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General Efficiency Balanced Block Designs with Unequal Block Sizes for Comparing Two Sets of Treatments

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SUMMARY

This article deals with General Efficiency Balanced (GEB) block designs with unequal block sizes for comparing treatments belonging to two disjoint sets, each set consisting of two or more treatments. GEB block designs with unequal block sizes (GEBUB) have been defined and some methods of construction of these designs have been given. A list of some of the GEBUB designs alongwith the parameters is also prepared.

Key words: BTIB design, GEB design, Balanced bipartite block design, Group divisible design, BIB design.

1. Introduction

In the existing literature on incomplete block designs, the property of balance has been considered in different contexts. Calinski [2] studied the notion and introduced a general definition of balance, called X^{-1}-balance. Variance balance (VB) and Efficiency balance (EB) become special cases of this definition. Das and Ghosh [5] introduced the concept of General Efficiency Balanced (GEB) designs and constructed several series of VB and EB designs as a subclass of GEB designs using the reinforcement technique. The GEB designs are a useful class of designs and their analysis is very simple because of the simplified form of the information matrix C of the design (for more details on GEB designs, see Das and Ghosh [5]). Gupta [6] gave an application of X^{-1}-balanced designs for estimating orthonormal treatment contrasts with some specified weights. Gupta showed that if the experimenter wishes to estimate the contrasts $P_1'\tau, ..., P'_{v-1}\tau$ with variances respectively in the ratio $w_1^{-1}, ..., w_{v-1}^{-1}$ and if $w_i^{-1} = \alpha P_i'S^{-1}P_i$, $i = 1, ..., v-1$, then GEB designs are appropriate, where α is a positive scalar.

The GEB designs are also an important class of designs for comparing a set of test treatments to a single control. Prasad [11] showed that a GEB design with $v+1$ treatments is also a Balanced Treatment Incomplete Block

(BTIB) design of Bechhofer and Tamhane [1] and vice-versa. Prasad [11] has also given some methods of construction of GEB block designs with equal and unequal block sizes for comparing a set of test treatments to a control.

There is yet another class of GEB designs with $v = v_1 + v_2$ treatments, where the v treatments are distributed in two disjoint sets with respective cardinality as v_1 and v_2. Kageyama and Mukerjee [8] have defined GEB designs for two sets of treatments as follows :

Definition 1.1. Suppose a design d in $v_1 + v_2$ treatments $(v_1, v_2 \geq 2)$ has a C matrix of the form

$$C_d = \begin{bmatrix} f_1 I_{v_1} - f_2 1_{v_1} 1'_{v_1} & -f_3 1_{v_1} 1'_{v_2} \\ -f_3 1_{v_2} 1'_{v_1} & f_4 I_{v_2} - f_5 1_{v_2} 1'_{v_2} \end{bmatrix} \tag{1.1}$$

where $f_1 = f_2 v_1 + f_3 v_2$, $f_4 = f_3 v_1 + f_5 v_2$. Then d is a GEB design if and only if $f_2 f_5 = f_3^2$.

It is easy to verify that the C matrix above is of the form $C = \delta[S - ss'/g]$ with $s' = [f_2 1'_{v_1}, f_3 1'_{v_2}]$, $g = f_2 v_1 + f_3 v_2 = f_1$ and $\delta = f_1/f_2$.

The concept of BTIB design when extended for comparing treatments belonging to two sets has been termed as Balanced Bipartite Block (BBPB) designs by Kageyama and Sinha [9]. In such type of designs the main interest is in estimating the treatment contrasts of the form $(\tau_i - \tau_m)$, where $\tau_i\ (i = 1, ..., v_1)$ is the i^{th} treatment belonging to the first set and $\tau_m\ (m = v_1 + 1, \ldots, v_1 + v_2)$ is the m^{th} treatment belonging to the second set. Since the structure of C of a GEB design in (1.1) is same as that of a BBPB design with a further condition that $f_3^2 = f_2 f_5$, therefore, it follows from this equivalence that a GEB design for two sets of treatments is also a BBPB design but the converse may not be true always. Hence, the GEB designs can also be used for comparing a set of test treatments to a set of control treatments. The same result can be extended for the block designs with unequal block sizes.

