Jour. Ind. Soc. Ag. Statistics
53(3), 2000 : 243–256

On Estimating Square of Population Mean and Population Variance

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(Received : June, 1991)

SUMMARY

Two classes of estimators, one for the square of the population mean μ^2 and other for the population variance σ^2 are proposed and their properties are studied under large sample approximations. Further, sub-classes of optimum estimators in the minimum mean squared sense are found and the results of some estimators considered by various authors are shown to be the special cases of this study.

Key words : Coefficient of variation, Class of estimators, Sub-class of optimum estimators, Efficiency.

1. Introduction

On account of the stability and the fairly accurate known value of coefficient of variation, it has got practical importance in many situations. Thus the problem of estimation of σ^2 reduces to the problem of estimation of μ^2 in such cases. Other instances where μ^2 may be of the parameter of interest may be seen in Govindarazulu and Sahai [2] and Upadhyaya and Singh [7].

Supposing $C^2 = \frac{\sigma^2}{\mu^2}$ to be exactly known, two unbiased estimators of μ^2 are

$$d = \frac{\bar{y}^2}{1+\frac{C^2}{n}} = \bar{y}^2\left[1 - \frac{C^2}{n+C^2}\right] \tag{1.1}$$

and

$$d^* = \frac{s^2}{C^2} \tag{1.2}$$

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where $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$ and $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$ are unbiased estimators of μ and σ^2 based on a random sample $y_1, y_2, ..., y_n$ on size n.

The relative variances of d and d^* are given by

$$RV(d) = E\left[\frac{d - \mu^2}{\mu^2}\right]^2 = \frac{2C^2 (2n + C^2)}{(n + C^2)^2} \tag{1.3}$$

and

$$RV(d^*) = E\left[\frac{d^* - \mu^2}{\mu^2}\right]^2 = \frac{2}{(n-1)} \tag{1.4}$$

As C^2 may not be known in practice, the estimators d and d^* are of little utility. The alternative in such type of situation is to estimate C^2 as follows (Srivastava [6]).

$$\hat{C}_1^2 = \frac{s^2}{\bar{y}^2}\left(1 - \frac{s^2}{n\bar{y}^2}\right)^{-1} \tag{1.5}$$

$$\hat{C}_2^2 = \frac{s^2}{\bar{y}^2}$$

The substitution of $\hat{C}_1^2$ and $\hat{C}_2^2$ in (1.1) and (1.2) leads to formulate some estimators for μ^2. Similarly some estimators for σ^2 are also formulated.

Proceeding on the lines of Srivastava [5] and keeping in view of the form the previous estimators considered by various authors, we propose the following generalized estimators :

(i) For the estimation of μ^2

$$t = \bar{y}^2 f\left(\frac{s^2}{n\bar{y}^2}\right) = \bar{y}^2 f(u) \; ; \; u = \frac{s^2}{n\bar{y}^2}$$

(ii) For the estimation of σ^2

$$t^* = s^2 f\left(\frac{s^2}{n\bar{y}^2}\right) = s^2 f(u) \; ; \; u = \frac{s^2}{n\bar{y}^2}$$

where f(u) is the function of u such that f(0) = 1 satisfying the validity conditions of Maclaurin's (Taylor's) series expansion.

It may be easily seen that, for the estimation of μ^2

(i) the minimum variance unbiased estimator (MVUE) $d_1 = \bar{y}^2 - \frac{s^2}{n}$

(ii) the estimators $d_2 = \bar{y}^2\left(1 + \frac{s^2}{n\bar{y}^2}\right)^{-1}$ (Das [1])

(iii) $d_3 = \bar{y}^2\left[1 + \frac{s^2}{n\bar{y}^2}\left(1 + \frac{s^2}{n\bar{y}^2}\right)\right]^{-1}$ (Pandey [3])

