2

Table 2. MBGDN-RC designs with parameters $v \le 10$, $r \le 10$, $p \le 10$, $q \le 10$ and $pq \le 20$

No.	v	b	p	q	r	λ_1	λ_2	Source	Method
1	6	4	2	3	4	0	1	SR18, YSD (3, 2, 1)	Method 5.1
2	6	3	3	6	9	3	4	R52	Method 5.2
3	8	8	2	3	6	0	1	R54, YSD (3, 2, 1)	Method 5.1
4	8	4	2	4	2	0	1	–	BMS 3.2.7
5	9	9	2	3	6	0	1	SR23, (3, 2, 1)	Method 5.1