We give here some methods of construction of GEB block designs with unequal block sizes (GEBUB) for making comparison between treatments belonging to two sets, each set consisting of two or more treatments. All these methods would reduce to proper GEB designs for two sets of treatments as particular case. We first define here GEBUB designs for two sets of treatments.

For a given block design, let $N = ((n_{hj}))$ denote a $(v_1 + v_2) \times b$ incidence matrix for $h = 1, ..., v = v_1 + v_2$ and $j = 1, ..., b$. N can be partitioned in the form $N = [N_1 : N_2 : ... : N_p]$, where $N_l, l = 1, ..., p$ is the incidence matrix of the l^{th} part of the design i.e. the one with b_l blocks of size k_l each. Let $\lambda_{lhh'}$ denote the number of times the pair of treatments h, h′ occur in the l^{th} part of the design. Also let

$$s_{im} = \sum_{l=1}^{p} \frac{\lambda_{lim}}{k_l}, \qquad i = 1, ..., v_1 ; \; m = v_1 + 1, ..., v_1 + v_2$$

$$s_{ii'} = \sum_{l=1}^{p} \frac{\lambda_{lii'}}{k_l}, \qquad i \neq i' = 1, ..., v_1$$

$$s_{mm'} = \sum_{l=1}^{p} \frac{\lambda_{lmm'}}{k_l} \qquad m \neq m' = v_1 + 1, ..., v_1 + v_2$$

Definition 1.2. A block design for comparing two sets of treatments is called a GEBUB design if

(i) for every i and m, $s_{im} = s_0$, for some constant s_0,

(ii) for every $i \neq i'$, $s_{ii'} = s_1$, for some constant s_1,

(iii) for every $m \neq m'$, $s_{mm'} = s_2$, for some constant s_2,

(iv) further $s_0^2 = s_1 s_2$ for constants s_0, s_1 and s_2.

As a result, the information matrix of the GEBUB design for treatment effects is given by

$$C = \begin{bmatrix} (a_1 + s_1)I_{v_1} - s_1 1_{v_1} 1'_{v_1} & -s_0 1_{v_1} 1'_{v_2} \\ -s_0 1_{v_2} 1'_{v_1} & (a_2 + s_2)I_{v_2} - s_2 1_{v_2} 1'_{v_2} \end{bmatrix} \tag{1.2}$$

which is positive semi-definite with zero row (column) sums. Here I_t is an identity matrix of order t, 1_t is a $t \times 1$ vector of ones and a_1 and a_2 are some scalar constants. If all the four conditions above are satisfied then the design is a GEBUB as well as a BBPB design with unequal block sizes (BBPBUB) design. However, if the fourth condition is not satisfied then the design is just a BBPBUB design.

The matrix C above can be expressed in the form $C = \delta[S - ss'/g]$ with

$$s' = [s_1 1'_{v_1}, s_0 1'_{v_2}], \qquad g = s'1, \qquad S = \begin{bmatrix} s_1 I_{v_1} & 0 \\ 0 & s_0 I_{v_2} \end{bmatrix} \text{ and}$$

$\delta = \dfrac{n - \text{trace}(NK^{-1}N')}{g - s's/g} = (a_1 + s_1)/s_1$. It is seen from the above definition that the C matrix of a GEBUB design is the same as that of a BBPBUB design, with a further condition that for a GEBUB design $s_0^2 = s_1 s_2$. Therefore, a GEBUB design is also a BBPBUB design, but the converse may not be true.

2. *Methods of Construction of GEBUB Designs*

In this section some methods of construction of GEBUB designs have been given along with proofs and examples wherever required.

2.1 Using Group Divisible (GD) Designs

Method 2.1.1. Let N_1 be the incidence matrix of a Semi Regular (SR) GD design with parameters $v_1 = mn, b_1, r_1, k_1, \lambda_{11}$ and $\lambda_{12} = \lambda_{11} + p$. Let N_2 be the incidence matrix of another design obtained from (m, n) GD association scheme by treating m groups as blocks of size n each. Taking p copies of N_2 we obtain N_2^* with parameters $v_1 = mn$, $b_2 = mp$, $r_2 = p$, $k_2 = n$, $\lambda_{21} = p$ and $\lambda_{22} = 0$. Now adding v_2 treatments i times in the blocks of the design N_1, where i is a positive integer such that $k_1 + iv_2 \neq k_2$ and taking the copies of N_1 and N_2^* in the ratio $\theta/\varphi = (k_1 + iv_2)/k_2$ so as to make s_1 constant, we obtain GEBUB design with incidence matrix

$$N^* = \begin{bmatrix} 1'_\theta \otimes N_1 & 1'_\phi \otimes N_2^* \\ i1_{v_2} 1'_{b_1\theta} & 0 \end{bmatrix}$$