(iv) the estimators $d_4 = \bar{y}^2\left[1 + \frac{s^2}{n\bar{y}^2}\left(1 - \frac{s^2}{n\bar{y}^2}\right)^{-1}\right]^{-1}$

$$d_5 = \bar{y}^2\left[1 + \frac{s^2}{n\bar{y}^2}\left\{1 + \frac{s^2}{n\bar{y}^2}\left(1 - \frac{s^2}{n\bar{y}^2}\right)^{-1}\right\}\right]^{-1}$$

$$d_6 = \bar{y}^2\left[1 + \frac{s^2}{n\bar{y}^2}\left\{1 + \frac{s^2}{n\bar{y}^2}\left(1 + \frac{s^2}{n\bar{y}^2}\right)\right\}\right]^{-1}$$

$$d_7 = \bar{y}^2\left[1 - \frac{s^2}{n\bar{y}^2}\left(1 - \frac{s^2}{n\bar{y}^2}\right)^{-1}\left\{1 + \frac{s^2}{n\bar{y}^2}\left(1 - \frac{s^2}{n\bar{y}^2}\right)^{-1}\right\}^{-1}\right]$$

by Singh [4]; are the special cases of proposed estimator t.

Further, for the estimation of σ^2, the

(i) estimators $d_1^* = s^2 - \frac{s^4}{n\bar{y}^2}$, $d_2^* = s^2\left(1 + \frac{s^2}{n\bar{y}^2}\right)^{-1}$

$$d_3^* = s^2\left[1 + \frac{s^2}{n\bar{y}^2}\left(1 + \frac{s^2}{n\bar{y}^2}\right)\right]^{-1}$$ by Pandey [3]

(ii) and the estimators

$$d_4^* = s^2\left[1 + \frac{s^2}{n\bar{y}^2}\left(1 - \frac{s^2}{n\bar{y}^2}\right)^{-1}\right]^{-1}$$

$$d_5^* = s^2\left[1 + \frac{s^2}{n\bar{y}^2}\left\{1 + \frac{s^2}{n\bar{y}^2}\left(1 - \frac{s^2}{n\bar{y}^2}\right)^{-1}\right\}\right]^{-1}$$

$$d_6^* = s^2\left[1+\frac{s^2}{n\bar{y}^2}\left\{1+\frac{s^2}{n\bar{y}^2}\left(1+\frac{s^2}{n\bar{y}^2}\right)\right\}\right]^{-1}$$

$$d_7^* = s^2\left[1-\frac{s^2}{n\bar{y}^2}\left(1-\frac{s^2}{n\bar{y}^2}\right)^{-1}\left\{1+\frac{s^2}{n\bar{y}^2}\left(1-\frac{s^2}{n\bar{y}^2}\right)^{-1}\right\}^{-1}\right]$$

$$d_8^* = s^2\left[1-\frac{s^2}{n\bar{y}^2}\left(1+\frac{s^2}{n\bar{y}^2}\right)^{-\alpha}\right] \text{ (for } \alpha \text{ being characterizing scalar)}$$

$$d_9^* = s^2\left[1+\frac{ks^2}{n\bar{y}^2}\left(1+\frac{gs^2}{n\bar{y}^2}\right)^{-1}\right]$$

(for k and g being characterizing scalars)

by Singh [4], are the special cases of the proposed estimator t^*.

2. *Bias and Mean Squared Error of t and t^**

In order to derive the bias and mean squared error (MSE) of t and t^* under lagre sample approximations, we write

$$\bar{y} = \mu + U \text{ and } s^2 = \sigma^2 + V$$

where U and V are of order $0(n^{-1/2})$ with $E(U) = E(V) = 0$. Assuming $\left|\frac{U}{\mu}\right| < 1$, expanding t in Taylor's (Maclaurin's) series about the point $u = 0$, with $f'(0)$, $f''(0)$, $f'''(0)$ and $f''''(0)$ being the first, second, third and fourth derivatives respectively and $u^* = hu$, $0 < h < 1$, we have

$$t = \bar{y}^2\left[f(0) + uf'(0) + \frac{u^2}{2!}f''(0) + \frac{u^3}{3!}f'''(0) + \frac{u^4}{4!}f''''(u^*)\right]$$

$$= \bar{y}^2\left[1+\frac{s^2}{n\bar{y}^2}f'(0) + \frac{1}{2n^2}\left(\frac{s^2}{\bar{y}^2}\right)^2 f''(0) + \frac{1}{6n^3}\left(\frac{s^2}{\bar{y}^2}\right)^3 f'''(0) + \frac{u^4}{4!}f''''(u^*)\right]$$

