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Minimaxity of the Pitman Estimator of Ordered Normal Means When the Variances are Unequal

Somesh Kumar and Divakar Sharma*
University of Jammu, Jammu (Tawi) - 180 001
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Summary

Let $X_1, X_2, \ldots, X_k$ be independent normal random variables with means $\theta_1, \theta_2, \ldots, \theta_k$ and the common variance unity. It is assumed that $\theta_1 \le \theta_2 \le \ldots \le \theta_k$. Kumar and Sharma [4] showed that the Pitman estimator δ_p of $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$, that is, the generalized Bayes estimator of $\underline{\theta}$ with respect to the uniform prior on the space $\{\underline{\theta} : \theta_1 \le \theta_2 \le \ldots \le \theta_k\}$, is minimax when the loss function is the sum of squared errors. In this paper we assume that the variances of the k populations are not necessarily the same and prove the minimaxity of the Pitman estimator with respect to a scale invariant loss function.

Key Words : Loss function, Admissible Estimators, Minimax.

Introduction

Let $X_{i1}, \ldots, X_{in}$ be a random sample from a population with density function $f(x-\theta_i)$, i = 1, 2 with respect to lebesgue measure. Under the assumption that $\theta_1 \le \theta_2$ Blumenthal and Cohen [1] obtained sufficient conditions for the admissibility and minimaxity of the Pitman estimator of $\underline{\theta} = (\theta_1, \theta_2)$. Cohen and Sackrowitz [2] considered the estimation of larger of two translation parameters. When the populations are normal they obtained a class of admissible estimators. They also proved the minimaxity and admissibility of the Pitman estimator with respect to the squared error loss function. Kumar and Sharma [3] generalized the results of Blumenthal and Cohen [1] to densities $f_1(x-\theta_1)$ and $f_2(x-\theta_2)$ where f_1 and f_2 may be distinct, and later considered the estimation of k ordered normal means with common variance unity and proved that the Pitman estimator of $\underline{\theta} = (\theta_1, \ldots, \theta_k)$ is minimax when the loss function is the sum of squared errors. In this paper we extend this result to the case when the variances of the k populations are unequal and the loss function is scale invariant.

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* Indian Institute of Technology, Kanpur - 208 016

2. *Result*

Let $X_1, X_2, \ldots, X_k$ be random variables with means $\theta_1, \theta_2, \ldots, \theta_k$, $\theta_1, \leq \theta_2 \leq \ldots \leq \theta_k$; and unequal but known variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$ and the loss in estimating $\underline{\theta} = (\theta_1, \ldots, \theta_k)$ by $\underline{a} = (a_1, \ldots, a_k)$ be

$$L(\underline{\theta}, \underline{a}) = \sum_{i=1}^{k} \frac{(\theta_i - a_i)^2}{\sigma_i^2}. \tag{2.1}$$

The Pitman estimator $\underline{\delta}_p = (\delta_{p1}, \ldots, \delta_{pk})$ is the generalized Bayes estimator of $\underline{\theta}$ with respect to the uniform prior on $\Omega = \{\theta : \theta_1 \leq \theta_2 \leq \ldots \leq \theta_k\}$ and is given by

$$\delta_{pi}(\underline{x}) = \frac{\int_\Omega \theta_i\, p(\underline{x}, \underline{\theta})\, d\underline{\theta}}{\int_\Omega p(\underline{x}, \underline{\theta})\, d\underline{\theta}}, \quad i = 1, 2, \ldots, k, \tag{2.2}$$

where

$$p(\underline{x}, \underline{\theta}) = \prod_{i=1}^{k} \frac{1}{\sigma_i} \varphi\left(\frac{x_i - \theta_i}{\sigma_i}\right), \tag{2.3}$$

with φ the standard normal probability density function. Following Kumar and Sharma [4] define

$$\alpha_i(\underline{x}) = \int_\Omega (\theta_i - x_i)\, p(\underline{x}, \underline{\theta})\, d\underline{\theta}, \tag{2.4}$$

$$D(\underline{x}) = \int_\Omega p(\underline{x}, \underline{\theta})\, d\underline{\theta} \tag{2.5}$$

and

$$\gamma_i(\underline{x}) = \frac{\alpha_i(\underline{x})}{D(\underline{x})}, \qquad i = 1, \ldots, k. \tag{2.6}$$

Then $$\frac{\partial D}{\partial x_i} = \frac{\alpha_i(\underline{x})}{\sigma_i^2}, \text{ and} \tag{2.7}$$

$$\delta_{pi}(\underline{x}) = x_i + \gamma_i(\underline{x}), \qquad i = 1, \ldots, k. \tag{2.8}$$

Now we are ready to prove.

Theorem 2.1 : The Pitman estimator $\underline{\delta}_p$ is minimax for $\underline{\theta}$ with respect to the loss function (2.1).

The risk of $\underline{\delta}_p$ is given by

$$R(\underline{\theta},\ \underline{\delta}_p) = \sum_{i=1}^{k} E\left\{\frac{(X_i + \gamma_i(\underline{X}) - \theta_i)^2}{\sigma_i^2}\right\}$$

$$= k + \sum_{i=1}^{k} E\left\{\frac{\gamma_i^2(\underline{X}) + 2(X_i - \theta_i)\,\gamma_i(\underline{X})}{\sigma_i^2}\right\} \tag{2.9}$$

Following a modification in Stein's identity [5], we have

$$\frac{1}{\sigma_i^2} E(X_i - \theta_i)\ \gamma_i(\underline{X}) = E\frac{\partial}{\partial X_i}\ \gamma_i(\underline{X}),\ i = 1, \ldots, k.$$

