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Construction of Binary and Non-Binary Variance Balanced Designs

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SUMMARY

The present paper gives simple and elegant methods of construction of variance balanced (VB) binary and non-binary designs with unequal block sizes and unequal number of replications using (a) two or more balanced incomplete block (BIB) designs having same number of treatments and (b) the concept of reinforced design. Besides getting new binary and non-binary VB designs which are not obtainable by the existing methods given by many authors. It has been shown that some of these methods turn out to be particular cases of the methods described in this paper.

Key words: BIB design, VB design, Kronecker product of matrices, Binary and non-binary designs, Reinforced design.

1. Introduction

Understanding significance of non-equireplicate VB designs, many have contributed in the construction of such designs. John [4] constructed binary and ternary VB designs starting with a BIB design. Kulshreshtha, *et al.* [5] generalized the method of John [4] giving VB binary and ternary designs with unequal replications and unequal block sizes starting with two BIB designs having same number of treatments. Das [1], explaining the significance, introduced reinforced incomplete block (IB) designs and gave reinforced BIB designs. K.R Nair pointed out that the reinforced BIB designs with no complete block added is a particular case of the intra- and inter- group BIB designs introduced by Nair and Rao [7]. Since then VB and efficiency balanced (EB) designs are constructed using reinforced IB designs. As a significant contribution, Das and Ghosh [2] introduced the concept of generalized efficiency balanced (GEB) designs unifying BIB designs, VB and EB designs and constructed several series of new VB and EB designs using the technique of reinforcement. Kageyama and Mukerjee [6] starting from BIB designs attempted to develop a unified theory of construction of VB, EB and GEB designs based on reinforcement, considering all possible choices of the parameters t, p, n,

u_1 and u_2 used by them in their methods of construction. Gujarathi and Pravender [3] gave construction of some new EB designs using the technique of reinforcement.

In the present paper, two methods of construction of improper VB designs with unequal number of replications are given by the reinforced IB designs introduced by Das [1] using BIB designs as basic designs. It is shown that besides constructing some new binary and non-binary VB designs by these methods, some of the above discussed methods turn out to be a particular case of our results and both of these facts are illustrated through examples.

2. *Notations*

We introduce below the notations used in the present paper :

$\otimes$: sign of Kronecker product

$1'_{\bar{p}} \otimes N$: $\bar{p}$ - replications of N

I_v : v-square identity matrix as an incidence matrix of a particular BIB design with $v = b, r = 1, k = 1, \lambda = 0$

$J_{v,b}$: $v \times b$ matrix of unities as an incidence matrix of a complete block design which is a trivial BIB design with $r = \lambda$

$1_{v,1}$: a single complete block as an incidence matrix of a trivial BIB design with $b = r = \lambda = 1$

$1'_{v,1}$: $1 \times v$ row vector of unities

3. *Methods of Construction*

Two general methods of construction of VB designs are given below. Next various types of reinforcement techniques are explained as a particular case of the method in the corollaries.

Theorem 3.1. Let N_1 and N_2 be incidence matrices of two BIB designs D_1 and D_2 whose parameters are respectively given by $v, b_1, r_1, k_1, \lambda_1$ and $v, b_2, r_2, k_2, \lambda_2$. Then the matrix N is given by

$$N = \begin{bmatrix} \bar{x}1'_{\bar{p}} \otimes N_1 & \bar{y}1'_{\bar{q}} \otimes N_2 \\ \bar{w}1'_{\bar{p}b_1} & \bar{z}1'_{\bar{q}b_2} \end{bmatrix}$$

is the incidence matrix of a VB design D with parameters $v' = v + 1$, $b' = \bar{p}b_1 + \bar{q}b_2$, $r' = [(\bar{x}\,\bar{p}r_1 + \bar{y}\,\bar{q}r_2)1'_v,\ \bar{w}\,\bar{p}b_1 + \bar{z}\,\bar{q}b_2]$, $k' = [(\bar{x}k_1 + \bar{w})1'_{\bar{p}b_1}$,