$$= (\mu+U)^2\left[1+\frac{1}{n}\frac{(\sigma^2+V)}{(\mu+U)^2}f'(0) + \frac{1}{2n^2}\left\{\frac{(\sigma^2+V)}{(\mu+U)^2}\right\}^2 f''(0)\right.$$

$$\left. + \frac{1}{6n^3}\left\{\frac{(\sigma^2+V)}{(\mu+U)^2}\right\}^3 f'''(0) + \frac{u^4}{4!}f''''(u^*)\right]$$

$$= \mu^2\left(1+\frac{U}{\mu}\right)^2\left[1+\frac{C^2}{n}\left(1+\frac{V}{\sigma^2}\right)\left(1+\frac{U}{\mu}\right)^{-2} f'(0)\right.$$

$$+\frac{C^4}{2n^2}\left(1+\frac{V}{\sigma^2}\right)^2 \cdot \left(1+\frac{U}{\mu}\right)^{-4} f''(0)$$

$$\left.+\frac{C^6}{6n^3}\left(1+\frac{V}{\sigma^2}\right)^3 \left(1+\frac{U}{\mu}\right)^{-6} f'''(0)+\frac{u^4}{4!} f''''(u^*)\right]$$

$$= \mu^2\left[\left(1+\frac{U}{\mu}\right)^2 + \frac{C^2}{n}\left(1+\frac{V}{\sigma^2}\right) f'(0) + \frac{C^4}{2n^2}\left(1+\frac{V}{\sigma^2}\right)^2 + \left(1+\frac{U}{\mu}\right)^{-2} f''(0)\right.$$

$$\left.+\frac{C^6}{6n^3}\left(1+\frac{V}{\sigma^2}\right)^3 \left(1+\frac{U}{\mu}\right)^{-4} f'''(0)+\frac{u^4}{4!} f''''(u^*)\right]$$

$$= \mu^2\left[1+\frac{2U}{\mu}+\frac{U^2}{\mu^2}+\frac{C^2}{n}\left(1+\frac{V}{\sigma^2}\right) f'(0)\right.$$

$$+\frac{C^4}{2n^2}\left(1-\frac{2U}{\mu}+\frac{2V}{\sigma^2}+\frac{3U^2}{\mu^2}+\frac{V^2}{\sigma^4}-\frac{4UV}{\mu\sigma^2}-\ldots\right) f''(0)$$

$$\left.+\frac{C^6}{6n^3}\left(1+\frac{V}{\sigma^2}\right)^3 \left(1+\frac{U}{\mu}\right)^{-4} f'''(0)+\frac{u^4}{4!} f''''(u^*)\right] \tag{2.1}$$

so that upto the terms of order $0(n^{-3})$, we have

$$E(t) = \mu^2+\mu^2\frac{C^2}{n}\left[1+f'(0)+\frac{C^2}{2n}f''(0)+\frac{C^4}{2n^2}\left\{\left[(3-4\theta)\right.\right.\right.$$

$$\left.\left.\left.+\frac{\gamma_2+2}{C^2}\right]f''(0)+\frac{f'''(0)}{3}\right\}\right] \tag{2.2}$$

where $\theta = \gamma_1/C$ and γ_1, γ_2 are the Pearson's measures of skewness and kurtosis of the population.

From (2.2), the bias of t upto terms of order $0(n^{-3})$ is

$$\text{Bias }(t) = E(t)-\mu^2$$

$$= \mu^2\frac{C^2}{n}\left[1+f'(0)+\frac{C^2}{2n}f''(0)+\frac{C^4}{2n^2}\left\{\left[(3-4\theta)+\frac{\gamma_2+2}{C^2}\right]f''(0)+\frac{f'''(0)}{3}\right\}\right] \tag{2.3}$$

and the relative bias (RB) is

$$RB(t) = \frac{\text{Bias}(t)}{\mu^2} = \frac{C^2}{n}\left[1 + f'(0) + \frac{C^2}{2n} f''(0) + \frac{C^4}{2n^2}\left\{\left[(3-4\theta) + \frac{\gamma_2 + 2}{C^2}\right] f''(0) + \frac{f'''(0)}{3}\right\}\right] \tag{2.4}$$

