LATTICE SAMPLING EXPERIMENTS IN FORESTRY

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1. INTRODUCTION

A lattice sample is a cluster sample in which the members forming the cluster are arranged according to a particular pattern τ. Thus an observation $L(n, \tau)$ for a lattice sample is denoted by the number of members (n) forming the lattice observation and its pattern. Thus, if we draw a sample of size m, *viz.* $(L_1, \ldots, L_m)$ from a population of lattices $\{L(n, \tau)\}$, it implies that the i-th lattice observation $L(n, \tau)$ comprises n nodal observations $(x_{i1}, \ldots, x_{in})$ $i=1, \ldots, m$. The problem dealt with in this paper is to determine theoretically the variance of lattice means $\bar{x}_i \left(= \sum_{j=1}^{n} x_{ij}/n \right)$ given the variance of individuals $\{x\}$ and spatial correlations of various orders (explained below) and compare with the experimental values of lattice group variances.

Experiments have been performed mainly on forestry models: trees have been grown in various lattice designs (see Fig. 2). Measurements have been made of diameter at waist height of trees (DBH). The purpose of the paper has been to study the effect of competition among neighbours, in terms of spatial correlations or various orders.

Though experiments reported here relate primarily to forestry, the theoretical approach developed for lattice sampling can be applied in a wide range of practical problems. For example, a gross sample (comprising say a number of small samples of 10 lb. each) used while sampling minerals like coal, ores, etc., may be regarded as a lattice sample without a pattern. The effect of pattern τ can however be studied while sampling from a coal seam within the mine.

The scheme of the paper is as follows: Section 2 gives a review of theoretical models, in related work ; Section 3 gives the model

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* Most of the work was done when this author (A.G.) was at Monash University, Australia.

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of lattice sampling and design of developing experiments; Section 4 gives the results of sampling experiments done in forestry: it also gives comparison of results with other models (c. f. the model of Fairfield Smith [1938]). Sakai et al [1968] also developed a model to take account of the shape and the size of the cluster: their model has been referred in Section 2 but detailed computations of the expected values of the group variances have not been shown. The theoretical justification of the model (given in sec. 2) is that it takes account both of n and τ; when there is no effect of τ, there is no advantage over Fairfield Smith's model—in fact it may, in many cases, be easier to compute graphically or otherwise smoothed values of group variance for various values of n.

2. Related Work

Consider a lattice sample $L(n, \tau)$ in which the members of the sample $(x_1, \ldots\ldots, x_n)$ are nodes in the lattice L with pattern τ. If x_i $(i=1, \ldots\ldots, n)$ are i. i. d. r. v. (independent identically distributed random variables) with population variance σ^2, then if the variance of the sample average $\bar{x}$ $(= \sum_{i=1}^{n} x_i/n)$ is given by

$$\text{(2.1)} \qquad \text{Var}(\bar{x}) = \sigma^2/n.$$

In (2.1) it is assumed that neither spatial correlation exists among x_i nor is there any effect due to the pattern τ. When there is correlation among neighbouring observations and/or the effect of the pattern τ significant, the formula (2.1) breaks down. A number of artifices has been used by various in experiments in agriculture, genetics, coal, sampling, etc.

Fairfield Smith [1938] suggested an empirical rule, in connection with homogeneity trials of agricultural groups, to express the variance of the lattice mean $(\bar{x})$ as

$$\text{(2.2)} \qquad \text{Var}(\bar{x}) = \sigma^2/n^b$$

where b is a constant lying between 0 and 1. Mahalanobis [1944] also applied (2.2) in crop-cutting experiments. Sakai et al [1968] suggested the following break-up of the total phenotypic variance in terms of the following model

$$\text{(2.3)} \qquad \text{Var}(\bar{x}) = \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n^b} + \frac{\sigma_3^2 T_n}{n}$$

where σ_1^2 is the variance due to the genetic factor, σ_2^2 the variance due to the environment factor, σ_3^2 the variance due to competition factor, and T_n a factor depending on the size and shape.