$(\bar{y}k_2 + \bar{z})1'_{\bar{q}b_2}]$, where $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ are positive integers so chosen that $\bar{p}$ and $\bar{q}$ are positive integers which satisfy

$$\frac{\bar{p}}{\bar{q}} = \frac{(r_2\bar{z} - \bar{y}\lambda_2)(\bar{x}k_1 + \bar{w})\bar{y}}{(\bar{x}\lambda_1 - \bar{w}r_1)(\bar{y}k_2 + \bar{z})\bar{x}}$$

Proof. Evidently the off-diagonal elements of $C = (c_{ij})$ matrix of D are:

$$c_{ij} = \frac{\bar{x}^2\bar{p}\lambda_1}{\bar{x}k_1 + \bar{w}} + \frac{\bar{y}^2\bar{q}\lambda_2}{\bar{y}k_2 + \bar{z}} \qquad i, j \leq v \text{ and } i \neq j$$

$$c_{ij} = \frac{\bar{x}\,\bar{w}\,\bar{p}r_1}{\bar{x}k_1 + \bar{w}} + \frac{\bar{y}\,\bar{z}\,\bar{q}r_2}{\bar{y}k_2 + \bar{z}} \qquad i \leq v \text{ and } j = v+1$$

By Rao [10] we get the required result upon equating off-diagonal elements.

Corollary 3.1.1. If in Theorem 3.1, $N_2 = J_{v,b_2}$ then the matrix N gives the incidence matrix of a VB design D with parameters

$$v' = v+1, \qquad b' = \bar{p}b_1 + \bar{q}b_2$$

$$r' = [(\bar{x}\,\bar{p}r_1 + \bar{y}\,\bar{q}b_2)1_v', \quad \bar{w}\,\bar{p}b_1 + \bar{z}\,\bar{q}b_2]$$

$$k' = [(\bar{x}k_1 + \bar{w})1'_{\bar{p}b_1}, (\bar{y}v + \bar{z})1'_{\bar{q}b_2}]$$

where $\bar{p}$ and $\bar{q}$ are determined by

$$\frac{\bar{p}}{\bar{q}} = \frac{(\bar{z} - \bar{y})(\bar{x}k_1 + \bar{w})b_2\bar{y}}{(\bar{x}\lambda_1 - \bar{w}r_1)(\bar{y}v + \bar{z})\bar{x}}$$

Corollary 3.1.2. If in Theorem 3.1, $N_2 = 1_{v,1}$ then the matrix N gives the incidence matrix of a VB design D with parameters

$$v' = v+1, \qquad b' = \bar{p}b_1 + \bar{q},$$

$$r' = [(\bar{x}\,\bar{p}\,r_1 + \bar{y}\,\bar{q})1_v', \quad \overline{wp}b_1 + \overline{zq}\,]$$

$$k' = [(\bar{x}k_1 + \bar{w})1'_{\bar{p}b_1}, \quad (\bar{y}v + \bar{z})1'_{\bar{q}}\,]$$

where $\bar{p}$ and $\bar{q}$ are determined by

$$\frac{\bar{p}}{\bar{q}} = \frac{(\bar{z} - \bar{y})(\bar{x}k_1 + \bar{w})\bar{y}}{(\bar{x}\lambda_1 - \bar{w}r_1)(\bar{y}v + \bar{z})\bar{x}}$$

Note that the VB design given by Corollaries 3.1.1 and 3.1.2 does not exist when $\bar{y} = \bar{z}$.

Corollary 3.1.3. If in Theorem 3.1, $N_2 = I_v$ then the matrix N gives the incidence matrix of a VB design D with parameters

$$v' = v + 1$$

$$b' = \bar{p}b_1 + \bar{q}v$$

$$r' = [(\bar{x}\,\bar{p}r_1 + \bar{y}\,\bar{q})1_v', \bar{w}\,\bar{p}b_1 + \bar{z}\,\bar{q}v]$$

$$k' = [(\bar{x}k_1 + \bar{w})1'_{\bar{p}b_1}, (\bar{y} + \bar{z})1'_{\bar{q}v}]$$

where $\bar{p}$ and $\bar{q}$ are determined by

$$\frac{\bar{p}}{\bar{q}} = \frac{\bar{z}\,\bar{y}(\bar{x}k_1 + \bar{w})}{(\bar{x}\lambda_1 - \bar{w}r_1)(\bar{y} + \bar{z})\bar{x}}$$

Remarks.