From (2.1) the mean squared error of t upto terms of order $0(n^{-3})$ is $MSE(t) = E(t-\mu^2)^2$

$$= \mu^4\left[\frac{4U^2}{\mu^2} + \frac{4U^3}{\mu^3} + \frac{U^4}{\mu^4} + \frac{C^4}{n^2}\left(1 + \frac{2V}{\sigma^2} + \frac{V^2}{\sigma^4}\right)\{f'(0)\}^2 + \frac{2C^2}{n}\left(\frac{2U}{\mu} + \frac{U^2}{\mu^2} + \frac{2UV}{\mu\sigma^2} + \frac{U^2V}{\mu^2\sigma^2}\right) f'(0) + \frac{C^4}{n^2}\left(\frac{2U}{\mu} - \frac{3U^2}{\mu^2} + \frac{4UV}{\mu\sigma^2}\right) f''(0) + \frac{C^6}{n^3} f'(0) f''(0)\right]$$

$$= \mu^4 \frac{C^2}{n}\left[4 + 4\theta\frac{C^2}{n} + \frac{C^2}{n}\left\{\frac{\gamma_2}{n} + 3 + (f'(0))^2 + 2f'(0)\right\} + 4\theta\frac{C^2}{n} f'(0) + 2\gamma_2\frac{C^2}{n^2} f'(0) + (\gamma_2 + 2)\frac{C^2}{n^2}\{f'(0)\}^2 + 4(\theta - 1)\frac{C^4}{n^2} f''(0) + \{1 + f'(0)\}\frac{C^4}{n^2} f''(0)\right] \tag{2.5}$$

and the relative mean squared error (RMSE) is

$$RMSE(t) = \frac{MSE(t)}{\mu^4} = \frac{C^2}{n}\left[4 + 4\theta\frac{C^2}{n} + \frac{C^2}{n}\left\{\frac{\gamma_2}{n} + 3 + (f'(0))^2 + 2f'(0)\right\} + 4\theta\frac{C^2}{n} f'(0) + 2\gamma_2\frac{C^2}{n^2} f'(0) + (\gamma_2 + 2)\frac{C^2}{n^2}\{f'(0)\}^2 + 4(\theta - 1)\frac{C^4}{n^2} f''(0) + \{1 + f'(0)\}\frac{C^4}{n^2} f''(0)\right] \tag{2.6}$$

For population having symmetric ($\theta = 0$) and mesokurtic ($\gamma_2 = 0$) distribution the values of relative bias and relative mean squared error further reduce to

$$RB(t) = \frac{C^2}{n}\left[1 + f'(0) + \frac{C^2}{2n} f''(0) + \frac{C^4}{2n^2}\left\{\left[3 + \frac{2}{C^2}\right] f''(0) + \frac{f'''(0)}{3}\right\}\right] \quad (2.7)$$

and

$$RMSE(t) = \frac{C^2}{n}\left[4 + \frac{C^2}{n}\{3 + (f'(0))^2 + 2f'(0)\} + 2\frac{C^2}{n^2}\{f'(0)\}^2 - 4\frac{C^4}{n^2} f''(0) + \{1 + f'(0)\}\frac{C^4}{n^2} f''(0)\right] \quad (2.8)$$

On the similar lines as above, expanding t^* in third order Taylor's (Maclaurin's) series above the point $u = 0$, for $u^* = hu$, $0 < h < 1$, we have

$$t^* = s^2\left[f(0) + uf'(0) + \frac{u^2}{2!} f''(0) + \frac{u^3}{3!} f'''(u^*)\right]$$

$$= s^2\left[1 + \frac{s^2}{n\bar{y}^2} f'(0) + \frac{1}{2n^2}\left(\frac{s^2}{\bar{y}^2}\right)^2 f''(0) + \frac{u^3}{3!} f'''(u^*)\right]$$

$$= (\sigma^2 + V)\left[1 + \frac{1}{n}\frac{(\sigma^2 + V)}{(\mu + U)^2} f'(0) + \frac{1}{2n^2}\left\{\frac{(\sigma^2 + V)}{(\mu + U)^2}\right\}^2 f''(0) + \frac{u^3}{3!} f'''(u^*)\right]$$

$$= \sigma^2\left(1 + \frac{V}{\sigma^2}\right)\left[1 + \frac{C^2}{n}\left(1 + \frac{V}{\sigma^2}\right)\left(1 + \frac{U}{\mu}\right)^{-2} f'(0) + \frac{C^4}{2n^2}\left(1 + \frac{V}{\sigma^2}\right)^2\left(1 + \frac{U}{\mu}\right)^{-4} f''(0) + \frac{u^3}{3!} f'''(u^*)\right]$$