In Fairfield Smith's model (2.2) the parameter b is estimated from the experimental data on Var $(\bar{x})$ from the following

$$\log V(\bar{x}) = \log \sigma^2 - b \log n \tag{2.4}$$

or

$$\log n\ V(\bar{x}) = \log \sigma^2 + (1-b) \log n \tag{2.5}$$

where $V(\bar{x}) = \text{Var}(\bar{x})$. In fact from experimental results (also from theoretical considerations) b lies on the range $(0, \infty)$. The formula (2.2), however, does not give a mathematical derivation of b by taking into consideration n and τ. Besides, we may clump together cluster samples of various patterns but the same size n to estimate b: in such a case, if the effect of pattern τ is significant, we may not get a good smoothed value for $V(\bar{x})$ from (2.4) or (2.5).

The model by Sakai et al is theoretically tenable, but in practice leads to cumbersome computations.

The model for the mean of a gross sample, each comprising smaller increments, taken primarily from bulk materials like coal or minerals, was given by Ghosal [19:8, 1962]. According to that model, if σ^2 be the variance of increments (a gross sample comprises n increment), then the variance of the gross sample $V(\bar{x})$ is given by (see eqn. (7), p. 364 in Ghosal [1962]):

$$V(\bar{x}) = A + \frac{\sigma^2}{n}(1+\phi_n) \tag{2.6}$$

where A is a constant ($\geqslant 0$), and ϕ_n is a measure of mutual spatial correlation among n increments in a gross sample. A somewhat akin model was also given by Quinouille [1950]. A model almost similar to (2.6) enables us to consider the effect of the pattern τ in lattice sample: this has been explained in section 3. In the context of the work on coal sampling, the effect of A (due to drawing of samples) was significant.

3. Model

In the forestry experiments reported in this paper we apply the following relation to determine the variance of lattice means, $V(\bar{x})$, from the knowledge of nodal variances:

$$V(\bar{x}) = \frac{\sigma^2}{n}(1+\phi_n) \tag{3.1}$$

with

$$\phi_n = \frac{2}{n}\sum_{k=1}^{w} t_k\, \theta_k, \tag{3.2}$$

where t_k is the number of times the inter node distance d_k appears within each lattice sample L (n, τ) $(i = l, \ldots, m)$ and θ_k are certain constants. We have w such that

$$\sum_{k=1}^{w} t_k = \binom{n}{2};$$

thus w depends both on n and the pattern τ.

Let us consider the lattices shown in Fig. 1. In Fig. 1 (*b*) for example, $n=3$; the nodes 1 and 2 have unit distance d_1 $(=1)$; nodes 2 and 3 have also distance d_1, while the distance between nodes 1 and 3 is $d_2 (= \sqrt{2})$

From (3.2) we get

$$\phi_3 = 2\ (2\theta_1 + \theta_2)/3$$

In Fig. 1 (*a*), $n=7$, $d_1 = 1$, $d_2 = \sqrt{2}$, $d_3 = 2$, $d_4 = \sqrt{5}$, $d_5 = 2\sqrt{2}$ (distance between nodes 1 and 7). We have

$$\phi_7 = 2\ (8\theta_1 + 6\theta_2 + 2\theta_3 + 4\theta_4 + \theta_5)/7.$$

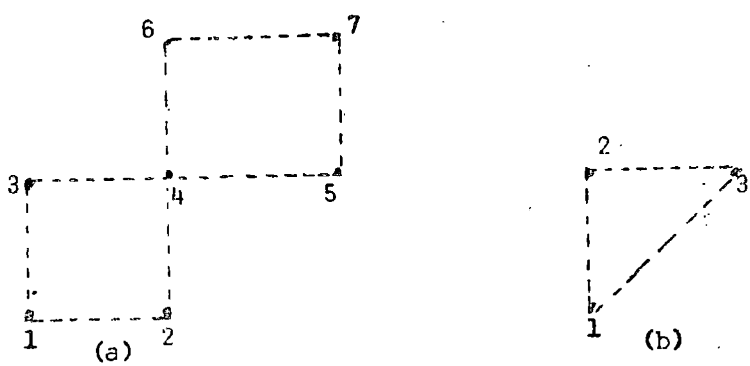

Fig. 1.