(1) In place of N_2, N_1 can also be selected as $N_1 = J_{v, b_1}$ or $N_1 = 1_{v,1}$ and VB design can be constructed by obtaining appropriate condition.

(2) Because of above discussed choices for N_1 and N_2, we have options to construct a design with minimum size.

(3) The method discussed in Theorem 3.1 gives construction of binary and non-binary designs is mentioned below in Table 3.1 as a sample of construction.

Table 3.1 : Construction of binary and non-binary VB designs

S. No.	Constants $\bar{x}$	$\bar{y}$	$\bar{w}$	$\bar{z}$	Type of VB design	Condition for variance balancedness
1	1	1	0	1	Binary	$\frac{\bar{p}}{\bar{q}} = \frac{(r_2 - \lambda_2)k_1}{\lambda_1(k_2 + 1)}$
2	2	1	1	0	Ternary	$\frac{\bar{p}}{\bar{q}} = \frac{-\lambda_2(2k_1 + 1)}{2k_2(2\lambda_1 - r_1)}$
3	2	1	3	0	4-ary	$\frac{\bar{p}}{\bar{q}} = \frac{-\lambda_2(2k_1 + 3)}{2k_2(2\lambda_1 - 3r_1)}$
4	2	1	3	4	5-ary	$\frac{\bar{p}}{\bar{q}} = \frac{(4r_2 - \lambda_2)(2k_1 + 3)}{(2\lambda_1 - 3r_1)(2k_2 + 8)}$

(4) (a) The design in Theorem 3.1 of Patel *et al.* [8] follows from Corollary 3.1.1 taking $\bar{x} = \bar{p} = \bar{q} = 1$, $\bar{y} = q$, $\bar{w} = p$, $\bar{z} = x$, $b_2 = n$, respectively.

(b) The design in Theorem 3.1 (ii) of Kageyama and Mukerjee [6] follows from Corollary 3.1.1 taking $b_2 = n$, $\bar{x} = \bar{p} = \bar{q} = 1$, $\bar{y} = u_1$, $\bar{w} = p$, $\bar{z} = u_2$ and renaming $b_1 = b$, $r_1 = r$, $k_1 = k$, $\lambda_1 = \lambda$, $N_1 = N_0$, respectively.

(c) The design in Section 2 of Kulshreshtha *et al.* [5] follows from Theorem 3.1 taking $\bar{x} = \bar{y} = 1$, $\bar{z} = 0$, $\bar{w} = k^*$, $\bar{p} = n$, $\bar{q} = m$, respectively.

(d) The design in Section 3 of John [4] follows from Theorem 3.1 taking $N_1 = I_v$, $\bar{p} = \bar{q} = \bar{x} = \bar{y} = 1$, $\bar{w} = k^*$, $\bar{z} = 0$, $k_2 = k$, $\lambda_2 = \lambda$, $b_2 = b$, $r_2 = r$, respectively.

(5) A VB design constructed by Theorem 3.1 is always a BIBD provided (a) $\bar{x} = \bar{y} = \bar{z} = 1, \bar{w} = 0$ and $k_1 = k_2 + 1$ (b) $\bar{x} = \bar{y} = 1, \bar{w} = 1, \bar{z} = 0$ and $k_2 = k_1 + 1$.

(6) Note that $\bar{p}$ and $\bar{q}$ may get large values as the number of treatments and/or block sizes increase and thus these designs might not find much use in the field of agriculture. However, they may be used in industrial field (cf. the discussion in [11]). Furthermore, from the necessity (not merely the desirability) of investigating various effects among many factors and the prevasion of high-speed and large-scale computers we may stand in need of VB designs with large values of v (number of treatments) and b (number of blocks). Hence, these VB designs may find some use even in the field of agriculture. Thus, the methods presented here will be of both statistical and combinatorial usefulness.