Using relations (2.4) – (2.7) we get

$$\frac{1}{\sigma_i^2} E(X_i - \theta_i)\ \gamma_i(\underline{X}) = E\left(-\frac{\gamma_i^2(\underline{X})}{\sigma_i^2} + \frac{1}{D(\underline{X})}\frac{\partial}{\partial X_i}\ \alpha_i(\underline{X})\right) \tag{2.10}$$

so that (2.9) is equivalent to

$$R(\underline{\theta},\ \underline{\delta}_p) = k + \sum_{i=1}^{k} E\left[\frac{1}{D(\underline{X})}\left(\frac{X_i\alpha_i(\underline{X})}{\sigma_i^2} + \frac{\partial}{\partial X_i}\alpha_i(\underline{X})\right) - \frac{\theta_i\gamma_i(\underline{X})}{\sigma_i^2}\right]. \tag{2.11}$$

The following two identities, similar to (2.12) and (2.14) in Kumar and Sharma [4], can be easily established

$$\frac{x_i\,\alpha_i(\underline{x})}{\sigma_i^2} + \frac{\partial}{\partial x_i}\ \alpha_i(\underline{X}) = 0 \qquad \text{for all } \underline{x}\ \varepsilon\ R^k, \tag{2.12}$$

$$\sum_{i=1}^{k} \frac{\partial}{\partial x_i}\ D(\underline{X}) = 0 \qquad \text{for all } \underline{x}\ \varepsilon\ R^k. \tag{2.13}$$

Next define

$$u_0 = x_1,\ \ \mu_0 = \theta_1,\ \ u_i = x_{i+1} - x_i,\ \ \mu_i = \theta_{i+1} - \theta_i,$$

$$i = 1, \ldots, k-1. \tag{2.14}$$

Then D $(\underline{x})$ of (2.5) can be written as

$$D^*(\underline{u}) = \int_{-\infty}^{u_{k-1}} \cdots \int_{-\infty}^{u_1} \int_{-\infty}^{\infty} \frac{1}{\sigma_1}\ \varphi\left(\frac{\mu_o}{\sigma_1}\right) \frac{1}{\sigma_2}\ \varphi\left(\frac{\mu_o + \mu_1}{\sigma_2}\right) \cdots$$

$$\frac{1}{\sigma_k} \varphi \left(\frac{\mu_0 + \ldots + u_{k-1}}{\sigma_k} \right) d\mu_0 \; d\mu_1 \ldots d\mu_{k-1} \tag{2.15}$$

where $\underline{u} = (u_1, \ldots, u_{k-1})$.

Using identity (2.13) one can prove

$$\frac{\partial}{\partial x_1} D(\underline{x}) = -\frac{\partial}{\partial u_1} D^*(\underline{u}), \quad \frac{\partial}{\partial x_k} D(\underline{x}) = \frac{\partial}{\partial u_{k-1}} D^*(\underline{u}) \tag{2.16}$$

and

$$\frac{\partial}{\partial x_i} D(\underline{x}) = \frac{\partial}{\partial u_{i-1}} D^*(\underline{u}) - \frac{\partial}{\partial u_i} D^*(\underline{u}), \quad i = 1, 2, \ldots, k-1.$$

Relations (2.6), (2.7), (2.12), (2.14) and (2.16) together with (2.11) now yield

$$R(\underline{\theta}, \underline{\delta}_p) = k - \sum_{i=1}^{k-1} \mu_i \; E\left(\frac{1}{D(\underline{X})} \; \frac{\partial}{\partial U_i} D^*(\underline{U}) \right). \tag{2.17}$$

The fact that μ_i's are non-negative for $\underline{\theta} \in \Omega$ and $D^*(\underline{u})$ is an increasing function of μ_i, $i = 1, \ldots, k-1$ implies

$$R(\underline{\theta}, \underline{\delta}_p) \leq k \qquad \text{for all } \underline{\theta} \in \Omega. \tag{2.18}$$

It can be seen that Theorem 2.1 of Kumar and Sharma [3] and the corollary following it remain true for the loss function (2.1) and so from the inequality (2.18) we conclude that $\underline{\delta}_p$ is minimax for estimating $\underline{\theta}$ when $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$.

Remark : As in the case of equal variances, the estimator $\underline{\delta}_p$ is inadmissible and in both the cases the problem of obtaining an improvement over $\underline{\delta}_p$ is open.

3. *Applications*

The above problem of estimating ordered parameters, when the ordering is known, arises in several agricultural, industrial, sociological and economic studies. In the following examples we describe some such situations.

Example 1 : Suppose we want to measure the effectiveness of a fertilizer on a crop. For this we consider an experimental design where Treatment 1 is to grow a crop using the fertilizer and Treatment 2 is simply to grow the crop. If θ_1 and θ_2 are the average yields of two treatments, it is natural to expect that $\theta_1 \geq \theta_2$. Also $\theta_1 - \theta_2$ will give a measure of effectiveness of the fertilizer.

Example 2 : In an intelligence test, in order to estimate the grasping power of students, the following experiment was conducted. The student carries out a set of calculations twice on the same calculating machine. It is clear that with the increased familiarity with data he will take less time in second calculations. Thus if θ_1 and θ_2 are the average speeds of both the calculations we will have $\theta_1 \geq \theta_2$ and $\theta_1 - \theta_2$ will be give a measure of grasping power.

Example 3 : In an industrial experiment, changes are made in the design of an automobile so as to increase fuel efficiency. If the changes are effected in k stages θ_i and denotes the average fuel consumption at ith stage, we have $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_k$.

As proved in the Theorem 2.1, the Pitman estimator is better when compared to the natural estimator $\underline{X} = (X_1, \ldots, X_k)$ in all these situations.

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