Next we deal with the problem of estimating θ_k, $k = 1, \ldots, j$ in 3.1. In the first step we prove that θ_k can be interpreted as spatial correlation.

We apply the super population approach Quinouille (1950), Cochran (1953)] as a general case. Let y_{ij} be the observed DBH for the nodal tree in the location co-ordinates (i, j) and let

$$E(y_{ij}) = \alpha_{ij},$$

$$E\ y_{ij} - \alpha_{ij})^2 = \sigma^2,$$

$$E(y_{ij} - \alpha_{ij})\ (y_{rs} - \alpha_{rs}) = \rho_{ji,\ rs}\sigma^2$$

where

(3·3) $\rho_{ij,\,rs}=\rho_d$

for all i, j, r, s such that $(i-r)^2+(j-s)^2=d^2$. In other words ρ_d is the spatial correlation between two nodes at a mutual distance d.

Consider a lattice observation $L(n, \tau)$ in which the nodal set is $(x_1,\ldots\ldots,x_n)$, where $x_1=y_{ij}$, $x_2=y_{rs}$, etc. Then we get $(\alpha_1=\alpha_{ij}$, $\alpha_2=\alpha_{rs}$, etc.).

$$E(x_1+\ldots\ldots+x_n)=\alpha_1+\ldots\ldots+\alpha_n$$
$$=n\overline{\alpha} \text{ (by defn.)}$$

The variance of $\sum_{i=1}^{n} x_i$ is given by

$$V\left(\sum_{i=1}^{w} x_i\right)=n\sigma^2+2\sigma^2\sum_{k=1}^{w} t_k\rho_k,$$

where t_k is the number of times the inter-node distance d_k appears within the lattice sample.

Hence the variance of the mean $(\bar{x})$ is given by (Var$=V$) :

$$V(\bar{x})=\frac{\sigma^2}{n}(1+\phi_n)$$

where

(3·4) $\phi_n=2\ \Sigma\ t_k\,\rho_k/n.$

Comparing with (3·2) we get, under the above conditions

(3·5) $\rho_k=\theta_k.$

For the ordinary population we have $E(y_{ij})=E(x)=\alpha$ for all i, j.

Validity of the model (3.1), therefore, depends on the condition that correlation between two nodes depends on the inter-nodal distance. The condition was found to hold true in the experiments done by the authors.

The design for the sampling experiments is given below. The full material (measurements of DBH of trees) was available for most of 13 plots reported in the paper. From this estimation was made of σ^2, ρ_1, ρ_2, ρ_3, etc.

Drawing of lattice samples of seven configurations (Fig. 2) was done on paper in the following manner : (*a*) a lattice population was constructed for each configuration, (*b*) lattice samples were drawn randomly. In a few bigger plots a master sample of trees gave estimates of σ^2 and spatial correlations ; it was partitioned in various ways to give lattice samples of various patterns.

4. Results of Sampling Experiments

The results reported in this paper relate to sampling experiments done in 16 plots of Pinus radiata of the Australian Paper Manufacturers Ltd. (APM). Seven types of lattice populations were considered (see Fig. 2).

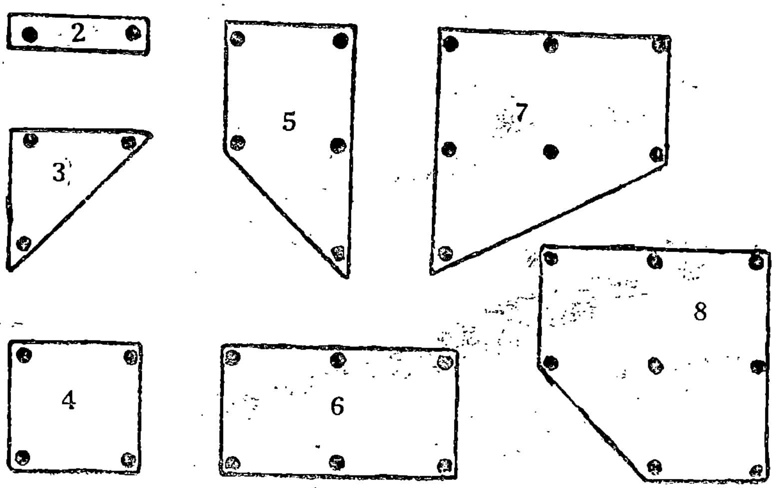

Fig. 2. Configuration of clusters of 2-8 trees, referred to in the study. For simulation studies, a specific shape and size of cluster would yield approximateld value of ϕ_n.