The parameters of the resulting design will be $v_1^* = v_1$, $v_2^* = v_2$, $r^* = [(r_1\theta + r_2\phi)1'_{v_1}, b_1\theta 1'_{v_2}]'$, $k^* = [(k_1 + iv_2)1'_{b_1\theta}, k_2 1'_{b_2\phi}]'$, $b^* = b_1\theta + b_2\phi$,

$s' = \left[\dfrac{\theta\lambda_{12}}{k_1 + iv_2} 1'_{v_1}, \dfrac{\theta r_1}{k_1 + iv_2} 1'_{v_2}\right]$ and $\delta = (v_1\lambda_{12} + v_2 r_1)/\lambda_{12}$. $\otimes$ denotes the Kronecker product of matrices.

Proof. For the design obtained above to be a GEBUB design, C is of the form (1.2) and $s_0^2 = s_1 s_2$

i.e.
$$\left(\frac{i\theta r_1}{k_1 + iv_2}\right)^2 = \frac{\theta\lambda_{12}}{k_1 + iv_2}\,\frac{i^2 b_1 \theta}{k_1 + iv_2}$$

$$\Rightarrow \qquad r_1^2 = \lambda_{12} b_1$$

i.e.,
$$r_1 k_1 - v_1 \lambda_{12} = 0 \quad (\text{since } v_1 r_1 = b_1 k_1)$$

which is the condition satisfied by SRGD design. Hence we start this method using a SRGD design.

Remark 2.1.1. (i) Here if $k_1 + iv_2 = k_2$, we obtain proper GEB designs for comparing two sets of treatments.

(ii) By augmenting the v_2 treatments i_1 times in N_1, where N_1 is the incidence matrix of any GD design and i_2 times in N_2^*, where i_1, i_2 are positive integers and $i_1 + i_2 \neq 0$, we obtain BBPBUB designs which are not GEBUB. For $v_2 = 1$, this method reduces to that of Parsad and Gupta [10].

Example 2.1.1. Consider a design SR1 (Clatworthy, [3]) with parameters $v_1 = 4$, $b_1 = 4$, $r_1 = 2$, $k_1 = 2$, $\lambda_{11} = 0$, $\lambda_{12} = 1$, $m = 2$, $n = 2$. Since $p = 1$, we take a single copy of the design obtained from (2, 2) association scheme with parameters $v_1 = 4$, $b_2 = 2$, $r_2 = 1$, $k_2 = 2$, $\lambda_{21} = 1$ and $\lambda_{22} = 0$. Now adding $v_2 = 2$ treatments once in the blocks of SR1 design and then taking the copies of the two designs in the ratio $\theta/\phi = 4/2 = 2/1$, we obtain the following GEBUB design

(1, 2, 5, 6); (3, 4, 5, 6); (4, 1, 5, 6); (2, 3, 5, 6); (1, 2, 5, 6); (3, 4, 5, 6); (4, 1, 5, 6); (2, 3, 5, 6); (1, 3); (2, 4)

The parameters of this design are $v_1^* = 4$, $v_2^* = 2$, $r^* = [5 1_4', 8 1_2']'$, $k^* = [4 1_8', 2 1_2']'$, $b^* = 10$, $s' = [1/2\, 1_4', 1_2']$ and $\delta = 8$.

In the above method since we take the copies of N_1 and N_2^* in the ratio θ/ϕ, therefore the design may require large number of experimental units. In order to keep the size of the design smaller, θ and ϕ have to be made as small as possible, preferably one. We give here the procedure which results in $\theta = \phi = 1$.

