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On NCm Type PBIB Designs

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SUMMARY

Twenty five highly efficient NCm type PBIB designs for number of treatments $v \leq 15$ and $r \leq 10$ have been listed alongwith their efficiency factors for all types of comparisons.

Keywords : Association scheme, Partially balanced incomplete block design, NCm type design, Efficiency factor.

1. Introduction

Adhikary [1] generalized the two class cyclic association scheme of Bose and Shimamoto [3] to higher associate classes. He gave certain conditions under which the division of the group of treatments into more than two subsets yield higher associate class cyclic scheme. He illustrated these by giving some examples of three-class schemes for some chosen values of the number of treatments. However, he did not propose any general method of dividing the group of treatments into subsets so that the stated conditions are met. It was left to Saha, Kulshreshtha and Dey [5] and Agrawal and Nair [2] to introduce general m-class cyclic association schemes with the names of NCm cyclic association scheme and reduced residue classes cyclic association scheme respectively. While the former is defined for v either a prime or power of a prime, the latter is defined for v a composite number. Saha *et al* [5] have presented a method of obtaining cyclic solutions of PBIB designs based on the NCm scheme. However, they have not listed the proposed PBIB designs.

In this paper we have taken up the problem of constructing NCm type PBIB designs not having more than three associate classes in the useful range of $5 \leq v \leq 15$. With these restrictions, the only association scheme of NCm type that exist are for $v = 5, 7, 9$ and 13 and of these the two-class association scheme for $v = 9$ coincides with the well-known Latin-square type scheme. While no new design for $v = 5$ or 9, which necessarily has two associate classes, seemed feasible, a number of new designs for $v = 7$ and $v = 13$ treatments having three associate classes could be constructed. Of these we present in

Section-3 a selection of twenty five highly efficient designs alongwith their efficiency factors for the three types of comparisons as also their overall efficiency factors.

2. *Some Preliminaries*

In this section we set out some results of Saha *et al* [5] and a few more results regarding the construction of NCm type PBIB designs. These will find use in the subsequent section.

Saha *et al* [5] have defined NCm association scheme as follows:

Definition 2.1: Let $v = ms + 1$ $(m, s \geq 2)$ be a prime or power of a prime and further let v and m be such $v - 1 = 0 \pmod m$ if v is even and $(v-1)/2 = 0 \pmod m$ if v is odd. Let x be a primitive element of GF(v) so that all the elements of GF(v) can be expressed as

$$0, x^0, x^1 \ldots x^{v-2}$$

Also let $\quad A_1 = (x^{qm}/0 \leq q \leq s-1)$

$$A_j = x^{j-1} A_1 = (x^{j-1+qm}/0 \leq q \leq s-1), 2 \leq j \leq m$$

Let us designate v elements of GF(v) as v treatments and define two distinct treatments $\alpha, \beta : \alpha, \beta \in GF(v)$, as ith associates of each other if $\alpha - \beta \in A_i, 1 \leq i \leq m$. Then this defines an m-class cyclic association scheme called the NCm association scheme.

Note that A_i, $1 \leq i \leq m$, is simply the ith associate class of treatment 0 and the ith associate class of any treatment α is given by $\alpha + A_i$, which shows that NCm scheme is of cyclic nature.

The following theorem can easily be proved.

Theorem 2.1: Among the s^2 differences u-v, $u \in A_i$, $v \in A_j$, each element of A_k occurs P^k_{ij} times (i, j, k = 1, 2, ..., m).

Using this theorem one can easily obtain the following theorems regarding the construction of NCm type PBIB designs.

Theorem 2.2: The set A_i any i, $1 \leq i \leq m$, is an initial block for the NCm type PBIB design with parameters:

$$v = ms + 1 = b,\ r = k = s,\ \lambda_k = P^k_{ii} \quad (k = 1, 2, \ldots, m)$$

Theorem 2.3: The set {0} U A_i for any i, $1 \le i \le m$, is an initial block for the NCm type PBIB design with parameters:

$$v = ms + 1 = b,\ r = k = s + 1,\ \lambda_i = P^i_{ii} + 2$$
$$\lambda_k = P^k_{ii} \quad (k \neq i\, ;\, k = 1, 2, \ldots, m)$$

Theorem 2.4: The set A_i U A_j $(i \neq j;\ i, j = 1, 2, \ldots, m)$ is an initial block for the NCm type PBIB design with parameters:

$$v = ms + 1 = b,\ r = k = 2s,\ \lambda_k = P^k_{ii} + P^k_{jj} + 2P^k_{ij} \qquad (k = 1, 2, \ldots, m)$$

Theorem 2.5: The set {0} U A_i U A_j $(i \neq j;\ i, j = 1, 2, \ldots, m)$ is an initial block for the NCm type PBIB design with parameters:

$$v = ms + 1 = b,\ r = k = 2s + 1$$
$$\lambda_i = P^i_{ii} + P^i_{jj} + 2P^i_{ij} + 2,\ \lambda_j = P^j_{ii} + P^j_{jj} + 2P^j_{ij} + 2$$
$$\lambda_k = P^k_{ii} + P_{jj} + 2P^k_{ij} \quad (k \neq i, j\, ;\, k = 1, 2, \ldots m)$$

The following construction method using more than one initial block is due to Saha *et al* [5].

Consider a basic initial block $I = (a_1, a_2, \ldots, a_k)$ consisting of k $(k < v)$ distinct elements of GF(v) and a set $M = (e_1, e_2, \ldots, e_t)$ of t distinct non zero elements of GF(v). We say a design D is generated by I and M and write D = [I, M] if it is obtained by developing (mod v) the blocks $I_j = e_j I = (e_j a_1,\ e_j a_2, \ldots, e_j a_k)$ mod (v), $1 \le j \le t$.