The manner in which forest plantation is square planted is depicted in Fig. 3 which however illustrates only with a total group of 25 trees. For any tree neighbours which lie at a specific distance can be joined by concentric shells. Thus shell 1 with radius d_1 (=1 say) with O as centre contains 4 nearest neighbours, shell 2 with radius d_2 (=2) contains 4 second nearest neighbours, etc. We can get a population of square lattices out of such a plantation in a number of different ways (one is shown in Fig. 3). With such a square plantation we also get all different types of lattice populations with configurations given in Fig. 2. In the experiment the population of a particular lattice configuration was so constructed that no individual tree belonged to two neighbouring lattice observations. Each plot comprised several acres, and the number of stems per acre varied from 500 to 1,050 with unit distance (d_1) ranging from 6·5 ft. to 9·25 ft. The size of each lattice population was finite (exceeding 300) and the sample size varied between 30 and 72. The characteristic measured was diameter at breast height (DBH).

[N.B.—If d_1 (unit distance)=1, then $d_2=\sqrt{2d_1}$, $d_3=2$, $d_4=\sqrt{5}$, $d_5=\sqrt{6}$, etc.]

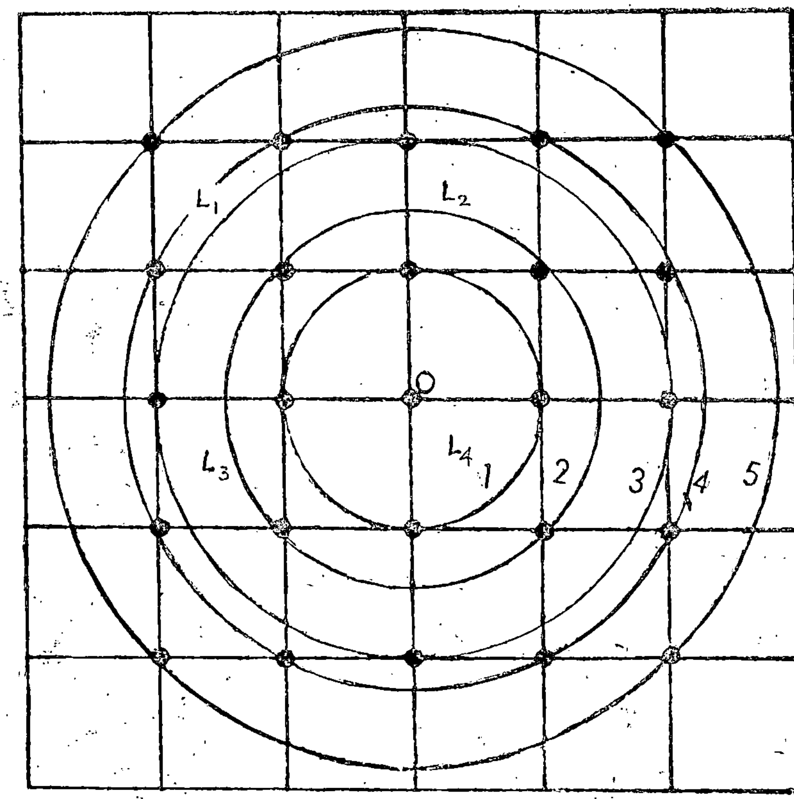

Fig. 3. Spatial configuration of various shells containing nearest, second nearest, etc. neighbours.