Let N_1 be the incidence matrix as defined earlier and N_2 be the incidence matrix of another design obtained from (m, n) GD association scheme by treating m groups as blocks of size n each with parameters $v_1 = mn$, $b_2 = m$, $r_2 = 1$, $k_2 = n$, $\lambda_{21} = 1$ and $\lambda_{22} = 0$. Let $(\lambda_{12} - \lambda_{11}) = p(>1)$. Then adding, $v_2 = (np - k_1) > 1$ treatments once in the blocks of the design N_1, we obtain a GEBUB design with parameters $v_1^* = mn$, $v_2^* = np - k_1$, $r^* = [(r_1 + 1)1'_{v_1}, b_1 1'_{v_2}]'$, $k^* = [(k_1 + v_2)1'_{b_1}, n1'_{b_2}]'$, $b^* = b_1 + m$, $s' = [\lambda_{12}/np1'_{v_1}, r_1/np1'_{v_2}]$, $\delta = (v_2 r_1 + v_1\lambda_{12})/\lambda_{12}$.

Remark 2.1.2. (i) If $p = 1$ and $v_2 = (n - k_1) > 1$, then the resulting design will have equal block sizes.

(ii) If $p \geq 1$ and $v_2 = np - k_1 = 1$, then the method reduces to constructing GEB designs with equal and unequal block sizes for comparing v treatments with a single control.

(iii) If $np = k_1$, *i.e.*, $v_2 = 0$, the designs become pairwise and variance balanced. Hence we can get GEBUB designs from these designs by augmentation.

(iv) If $k_1 > np$, then general method of construction of GEBUB designs may be used.

A list of some of the GEBUB designs obtained through this method with $\theta = \phi = 1$ and average replication of two sets of treatments ≤ 15 is given in Table 1.

Method 2.1.2. Let N be the incidence matrix of a regular (RGD) design with parameters $v_1 = 2n, b_1, r_1, k_1(\neq n)$, $m = 2$, n, λ_{11}, λ_{12} and N_1 be the incidence matrix of a randomised complete block design (RBD) i.e. $N_1 = 1_n 1'_{r_2}$ with parameters $v_2 = n$, r_2 such that $(\lambda_{12}^2 - \lambda_{11}^2)/\lambda_{11} = r_2$, an integer, the n treatments of RBD being the treatments of any of the groups of the (2, n) association scheme. Then taking the copies of N and N_1 in the ratio $\theta/\phi = k_1/n$, we get

$$N^* = \begin{bmatrix} 1'_\theta \otimes N & \vdots & 0 \\ & \vdots & \\ & \vdots & 1'_\phi \otimes N_1 \end{bmatrix}$$

Table 1 : GEBUB designs obtained through Method 2.1.1 with $\theta = \phi = 1$

v_1^*	v_2^*	r_1^*	r_2^*	k_1^*	k_2^*	b_1^*	b_2^*	Reference design	s-vector
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
4	2	5	8	4	2	8	2	SR2	$[1/2 1_4', 1_2']$
4	4	7	12	6	2	12	2	SR3	$[1/2 1_4', 1_4']$
4	6	9	16	8	2	16	2	SR4	$[1/2 1_4', 1_6']$
4	8	11	20	10	2	20	2	SR5	$[1/2 1_4', 1_8']$
6	4	7	18	8	3	18	2	SR7	$[1/3 1_6', 1_4']$
6	3	7	12	6	2	12	3	SR20	$[1/2 1_6', 1_3']$
6	5	9	16	8	2	16	3	SR21	$[1/2 1_6', 1_5']$
6	7	11	20	10	2	20	3	SR22	$[1/2 1_6', 1_7']$
8	2	7	12	6	2	12	4	SR37	$[1/2 1_8', 1_2']$
8	4	9	16	8	2	16	4	SR39	$[1/2 1_8', 1_4']$
8	6	11	20	10	2	20	4	SR40	$[1/2 1_8', 1_6']$
9	3	7	18	6	3	18	3	SR24	$[1/3 1_9', 1_3']$
10	3	9	16	8	2	16	5	SR54	$[1/2 1_{10}', 1_3']$
10	5	11	20	10	2	20	5	SR55	$[1/2 1_{10}', 1_5']$
12	2	7	18	6	3	18	4	SR42	$[1/3 1_{12}', 1_2']$
12	2	9	16	8	2	16	6	SR69	$[1/2 1_{12}', 1_2']$
12	4	11	20	10	2	20	6	SR70	$[1/2 1_{12}', 1_4']$
14	3	11	20	10	2	20	7	SR83	$[1/2 1_{14}', 1_3']$
16	2	11	20	10	2	20	8	SR93	$[1/2 1_{16}', 1_2']$