Let f_i denote the number of differences arising from I, belonging to the set A_i, so that

$$\sum_{i=1}^{m} f_i = k\,(k-1)$$

Then if I is such that not all f_i's are equal, the following theorem can be proved.

Theorem 2.6: When v is an odd prime or power of a prime such that an NCm association scheme with the parameters $n_i = s$, P^i_{jk} exists, the design $D = [I, M], M = (x^0, x^m, x^{2m}, \ldots, x^{m(s-2)/2})$ is an NCm type PBIB design with parameters

$$b = sv/2,\ r = sk/2,\ k,\ \lambda_i = f_i/2, 1 \le i, j, k \le m$$

Table 3.1 : NCm-type PBIB designs having three associate classes
$v = 7, 13, r \leq 10$

v = 7											
Design No.	k	r	Initial Blocks	λ_1	λ_2	λ_3	E_1	E_2	E_3	E_0
1		3	(0, 1, 6)	2	0	1	0.8391813	0.5797981	0.6784871	0.6833334
2		6	J (0, 1, 2) (0, 1, 3)	3	1	2	0.8236582	0.7014894	0.7557283	0.7570282
3	3	6	J (0, 1, 2) (0, 1, 4)	3	2	1	0.7925985	0.7402104	0.7075215	0.7451517
4		9	J (0, 1, 2) (0, 1, 3) (0, 1, 4)	4	3	2	0.8062419	0.7695112	0.7375878	0.7700955
5		9	J (0, 1, 2) (0, 1, 3) (0, 1, 5)	4	2	3	0.8108092	0.7316627	0.7683294	0.7689129
6		4	(1, 2, 5, 6)	2	3	1	0.8501935	0.9266154	0.7891748	0.8516566
7	6	8	J (0, 1, 2, 3) (0, 1, 2, 4)	5	4	3	0.9007871	0.8696856	0.8417007	0.8700562
8		8	J (0, 1, 2, 3) (0, 1, 2, 4)	5	3	4	0.9037989	0.8380481	0.8691262	0.8694966
9	5	5	(0, 1, 2, 5, 6)	4	3	3	0.9582683	0.9166041	0.9182677	0.9306442
10		10	(0, 1, 2, 5, 6) (0, 2, 3, 4, 5)	7	6	7	0.9395754	0.9195748	0.9391408	0.9326689

(........ contd.)

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Design No.	k	r	Initial Blocks	λ_1	λ_2	λ_3	E_1	E_2	E_3	E_0
				v = 13						
11	3	6	(0, 1, 2) (0, 5, 10)	2	1	0	0.7362918	0.6956376	0.6406154	0.6885965
12		6	(0, 1, 2) (0, 5, 12)	2	0	1	0.7413543	0.6219139	0.6867761	0.6798247
13		4	(1, 5, 8, 12)	0	1	2	0.6996529	0.7726228	0.8340233	0.7648027
14	4	8	(0, 1, 2, 3) (0, 2, 5, 10)	3	3	0	0.8067788	0.8183042	0.7160160	0.7775735
15		8	(0, 1, 2, 4) (0, 2, 5, 8)	2	3	1	0.8034053	0.8345449	0.7689735	0.8014115
16		8	(0, 1, 2, 5) (0, 1, 3, 8)	3	2	1	0.8319353	0.8046179	0.7734491	0.8026213
17		5	(0, 1, 2, 8, 12)	2	1	2	0.8760872	0.8343687	0.8741868	0.8611113
18		10	(0, 1, 2, 6, 7) (0, 4, 5, 9, 10)	5	1	4	0.8855205	0.7996654	0.8617839	0.8474128
19		10	(0, 1, 2, 3, 6) (0, 2, 4, 5, 10)	4	4	2	0.8741875	0.8760872	0.8343688	0.8611160
20	5	10	(0, 1, 2, 3, 7) (0, 2, 5, 9, 10)	4	3	3	0.8786256	0.8586466	0.8593943	0.8653583
21		10	(0, 1, 2, 3, 8) (0, 1, 2, 5, 10)	5	3	2	0.8801893	0.8474072	0.8303836	0.8521628
22		10	(0, 1, 2, 5, 6) (0, 4, 5, 10, 12)	5	2	3	0.9239722	0.8833751	0.9002078	0.9022116
23	5	10	(0, 1, 2, 8, 9) (0, 1, 5, 6, 10)	6	1	3	0.8959061	0.7905863	0.8388235	0.8395786
24	8	8	(0, 1, 2, 3, 5, 8, 10, 11, 12)	4	5	5	0.9367236	0.9523400	0.9526003	0.9471624
25	9	9	(0, 1, 2, 3, 5, 8, 10, 11, 12)	6	7	5	0.9618280	0.9741692	0.9490244	0.9615641

3. *List of Designs and their Efficiency Factors*

Restricting ourselves to the consideration of NCm type PBIB designs with $v \leq 15$, $r \leq 10$ and having not more than three associate classes, we find that while no new design with two associate classes seems feasible, a number of new designs having three associate classes for $v = 7$ and $v = 13$ treatments can be constructed. Table 3.1 gives the initial blocks of twenty five such designs together with their efficiency factors for the three types of comparisons as well as their overall efficiency factors. Further, most of the designs appearing in Table 3.1, barring a few constructed by trial and error, have been constructed using the method outlined in Section 2. Following John [4] we regard two designs to be equivalent if one can be obtained from the other by relabelling of the v treatments. Table 3.1 omits all those designs which are equivalent or which can be obtained by duplicating smaller designs. Some of the designs listed here also appear as CIB designs in John [4]. They have been indicated by writing J before their initial blocks.

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