$$= \sigma^2\left[\left(1 + \frac{V}{\sigma^2}\right) + \frac{C^2}{n}\left(1 + \frac{V}{\sigma^2}\right)^2\left(1 + \frac{U}{\mu}\right)^{-2} f'(0) + \frac{C^4}{2n^2}\left(1 + \frac{V}{\sigma^2}\right)^3\left(1 + \frac{U}{\mu}\right)^{-4} f''(0) + \frac{u^3}{3!} f'''(u^*)\right]$$

$$= \sigma^2\left[1 + \frac{V}{\sigma^2} + \frac{C^2}{n}\left(1 - \frac{2U}{\mu} + \frac{2V}{\sigma^2} + \frac{3U^2}{\mu^2} + \frac{V^2}{\sigma^4} - \frac{4UV}{\mu\sigma^2} - \ldots\right) f'(0) + \frac{C^4}{2n^2}\left(1 + \frac{V}{\sigma^2}\right)^3\left(1 + \frac{U}{\mu}\right)^{-4} f''(0) + \frac{u^3}{3!} f'''(u^*)\right] \quad (2.9)$$

so that upto the terms of order $0(n^{-2})$, we have

$$E(t^*) = \sigma^2 + \sigma^2 \frac{C^2}{n}\left[f'(0) + \frac{1}{n}\left\{[(3-4\theta)C^2 + (\gamma_2+2)]\, f'(0) + \frac{C^2}{2} f''(0)\right\}\right] \quad (2.10)$$

From (2.10), the bias of t^* upto terms of order $0(n^{-2})$ is

$$\text{Bias}\,(t^*) = E(t^*) - \sigma^2$$

$$= \sigma^2 \frac{C^2}{n}\left[f'(0) + \frac{1}{n}\left\{[(3-4\theta)C^2 + (\gamma_2+2)]\, f'(0) + \frac{C^2}{2} f''(0)\right\}\right] \quad (2.11)$$

and the relative bias (RB) is

$$RB(t^*) = \frac{\text{Bias}\,(t^*)}{\sigma^2}$$

$$= \frac{C^2}{n}\left[f'(0) + \frac{1}{n}\left\{[(3-4\theta)C^2 + (\gamma_2+2)]\, f'(0) + \frac{C^2}{2} f''(0)\right\}\right] \quad (2.12)$$

From (2.9) the mean squared error of t^* upto terms of order $0(n^{-2})$ is

$$MSE(t^*) = E(t^* - \sigma^2)^2$$

$$= \sigma^4\left[\frac{V^2}{\sigma^4} + \frac{C^4}{n^2}\left(1 - \frac{4U}{\mu} + \frac{4V}{\sigma^2}\right)\{f'(0)\}^2 + \frac{2C^2}{n}\left(\frac{V}{\sigma^2} - \frac{2UV}{\mu\sigma^2} + \frac{2V^2}{\sigma^4}\right) f'(0)\right]$$

$$= \sigma^4\left[\frac{\gamma_2+2}{n} + \frac{C^4}{n^2}\{f'(0)\}^2 + 4\frac{C^2}{n^2}\left\{(\gamma_2+2) - \theta C^2\right\} f'(0)\right] \quad (2.13)$$

and the relative mean squared error (RMSE) is

$$RMSE(t^*) = \frac{MSE(t^*)}{\sigma^4}$$

$$= \left[\frac{\gamma_2+2}{n} + \frac{C^4}{n^2}\{f'(0)\}^2 + 4\frac{C^2}{n^2}\left\{(\gamma_2+2) - \theta C^2\right\} f'(0)\right] \quad (2.14)$$

For population having symmetric ($\theta = 0$) and mesokurtic ($\gamma_2 = 0$) distribution, the values of relative bias and relative mean squared error further reduce to

$$RB(t^*) = \frac{C^2}{n}\left[f'(0) + \frac{1}{n}(3C^2+2)\, f'(0) + \frac{C^2}{2} f''(0)\right] \quad (2.15)$$

and

$$RMSE(t^*) = \left[\frac{2}{n} + \frac{C^4}{n^2}\{f'(0)\}^2 + \frac{8C^2}{n^2} f'(0)\right] \quad (2.16)$$