Example 3.1.1. Consider two BIB designs D_1 and D_2 having parameters $v = 7$, $b_1 = 7$, $r_1 = 3$, $k_1 = 3$, $\lambda_1 = 1$ and $v = 7$, $b_2 = 21$, $r_2 = 6$, $k_2 = 2$, $\lambda_2 = 1$, respectively. Using Theorem 3.1 with $\bar{x} = 2$, $\bar{y} = \bar{w} = 1$, $\bar{z} = 0$, giving $\bar{p} = 7$ and $\bar{q} = 4$, we get a new ternary VB design with parameters $v' = 8$, $b' = 133$, $r' = [66 1_7', 49]$, $k' = [7 1_{49}', 2 1_{84}']$.

Note that (a) this design cannot be obtained by the method of Kulshreshtha, *et al.* [5] (b) if we take $\bar{x} = \bar{y} = 1, \bar{w} = 2, \bar{z} = 0$ with above mentioned two BIB designs as basic designs then we get $\bar{p} = 1, \bar{q} = 2$ which gives us a VB design with parameters $v' = 8$, $b' = 49$, $r' = [15 1_7', 14]$, $k' = [5 1_7', 2 1_{42}']$

which is also obtainable by Kulshreshtha *et al.* [5] taking $k^* = 2$ and above considered two BIB designs as basic designs.

Example 3.1.2. Consider two BIB designs D_1 and D_2 having parameters $v = 6$, $b_1 = 6$, $r_1 = 5$, $k_1 = 5$, $\lambda_1 = 4$ and $v = 6$, $b_2 = 13$, $r_2 = 13$, $k_2 = 6$, $\lambda_2 = 13$, respectively. Using Corollary 3.1.1 with $\bar{x} = \bar{w} = \bar{z} = 1$, $\bar{y} = 2$, giving $\bar{p} = 12$ and $\bar{q} = 1$, we get a new ternary VB design with parameters $v' = 7$, $b' = 85$, $r' = [861'_6, 85]$, $k' = [61'_{72}, 131'_{13}]$. However, we cannot obtain above constructed design by the method of Kageyama and Mukerjee [6] taking any set of values for u_1, u_2, p with $b_2 = n$ and above mentioned designs as basic designs.

Example 3.1.3. Consider two BIB designs D_1 and D_2 having parameters $v = 5$, $b_1 = 10$, $r_1 = 4$, $k_1 = 2$, $\lambda_1 = 1$ and $v = 5$, $b_2 = 1$, $r_2 = 1$, $k_2 = 5$, $\lambda_2 = 1$ respectively. Using Corollary 3.1.1 with $\bar{x} = \bar{w} = 1$, $\bar{z} = 0$, $\bar{y} = 5$, giving $\bar{p} = 1$ and $\bar{q} = 1$, we get a ternary VB design having parameters $v' = 6$, $b' = 11$, $r' = [91'_5, 10]$, $k' = [31'_{10}, 25]$ which is same as the design obtained by Kageyama and Mukerjee [6] method, taking BIBD (5, 10, 4, 2, 1) with $p = 1$, $u_1 = 5$, $u_2 = 0$ and $n = 1$.

Theorem 3.2. Let N_1, N_2 and N_3 be incidence matrices of three BIB designs D_1, D_2 and D_3 whose parameters are respectively given by $v, b_1, r_1, k_1, \lambda_1$; $v, b_2, r_2, k_2, \lambda_2$ and $v, b_3, r_3, k_3, \lambda_3$. Then the matrix N given by

$$N = \begin{bmatrix} \bar{x}1'_{\bar{p}} \otimes N_1 & \bar{y}1'_{\bar{q}} \otimes N_2 & \bar{s}1'_{\bar{m}} \otimes N_3 \\ \bar{w}1'_{\bar{p}b_1} & \bar{z}1'_{\bar{q}b_2} & \bar{n}1'_{\bar{m}b_3} \end{bmatrix}$$

is the incidence matrix of a VB design D with parameters:

$$v' = v+1, b' = \bar{p}b_1 + \bar{q}b_2 + \bar{m}b_3$$

$$r' = [(\bar{x}\,\bar{p}r_1 + \bar{y}\,\bar{q}r_2 + \bar{s}\,\bar{m}r_3)1'_v, \ \bar{w}\,\bar{p}b_1 + \bar{z}\,\bar{q}b_2 + \bar{n}\,\bar{m}b_3]$$