Results of the experiments have been presented in Table 1. Only ρ_1 and ρ_2 (spatial correlations of first two orders) have been reported, because in most of the cases ρ_r ($r>2$) was not significant. In one or two cases ρ_3 (spatial correlation at distance $d_3=2$) was also slgnificant. The expected variance of the group (lattice) means was calculated with estimated values of ρ_1 and ρ_2 only ; in most of the cases spatial correlation of higher orders were very small Results of observed and expected values of the group means variances showed the validity of the assumption (particularly for lattice samples of orders 2, 3, 4). Differerce was more marked for lattices of higher orders ($n\geqslant6$) : this could be explained more by the fact that for higher order lattices it may be necessary to include terms ρ_3 and ρ_4 in calculating the expected variances (in plot no. 13, inclusion of ρ_3 and ρ_4 improved estimated variances in V_6, V_7). Close concordance of the expected and observed values was marked for most of the values. The test applied to test the difference between the obs erved and expected variances was χ^2 test.

TABLE 1

Estimated and observed lattice mean variances

Plot No.	σ^2	ρ_1	ρ_2	V_2		V_3		V_4		V_5		V_6		V_7		V_8	
				Obs	*Exp*	*Obs*	*Exp*	*Obs*	*Exp*	*Obs*	*Exp*	*Obs*	*Exp*	*Obs*	*Exp*	*Obs*	*Exp*
1.	.302	—·0236	—·0211	·1549	·1474	·0825	·0961	·0679	·0703	·0580	·0560	·0510	·0461	·0448	·0395	·0405	·0341
2.	·437	·1625	·0260	·2416	·2540	·1724	·1791	·1396	·1476	·1212	·1185	·0976	·1030	·0839	·0879	·0847	·0793
3.	·538	—·0341	·1993*	·2785	·2598	·2188	·1950	·1736	·1521	·1546	·1260	·1304	·1064	·1160	·0927	·1172	·0850
4.	·343	—·1670*	—·0423	·1399	·1429	·0899	·0857	·0584	·0535	·0634	·0422	·0379	·0317	·0354	·0273	·0328	·0218
5.	·746	·0694	—·2254**	·3516	·3989	·1724	·2343	·1676	·1703	·1564	·1296	·1155	·1071	·1032	·0892	·0824	·0726
6.	·635	·0333	·0073	·2555	·3281	·1970	·2221	·1594	·1705	·1325	·1366	·0986	·1151	·0990	·0986	·0903	·0870
7.	·255	—·0717	—·1151	·1210	·1184	·0625	·0704	·0372	·0473	·0335	·0366	·0226	·0289	·0231	·0245	·0174	·0197
8.	·595	·1770	·1785*	·3313	·3502	·2554	·2687	·2402	·2280	·1983	·1866	·1500	·1637	·1019	·1411	·0964	·1305
9.	·477	·0922	·1428	·2534	·2605	·1690	·1937	·1475	·1583	·1106	·1293	·0701	·1117	·0556	·0964	·0599	·0883
10.	·255	·0368	·2584*	·1405	·1322	·1181	·1038	·0865	·0849	·0656	·0706	·0695	·0608	·0624	·0529	·0431	·0492
11.	·545	·0535	·0736	·3372	·2871	·2036	·2035	·1617	·1609	·1219	·1303	·1382	·1111	·1051	·0956	·1012	·0860
12.	·404	·0443	—·1450	·2274	·2109	·1493	·1296	·0897	·0953	·0577	·0739	·0717	·0513	·0609	·0516	·0459	·0433
13.	·325	·0931	·1515	·1734	·1776	·1393	·1327	·1143	·1087	·0958	·0889	·1021	·0769	·0873	·0664	·0783	·0609
14.	·469	—·1381	·1932*	·2173	·2021	·1596	·1477	·1131	·1075	·0833	·0896	·0805	·0731	·0628	·0643	·0542	·0582
15.	·568	—·1152	—·2313	·1758	·2513	·1033	·1311	·0711	·0764	·0766	·0559	·0477	·0400	·0530	·0330	·0226	·0218
16.	·226	·0466	—·2493*	·1042	·1183	·0823	·0675	·0542	·0477	·0377	·0359	·0330	·0292	·0349	·0242	·0179	·0192

* Significant at Prob. Level ·05. ** Significant at Prob. Level ·01.