3. Concluding Remarks

(1) For symmetric and mesokurtic population, (2.4) and (2.6) reduce to

$$RB(t) = \frac{C^2}{n}\left[1 + f'(0) + \frac{C^2}{2n} f''(0) + \frac{C^4}{2n^2}\left\{\left[3 + \frac{2}{C^2}\right] f''(0) + \frac{f'''(0)}{3}\right\}\right] \tag{3.1}$$

and
$$RMSE(t) = \frac{C^2}{n}\left[4 + \frac{C^2}{n}\{3 + (f'(0))^2 + 2f'(0)\} + 2\frac{C^2}{n^2}\{f'(0)\}^2 - \frac{4C^4}{n^2} f''(0) + \{1 + f'(0)\}\frac{C^4}{n^2} f''(0)\right] \tag{3.2}$$

From the expression of RB(t) in (2.4) or (3.1), we see that there exists a sub-class of estimators with relative bias zero, if for any member in the sub-class, we have first, second and third derivatives (with respect to u at u = 0)

$$f'(0) = -1, f''(0) = 0 \text{ and } f'''(0) = 0 \tag{3.3}$$

respectively. For example, for the minimum variance unbiased estimator d_1 and the estimators d_4, d_5, d_6 and d_7 by Singh [4], we have $f'(0) = -1$, $f''(0) = 0$ and $f'''(0) = 0$ that is why these estimators have their relative bias equal to zero.

(2) From (3.2), for symmetric and mesokurtic population, RMSE(t) upto order $0(n^{-2})$ is minimized for $f'(0) = -1$ and substituting $f'(0) = -1$ in (3.2), we get RMSE(t) upto order $0(n^{-3})$ to be

$$RMSE(t) = \frac{2C^2}{n}\left[2 + \left(\frac{n+1}{n^2}\right)C^2\right] - \frac{4C^6}{n^3} f''(0) \tag{3.4}$$

$$= RMSE(d_i) - \frac{4C^6}{n^3} f''(0); \quad i = 2, 3, ..., 7$$

showing that there exists a sub-class of estimators having less RMSE or MSE than the estimators by Das [1], Pandey [3] and Singh [4] if for any member in the sub-class, we have $f'(0) = -1$ and $f''(0) > 0$. For example, for the estimator

$$d = \bar{y}^2\left[1 + \frac{s^2}{n\bar{y}^2}\left(1 + g\frac{s^2}{n\bar{y}^2}\right)^{-1}\right]^{-1} \tag{3.5}$$

$$= \bar{y}^2\left[1 + u(1 + gu)^{-1}\right]^{-1}$$

belonging to the class represented by t, we have $f'(0) = -1$ and $f''(0) = 2(1+g)$ so that the relative mean squared error of d is

$$\text{RMSE}(d) = \text{RMSE}(d_i) - \frac{8(1+g)C^6}{n^3}; \; i = 2, 3, \ldots, 7$$

showing that upto terms of order $0(n^{-3})$, for suitably chosen value of $g > -1$, d is better in the sense of having lesser RMSE or MSE than the estimators by Das [1], Pandey [3] and Singh [4].

(3) From (2.6), RMSE(t) upto terms of order $0(n^{-3})$ attains its minimum value for

$$f'(0) = -\frac{\left\{1 + 2\theta + \frac{\gamma_2}{n} + \frac{C^2}{2n} f''(0)\right\}}{\left(1 + \frac{\gamma_2 + 2}{n}\right)} \tag{3.6}$$

and the minimum RMSE (t) is given by

$$\text{RMSE}(t)_{min} =$$

$$= \frac{C^2}{n}\left[4 + \frac{C^2}{n}\left\{3 + 4\theta + \frac{\gamma_2}{2} - \frac{\left(1 + 2\theta + \frac{\gamma_2}{n} + \frac{C^2}{2n} f''(0)\right)^2}{2\left(1 + \frac{\gamma_2 + 2}{n}\right)}\right\} + \frac{C^4}{n^2}(4\theta - 3) f''(0)\right] \tag{3.7}$$

showing that there exists a set of optimum estimators (in the sense of having minimum RMSE or MSE) satisfying (3.6) and the minimum RMSE is given by (3.7).