$$k' = [(\bar{x}k_1 + \bar{w})1'_{\bar{p}b_1}, \ (y'k_2 + \bar{z})1'_{\bar{q}b_2}, \ (\bar{s}k_3 + \bar{n})1'_{\bar{m}b_3}]$$

where $\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{s}, \bar{n}$ are positive integers and positive integers $\bar{p}, \bar{q}$ and $\bar{m}$ are determined by

$$\frac{\bar{p}\,\bar{x}(\bar{x}\,\lambda_1 - \bar{w}r_1)}{\bar{x}k_1 + \bar{w}} + \frac{\bar{y}\,\bar{q}(\bar{y}\lambda_2 - \bar{z}r_2)}{\bar{y}k_2 + \bar{z}} + \frac{\bar{s}\,\bar{m}(\bar{s}\,\lambda_3 - \bar{n}r_3)}{\bar{s}k_3 + \bar{n}} = 0$$

Proof. Can be proved on similar lines of the proof of Theorem 3.1.

Corollary 3.2.1. If in theorem 3.2, $N_3 = J_{v,b_3}$ then the matrix N gives the incidence matrix of a VB design D with parameters:

$$v' = v+1,\ b' = \bar{p}b_1 + \bar{q}b_2 + \bar{m}b_3$$

$$r' = [(\bar{x}\,\bar{p}r_1 + \bar{y}\,\bar{q}r_2 + \bar{s}\,\bar{m}b_3)1_v',\ \bar{w}\,\bar{p}b_1 + \bar{z}\,\bar{q}b_2 + \bar{n}\,\bar{m}b_3]$$

$$k' = [(\bar{x}k_1 + \bar{w})1'_{\bar{p}b_1},\ (\bar{y}k_2 + \bar{z})1'_{\bar{q}b_2},\ (\bar{s}v + \bar{n})1'_{\bar{m}b_3}]$$

where $\bar{p}, \bar{q}$ and $\bar{m}$ are determined by

$$\frac{\bar{p}\,\bar{x}(\bar{x}\lambda_1 - \bar{w}r_1)}{\bar{x}k_1 + \bar{w}} + \frac{\bar{y}\,\bar{q}(\bar{y}\lambda_2 - \bar{z}r_2)}{\bar{y}k_2 + \bar{z}} + \frac{\bar{s}\,\bar{m}(\bar{s}v - \bar{n}b_3)}{\bar{s}v + \bar{n}} = 0$$

Note that for $b_3 = v$, $\bar{s} = \bar{n}$ or $\bar{s} = 0$, the condition for variance balancedness of this corollary reduces to the condition of Theorem 3.1.

Corollary 3.2.2. If in Theorem 3.2., $N_3 = 1_{v,1}$ then the matrix N gives the incidence matrix of a VB design D with parameters:

$$v' = v+1, b' = \bar{p}b_1 + \bar{q}b_2 + \bar{m}$$

$$r' = [(\bar{x}\,\bar{p}r_1 + \bar{y}\,\bar{q}r_2 + \bar{s}\,\bar{m})1_v',\ \bar{w}\,\bar{p}b_1 + \bar{z}\,\bar{q}b_2 + \bar{n}\,\bar{m}]$$

$$k' = [(\bar{x}k_1 + \bar{w})1'_{\bar{p}b_1},\ (\bar{y}k_2 + \bar{z})1'_{\bar{q}b_2},\ (\bar{s}v + \bar{n})1_{\bar{m}}']$$

where $\bar{p}, \bar{q}$ and $\bar{m}$ are determined by

$$\frac{\bar{p}\,\bar{x}(\bar{x}\lambda_1 - \bar{w}r_1)}{\bar{x}k_1 + \bar{w}} + \frac{\bar{y}\,\bar{q}(\bar{y}\lambda_2 - \bar{z}r_2)}{\bar{y}k_2 + \bar{z}} + \frac{\bar{s}\,\bar{m}(\bar{s} - \bar{n})}{\bar{s}v + \bar{n}} = 0$$

Note that for $\bar{s} = \bar{n}$ or $\bar{s} = 0$, the condition for variance balancedness of this corollary reduces to the condition of Theorem 3.1.