which is the incidence matrix of a GEBUB design with parameters $v_1^* = v_1 - v_2 = n$, $v_2^* = n$, $r^* = [r_1\theta 1_n', (r_1\theta + r_2\phi)1_n']'$, $k^* = [k_1 1_{\theta b_1}', n1_{\phi r_2}']'$, $b^* = \theta b_1 + \phi r_2$, $s' = \left[\frac{\theta\lambda_{11}}{k_1} 1_n', \frac{\theta\lambda_{12}}{k_1} 1_n'\right]$ and $\delta = n(\lambda_{11} + \lambda_{12})/\lambda_{11}$. This method would result in few designs as the condition $\lambda_{11} > 0$ and $\lambda_{12} > \lambda_{11}$ restricts the number of designs.

Remark 2.1.3. (i) Here if $k_1 = n$, then the method reduces to constructing proper GEB designs.

(ii) If instead of an RBD, we take N_1 as the incidence matrix of a balanced incomplete block (BIB) design with $\lambda = (\lambda_{12}^2 - \lambda_{11}^2)/\lambda_{11}$, it also results in a GEBUB design.

Example 2.1.2. Consider a GD design R24 (Clatworthy, [3]) with parameters $v_1 = 6$, $b_1 = 24$, $r_1 = 8$, $k_1 = 2$, $m = 2$, $n = 3$, $\lambda_{11} = 1$, $\lambda_{12} = 2$. Here $(\lambda_{12}^2 - \lambda_{11}^2)/\lambda_{11} = 3$. Therefore, taking an RBD with $v_2 = r_2 = 3$, where the three treatments of RBD can be any of the three treatments of the (2, 3) association scheme and taking the copies of R24 design and RBD in the ratio $\theta/\phi = 2/3$, we get a GEBUB design with parameters $v_1^* = 3$, $v_2^* = 3$, $r^* = [161_3', 251_3']'$, $k^* = [21_{48}', 31_9']'$, $b^* = 57$, $s' = [1_3', 21_3']$ and $\delta = 9$.

2.2. Using BIB designs

Method 2.2.1. This method is derived from the method of Corsten [4]. If there exists a resolvable BIB design with parameters v_1, b_1, $r_1 = \alpha p$, k_1, λ_1 such that $v_1/k_1 = b_1/r_1 = m$, then adding p new treatments to the blocks of this design in such a way that each of the blocks in α replications get the same treatment and further $b_2 = \alpha^2 p/(k_1 + 1)\lambda_1$ blocks of p new treatments when added in the design, results in a GEBUB design with parameters $v_1^* = v_1$, $v_2^* = p$, $r^* = [r_1 1_{v_1}', (m\alpha + b_2)1_p']'$, $k^* = [(k_1 + 1)1_{b_1}', p1_{b_2}']'$, $b^* = b_1 + b_2$,

$$s' = \left[\frac{\lambda_1}{k_1 + 1} 1_{v_1}', \frac{\alpha}{k_1 + 1} 1_p'\right] \text{ and } \delta = (v_1\lambda_1 + p\alpha)/\lambda_1.$$

Example 2.2.1. Let us take a resolvable BIB design with parameters $v_1 = 9$, $b_1 = 12, r_1 = 4, k_1 = 3, \lambda_1 = 1$. For $r_1 = 4$, suppose $\alpha = 2$ and $p = 2$. Then following the above procedure, the GEBUB design obtained is as follows

(1, 2, 3, 10); (4, 5, 6, 10); (7, 8, 9, 10); (1, 4, 7, 10); (2, 5, 8, 10); (3, 6, 9, 10); (1, 6, 8, 11); (2, 4, 9, 11); (3, 5, 7, 11); (1, 5, 9, 11); (2, 6, 7, 11); (3, 4, 8, 11); (10, 11); (10, 11).

with parameters $v_1^* = 9$, $v_2^* = 2$, $r* = [41_9', 81_2']'$, $k^* = [41_{12}', 21_2']'$, $b^* = 14$, $s' = [1/4, 1_9', 1/2\, 1_2']$ and $\delta = 13$.

Table 2 gives a list of some of the GEBUB designs constructed through this method with average replication of two sets of treatments ≤ 15.