(4) From (3.6) and (3.7), for symmetric and mesokurtic population, considering the terms of order $0(n^{-3})$, the RMSE (t) attains its minimum value for

$$f'(0) = -\frac{\left\{1 + \frac{C^2}{2n} f''(0)\right\}}{\left(1 + \frac{2}{n}\right)} \tag{3.8}$$

and the minimum RMSE (t) is given by

$$\text{RMSE}(t)_{\min} = \frac{C^2}{n}\left[4 + \frac{C^2}{n}\left\{3 - \frac{\left(1 + \frac{C^2}{2n} f''(0)\right)^2}{2\left(1 + \frac{2}{n}\right)}\right\} - \frac{3C^4}{n^2} f''(0)\right]$$

$$= \frac{C^2}{n}\left(4 + \frac{3C^2}{n}\right) - \frac{C^2}{n}\left[\frac{\frac{C^2}{n}\left\{1 + \frac{C^3}{2n} f''(0)\right\}^2}{2\left(1 + \frac{2}{n}\right)} - \frac{3C^4}{n^2} f''(0)\right]$$

$$= \text{RMSE}(\overline{y}^2) - \frac{C^2}{n}\left[\frac{\frac{C^2}{n}\left\{1 + \frac{C^2}{2n} f''(0)\right\}^2}{2\left(1 + \frac{2}{n}\right)} - \frac{3C^4}{n^2} f''(0)\right] \quad (3.9)$$

where $\text{RMSE}(\overline{y}^2) = \frac{C^2}{n}\left(4 + \frac{3C^2}{n}\right)$ is the relative mean squared error of the estimator $\overline{y}^2$ of μ^2. (3.8) and (3.9) show that, if C^2 is known, we can get a set of estimators satisfying (3.8) with $f''(0) > 0$ such that any member of the set has less RMSE (or MSE) than that of the estimator $\overline{y}^2$ of μ^2. For example, if we consider the estimator (g being nonstochastic characterizing scalar)

$$t = \overline{y}^2\left[1 + \frac{s^2}{n\overline{y}^2}\left(1 + g\frac{s^2}{n\overline{y}^2}\right)^{-1}\right]^{-1}$$

$$= \overline{y}^2\left[1 + u\,(1 + gu)^{-1}\right]^{-1} \quad (3.10)$$

we have $f'(0) = -1$ and $f''(0) = 2(1+g)$, and putting these values in (3.8), we get

$$-1 = -\frac{\left\{1 + 2\,(1+g)\frac{C^2}{2n}\right\}}{\left(1 + \frac{2}{n}\right)}$$

or

$$g = \left(\frac{2}{C^2} - 1\right) \quad (3.11)$$

so that t_M with this value of g has $f''(0) = 2(1+g) = \frac{4}{C^2} > 0$ giving the relative mean squared error from (3.9) to be

$$\text{RMSE}(t_M) = \text{RMSE}(\overline{y}^2) - \frac{C^4}{2n^3}(n+26) \tag{3.12}$$

showing that RMSE (or MSE) of t_M is always less than that of the maximum likelihood estimator $\overline{y}^2$ in normal parent.

(5) For symmetric and mesokurtic population (2.12) and (2.14) reduce to

$$\text{RB}(t^*) = \frac{C^2}{n}\left[f'(0) + \frac{1}{n}(3C^2+2)\, f'(0) + \frac{C^2}{2} f''(0)\right] \tag{3.13}$$

and $$\text{RMSE}(t^*) = \frac{2}{n}\left[1 + \frac{C^4 \{f'(0)\}^2 + 8C^2 f'(0)}{2n}\right] \tag{3.14}$$

(6) From (2.14), RMSE (t^*) is minimized for

$$f'(0) = -\frac{2\{(\gamma_2+2) - \theta C^2\}}{C^2} \tag{3.15}$$

and the minimum RMSE is given by

$$\text{RMSE}(t^*)_{\min} = \left[\frac{\gamma_2+2}{n} - \frac{4\{(\gamma_2+2) - \theta C^2\}^2}{n^2}\right] \tag{3.16}$$

showing that there exists a set of optimum estimators satisfying (3.15) with minimum RMSE given by (3.16).