N.B.—Expected variances are calculated on the basis of formula (3.1).

[For example, for plot 1 and lattice L_2, we test $H(V=0{\cdot}1472)$ on $m-1$ degrees of freedom. Since our $m=20$, our test criterion was $\chi^2=0{\cdot}1549/0{\cdot}1472=1{\cdot}052$, which was not significant at $P=$ ·05].

A plot of actual and expected values of $nV(n)$, where $V_n(\bar{x})$ represents the variance of the lattice mean of size n, gives a visual basis of comparison. Fig. 4 gives the expected values of $\log nV_n(\bar{x})$ by formula (3.1), as also by Fairfield Smith's form (2.5)

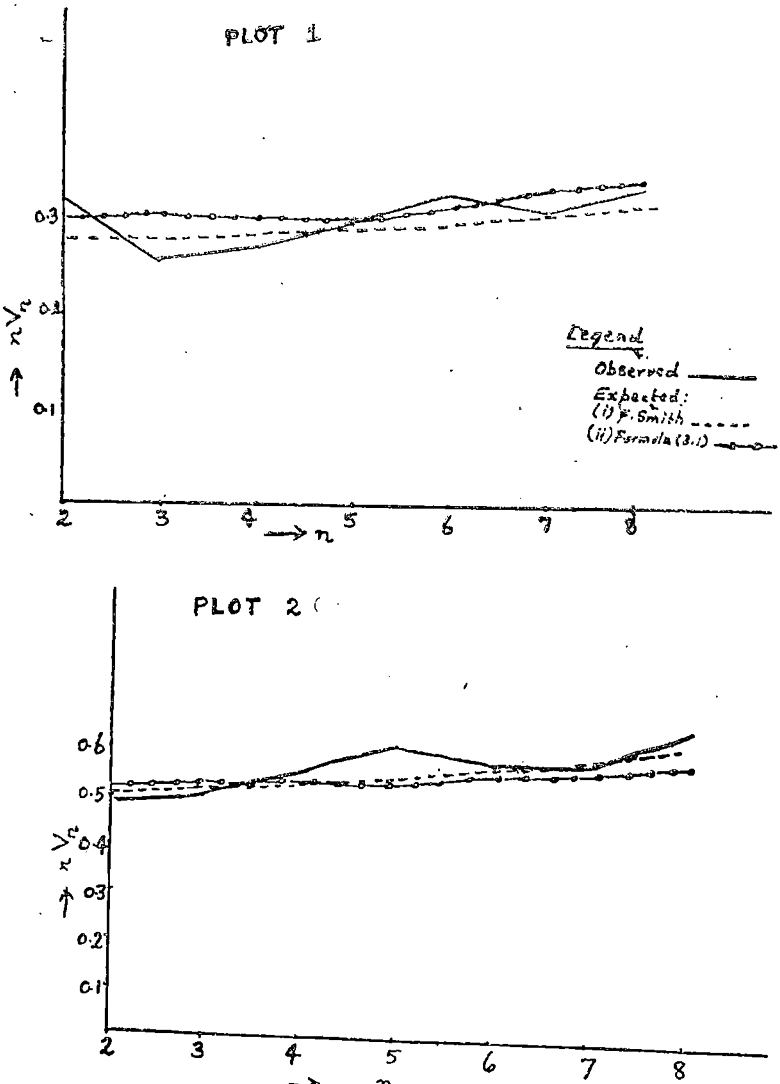

Fig. 4. Comparison of nV_n between observed and expected variances (Expected values are calculated on the basis of two formalse)

against the actual values (for plots 1 and 2). Both sets of formulae (2.5) and (3.1) give fairly concordant results : whatever differences we get we are not in a position to state definitely whether they are due to the effect of the lattice pattern, though this is a possibility. The authors have made a few pilot experiments to indicate the effect of the pattern in lattices with the same number of nodes : the effect of the lattice pattern appears to be significant in most of the cases (results of these experiments will be published in a subsequent communication). The conditions under which the formula (3.1) and Fairfield Smith's scheme lead to the same results as have been discussed earlier in Section 2 : is that the effect of pattern τ is absent.