Table 2 : Some GEBUB designs using Method 2.2.1.

v_1^*	v_2^*	r_1^*	r_2^*	k_1^*	k_2^*	b_1^*	b_2^*	s-vector
9	2	4	8	4	2	12	2	$[1/4\, 1_9', 1/21_2']$
10	9	9	8	3	9	45	3	$[1/3\, 1_{10}', 1/3\, 1_2']$
21	5	10	19	4	5	70	5	$[1/41_{21}', 1/21_2']$
25	3	6	12	5	3	30	2	$[1/51_{25}', 2/51_2']$
36	14	14	7	7	14	84	1	$[2/71_{36}', 1/71_2']$
49	2	8	32	8	2	56	4	$[1/81_{49}', 1/21_2']$
49	4	8	16	8	4	56	2	$[1/81_{49}', 1/41_2']$
64	3	9	27	9	3	72	3	$[1/91_{64}', 1/31_2']$
81	5	10	20	10	5	90	2	$[1/101_{81}', 1/51_2']$

Remark 2.2.1. A BIB (v, b, r, k, λ) design when augmented by v_2 treatments in each block once and adding t more blocks of v treatments such that $t = (r^2 - b\lambda)v/b(k + v_2)$, will always yield a GEBUB design with parameters $v_1^* = v, v_2^* = v_2, r^* = [(r+t)1'_{v_1}, b1'_{v_2}]'$, $k^* = [(k+v_2)1_b', v1_t']$, $b^* = b+t$, $s' = \left[\frac{\lambda}{k+v_2} 1_{v_1}', \frac{r}{k+v_2} 1_{v_2}'\right]$ and $\delta = (v\lambda + v_2 r)/\lambda$. For example, the BIB (6, 20, 10, 3, 4) design as reported by Kageyama(K3) [7] would yield a proper GEB design fọr $t = 1$ and $v_2 = 3$.

2.3 Designs obtained through merging of treatments

Method 2.3.1. Consider a binary VB block design with $v + \alpha p$ treatments indexed as $1, ..., v, v+1, ..., v+\alpha+1, ..., v+2\alpha, .., v+\alpha(p-1)+1, ..., v+\alpha p$ and other parameters as $r' = (r_1', r_2')$, where $r_1' = (r_1, ..., r_v)$ and $r_2' = (r_{v+1}, ..., r_{v+\alpha}, ..., r_{v+\alpha p})$, k' and unique non-zero eigenvalue $\beta = (n-b)/(v+\alpha p-1)$ with multiplicity $v + \alpha p - 1$. Then the design obtained by merging the treatments $v+1, ..., v+\alpha$ as one treatment, say A; $v+\alpha+1, ..., v+2\alpha$ treatments as another treatment, say B and so on; $v+\alpha(p-1)+1, ..., v+\alpha p$ as P, we get a GEBUB design with parameters $v_1^* = v$, $v_2^* = p$, $b^* = b$, $r^{*\prime} = (r_1', r_p')$ where $r_p' = (r_{v+1} + ... + r_{v+\alpha}, r_{v+\alpha+1} + ... + r_{v+2\alpha}, ..., r_{v+\alpha(p-1)+1} + ... + r_{v+\alpha p})$; k', $s' = \left[\frac{\beta}{v+\alpha p} 1_v', \frac{\alpha\beta}{v+\alpha p} 1_p'\right]$ and $\delta = v + \alpha p$. This method would result in number of GEBUB designs by choosing appropriate VB design.

Example 2.3.1. Consider a binary VB block design with v = 7, b = 9, $r = (4, 4, 4, 4, 4, 4, 6)'$, $k = (6 1_3', 2 1_6')'$. The block contents are given as (1, 2, 3, 4, 5, 6); (1, 2, 3, 4, 5, 6); (1, 2, 3, 4, 5, 6); (1, 7); (2, 7); (3, 7); (4, 7); (5, 7); (6, 7). Now if we merge treatments 4 and 5 to A and treatments 6 and 7 to B, we get a GEBUB design with blocks as (1, 2, 3, A, A, B); (1, 2, 3, A, A, B); (1, 2, 3, A, A, B); (1, B); (2, B); (3, B); (A, B); (A, B); (B, B) and with parameters $v_1^* = 3$, $v_2^* = 2$, $r^* = (4 1_3', 8, 10)'$, $k^* = (6 1_3', 2 1_6')'$, $b^* = 9$, $s' = [1/2\ 1_3', 1_2']$ and $\delta = 7$.

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