(7) For symmetric and mesokurtic population (3.15) and (3.16) respectively reduce to

$$f'(0) = -\frac{4}{C^2} \tag{3.17}$$

and $$\text{RMSE}(t^*)_{\min} = \frac{2}{n}\left(1 - \frac{8}{n}\right) \tag{3.18}$$

showing that, if C^2 is known, we get a set of optimum estimators (in sense of having minimum RMSE or MSE) satisfying (3.17) with the minimum RMSE given by (3.18). Further, for normal population, it is to be noted that the minimum RMSE given by (3.18) is always less than the RMSE of the maximum likelihood estimator $\frac{(n-1)}{n} s^2$ of σ^2, showing the existence of a set of optimum estimators [satisfying (3.17)] better than the maximum likelihood estimator of σ_2 for normal parent. For example, for the estimator

$$t_M^* = s^2\left[1 + k\frac{s^2}{n\bar{y}^2}\left(1 + \frac{s^2}{n\bar{y}^2}\right)^{-1}\right]^{-1}$$

$$= s^2\left[1 + ku\,(1+u)^{-1}\right]^{-1} \tag{3.19}$$

$$f'(0) = -k \tag{3.20}$$

which is when equated to $-\frac{4}{C^2}$ satisfying (3.17), that is $f'(0) = -k = -\frac{4}{C^2}$, we get $k = \frac{4}{C^2}$; and thus putting $k = \frac{4}{C^2}$ in (3.19), we get the estimator

$$t_M^{**} = s^2\left[1 + \frac{4}{C^2}\frac{s^2}{n\bar{y}^2}\left(1 + \frac{s^2}{n\bar{y}^2}\right)^{-1}\right]^{-1} \tag{3.21}$$

which attains the minimum RMSE given by (3.18).

Similarly, we may consider the estimator $s^2\left[1 - ku(1 - gu^{\alpha})\right]$ and find the operational optimum values of k, g and α.

(8) Comparative study of the generalized estimator t^* may be made with other estimators depending on the availability of the information regarding range of C also. For example, if we compare the generalized estimator t^* with the estimator d_2^*, d_3^* by Pandey [3], we have

$$\text{RMSE}(t^*) = \frac{2}{n}\left[1 + \frac{C^2\{f'(0)\}\{f'(0)\,C^2 + 8\}}{2n}\right] \quad \text{(from 3.14)} \tag{3.22}$$

and
$$\text{RMSE}(d_2^*) = \text{RMSE}(d_3^*) = \frac{2}{n}\left[1 + \frac{C^2}{2n}(C^2 - 8)\right] \tag{3.23}$$

From (3.22) and (3.23), we have

$$\text{RMSE}(t^*) < \text{RMSE}(d_2^*) = \text{RMSE}(d_3^*)$$

if
$$\{f'(0) + 1\}\left[f'(0)\,C^2 - (C^2 - 8)\right] < 0 \tag{3.24}$$

giving

$$\left.\begin{aligned} &\text{either} \quad -1 < f'(0) < \frac{C^2 - 8}{C^2} \\ &\text{or} \quad \frac{C^2 - 8}{C^2} < f'(0) < -1 \end{aligned}\right\} \tag{3.25}$$

which are the general efficiency conditions for the proposed estimator t^* to be better than the estimators d_2^* and d_3^* by Pandey [3]. It is to be mentioned here that efficiency conditions of the estimator

$$d_9^* = s^2\left[1+\frac{ks^2}{n\bar{y}^2}\left(1+g\frac{s^2}{n\bar{y}^2}\right)^{-1}\right] = s^2\left[1+ku\,(1+gu)^{-1}\right]$$

obtained by Singh [4], to be better than the estimator d_2^* and d_3^* by Pandey [3] are

$$\left.\begin{array}{ll} & -1 < k < \dfrac{C^2-8}{C^2} \\ \text{or} & \dfrac{C^2-8}{C^2} < k < -1 \end{array}\right\} \qquad (3.26)$$

which may be easily seen to be the special case of (3.25), since for the estimator d_9^*, the value of $f'(0) = k$.

(9) All the results obtained by various authors for the estimators given in Section 1 may be easily seen to be special cases of the present study.

ACKNOWLEDGEMENT

The authors wish to thank the refree for the valuable suggestions.

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