5. Concluding Remarks

The main contribution of the paper is to give a method of determining the variance of a lattice sample by taking into consideration the effect of pattern and the number of nodes in the lattice. This has been done by applying the formula (3.1) which takes into consideration spatial correlation among observations.

It is possible to simulate the variance of lattice means for various patterns before actual experimentation is done. For example, if we get the lattice configuration for L_3 (with three nodes) in the form of three nodes in a line (as shown in Fig. 5) theoretically the variance $V_3\ (\bar{x})$ will be

$$V_3\ (\bar{x}) = \frac{\sigma^2}{3}(1+4/3\rho_1+2/3\rho_3)$$

Fig. 5. Three Nodes in a line.

For example, in Plot 13, $V_3(\bar{x})$ for this lattice would be 0.1160 as against 0.120, of the actual experimental value, on the basis of 10 samples, and 0.1320 for triangular lattice given in Table 1.

We can thus simulate $V_n\ (\bar{x})$ for various configurations of lattice L_n. Further experimental result on the different lattice patterns with same n will be reported subsequently.

The usefulness of such exercises lies in the fact that we get an insight into the optimum manner in which plantation may be made. If n (number of nodes) is prefixed and ρ_1, ρ_2 are positive, the optimum lattice leading to the minimum variance $V_n(\bar{x})$ is one in which nodes are arranged in a line (on the assumption that high order space correlation tends to zero). If ρ_1 is negative (or both ρ_1 and ρ_2 are negative), large lattice (large n) with pattern involving large coefficients of ρ_i and ρ_2 leads to small value of $nV_n\ (\bar{x})$. It may be mentioned that it is desirable to use correlation of higher orders : there may be situations in which ρ_3 and ρ_4 are high.

SUMMARY

The paper reports results of an experiment in Forestry in which a sample is taken from a population of lattices (of trees), where each lattice $L(n, \tau)$ comprises n trees grown in a specific pattern τ. The variance of the lattice mean $\bar{x}_n$ on the basis of m samples $(L_1, \ldots\ldots, L_m)$ is derived theoretically in terms of variance of nodes (σ^2), lattice size (n) and the pattern τ. The theoretical results have been compared with experimental values. Lattice sampling can be applied, on similar lines, in a wide range of practical studies.

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Appendix A

Computation of variance of lattice means (theoretically)

The method of deriving the variance of lattice mean $[V_n(\bar{x})]$ for a lattice of n nodes and pattern τ has been explained in Section 2. For the lattices of size and shape given in Fig. 2 ($n=2,...,8$), the variance $V_n(\bar{x})$ is given in Table A (the definitions of σ^2 and ρ_r, $r=1, 2,......$, have been given in Sections 1 and 2), neglecting P_r, $r \geqslant 3$.

Theoretical Variance of $\bar{x}$

$[V_n(\bar{x}) = \sigma^2 (1+\phi_n)/n]$.

n	ϕ_n
2	$1+\rho_1$
3	$(4/3)\rho_1+(2/3)\rho_2$
4	$2\rho_1+\rho_2$
5	$2\rho_1+(6/5)\rho_2+(2/5)\rho_3+(2/5)\rho_4$
6	$(14/6)\rho_1+(8/6)\rho_2+(4/6)\rho_3+(4/6)\rho_4$
7	$(16/7)\rho_1+(10/7)\rho_2+\rho_3+(8/6)\rho_4+(2/6)\rho_5$
8	$(5/2)\rho_1+(7/4)\rho_2+\rho_3+(12/8)\rho_4+(1/4)\rho_5$

In the above ρ_i=Spatial correlation between inter-nodal distance d_i, $i=1, 2,......$

$$d_1=1,\ d_2=\sqrt{2},\ d_3=2,\ d_4=\sqrt{5},\ d_5=2\sqrt{2}.$$