Corollary 3.2.3. If in Theorem 3.2, $N_3 = I_v$ then the matrix N gives the incidence matrix of a VB design D with parameters:

$$v' = v+1, b' = \bar{p}b_1 + \bar{q}b_2 + \bar{m}v$$

$$r' = [(\bar{x}\,\bar{p}r_1 + \bar{y}\,\bar{q}r_2 + \bar{s}\,\bar{m})1_v',\ \bar{w}\,\bar{p}b_1 + \bar{z}\,\bar{q}b_2 + \bar{n}\,\bar{m}v]$$

$$k' = [(\bar{x}k_1 + \bar{w})1'_{\bar{p}b_1},\ (\bar{y}k_2 + \bar{z})1'_{\bar{q}b_2},\ (\bar{s} + \bar{n})1'_{\bar{m}v}]$$

where $\bar{p}, \bar{q}$ and $\bar{m}$ are determined by

$$\frac{\bar{p}\,\bar{x}(\bar{x}\lambda_1 - \bar{w}r_1)}{\bar{x}k_1 + \bar{w}} + \frac{\bar{y}\,\bar{q}(\bar{y}\lambda_2 - \bar{z}r_2)}{\bar{y}k_2 + \bar{z}} = \frac{\bar{s}\,\bar{m}\,\bar{n}}{\bar{s} + \bar{n}}$$

Note that for $\bar{s} = 0$ or $\bar{n} = 0$, the condition for variance balancedness of this corollary reduces to the condition of Theorem 3.1.

Remarks.

(1) In place of N_3, N_1 and N_2 can also be selected as '$N_1 = J_{v, b_1}$ or $N_2 = J_{v, b_2}$', '$N_1 = 1_{v, 1}$ or $N_2 = 1_{v, 1}$' and '$N_1 = I_v$ or $N_2 = I_v$' and VB design can be constructed by obtaining appropriate condition.

(2) Because of above discussed choices for N_1, N_2 and N_3 we have options to construct a design with minimum size.

(3) In place of D_1 and D_2 taken as pairwise balanced designs by Pal and Pal [9], we can take any two of the three designs D_1, D_2, D_3 as pairwise balanced designs.

(4) The method discussed in Theorem 3.2 gives construction of binary and non-binary designs by appropriate assignment of positive integers to positive integer constants $\bar{x}, \bar{y}, \bar{s}, \bar{w}, \bar{z}, \bar{n}$.

(5) (a) The design in Method 3.2 of Pal and Pal [9] follows from Theorem 3.2 taking N_2 as pairwise balanced design and $N_3 = I_v$, $\bar{w} = e$, $\bar{z} = 0$, $\bar{n} = k_2 - 1$, $\bar{m} = m_2$, $k_1 = k_1 - e$ (block size of Pal and Pal [9]), $\bar{x} = \bar{p} = \bar{y} = \bar{s} = 1$, $\bar{q} = m_1$, $r_1 = r$, respectively.

(b) The design in Method 3.3 of Pal and Pal [9] follows from Theorem 3.2 taking N_1 and N_2 as incidence matrices of pairwise balanced designs and $N_3 = I_v$, $\bar{w} = \bar{z} = 0$, $\bar{n} = k_2 - 1$, $\bar{m} = m_2$, $\bar{p} = m_1$, $\bar{x} = \bar{y} = \bar{q} = \bar{s} = 1$, respectively.

Example 3.2.1. Consider three BIB designs D_1, D_2 and D_3 having parameters $v = 16$, $b_1 = 16$, $r_1 = 6$, $k_1 = 6$, $\lambda_1 = 2$; $v = 16$, $b_2 = 24$, $r_2 = 9$, $k_2 = 6$, $\lambda_2 = 3$ and $v = 16$, $b_3 = 48$, $r_3 = 15$, $k_3 = 5$, $\lambda_3 = 4$, respectively. Using Theorem 3.2 with $\bar{x} = \bar{y} = 1$, $\bar{s} = \bar{n} = \bar{p} = \bar{m} = 1$, $\bar{q} = 3$ and $\bar{w} = \bar{z} = 0$, we get a new BIB design with parameters $v' = 17$, $b' = 136$, $r' = 48$, $k' = 6$ and pairing parameter λ' (say) $= 15$.

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