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Error Estimation in A Mixed Anova Model Using Two Preliminary Tests of Significance

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Summary

The paper presents an estimation procedure involving two preliminary tests of significance for estimation of the true error variance in the analysis of variance mixed model corresponding to a Conditionally Specified Inference procedure. The bias and mean square error of this estimation procedure have been studied. The estimator of the true error variance has been compared with the usual estimator as regards bias, mean square error and relative efficiency. The proposed estimator is found to be more efficient than the usual estimator when the first and the second doubtful errors are not significantly different from each other. Even in case of significance when the true error is three times or more than the first doubtful error the proposed estimator is more efficient. If the degrees of freedom of first and/orsecond doubtful errors are ensured to be high at the stage of planning of experiment, the proposed estimator becomes more efficient.

Key Words : Preliminary Test of Significance, Conditionally Specified inference, Efficiency,

Introduction

The study pertains to a conditionally specified inference procedure for which detailed bibliography may be seen in Han, Rao and Ravichandran [3]. It relates to a experimental design model for a split plot in time experiment in which some of the factors are fixed and the remaining random. These experiments are analogous to usual split plot experiments and are characterised mainly by the features that observations made are on the same whole unit over a period of time. Such situations arise frequently in experiments on forage crops (Steel and Torrie [5]), or with perennial and semi-perennial plants such as orchard and plantation crops like sugarcane, bananas, tropical fodder grasses etc. Considering a mixed model situation, one is interested in an estimator of the error variance when uncertainties regarding the parameters involved in the model specification exist.

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Ali and Srivastava [2] considered the following conditionally specified mixed AVOVA model corresponding to above mentioned split plot in time experiment having frequent use in forage crops,

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \tau_k + (\alpha\tau)_{ik} + (\beta\tau)_{jk} + \varepsilon_{ijk} \quad (1.1)$$

where, Y_{ijk} = Yield on the k^{th} cutting of the j^{th} variety in the i^{th} block, $i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, s$; $k = 1, 2, \ldots, t$; μ is the true mean effect, α_j is the random block effect and β_j, τ_k are the fixed effects of varieties and cuttings respectively. The cuttings effect, i.e. τ_k, is of main interest for which the abridged ANOVA table is as follows.

Table 1. Mixed model abridged ANOVA for a split-plot in time experiment

Source of variation	Degrees of freedom	Mean squares	Expected mean squares
Treatments (Cuttings)	$n_4 = t - 1$	V_4	$\sigma_4^2 = \sigma_\varepsilon^2 + s\,\sigma_{\alpha\tau}^2 + rs\,\sigma_\tau^2 = \sigma_3^2 (1 + s\,\lambda_4/n_4)$
True Error (Cuttings x Block)	$n_3 = (t-1)(r-1)$	V_3	$\sigma_3^2 = \sigma_\varepsilon^2 + s\sigma_{\alpha\tau}^2$
Doubtful Error II (Cuttings x Varieties)	$n_2 = (t-1)(s-1)$	V_2	$\sigma_2^2 = \sigma_\varepsilon^2 + r\,[\sigma_{\beta\tau}^2] = \sigma_1^2 (1 + 2\,\lambda_2/n_2)$
Doubtful Error I (Cuttings x Variety x Block)	$n_1 = (t-1)(s-1)(r-1)$	V_1	$\sigma_1^2 = \sigma_\varepsilon^2$

In Table 1 λ_2 and λ_4 are the non-centrality parameters. The model (1.1) applies to any three-way cross classification layout where any two factors may be fixed effects and the third being random.

The problem to be solved here is to find an estimate of σ_3^2, the true error variance, pertaining to the estimation situation arising out of the test proposed by Ali and Srivastava, where the doubtful condition that $(\alpha\tau)_{ik}$ and/or $(\beta\tau)_{jk}$ may equal to zero, i.e., $\sigma_{\alpha\tau}^2$ and/or $\sigma_{\beta\tau}^2$ may equal to zero (Table 1), exists. In other words, where σ_3^2 and/or $\sigma_2^2 \geq \sigma_1^2$. When $\sigma_3^2 \neq \sigma_2^2 \neq \sigma_1^2$, the usual never pool estimate of σ_3^2 is V_3.

To resolve the doubtful conditions Ali and Srivastava considered the preliminary tests $H_{01}: \sigma_3^2 = \sigma_1^2$ (i.e., $\theta_{31} = 1.0$) vs $H_{11}: \sigma_3^2 > \sigma_1^2$ ($\theta_{31} > 1.0$) and $H_{02}: \sigma_2^2 = \sigma_1^2$ (i.e. $\lambda_2 = 0$) vs $H_{12}: \sigma_2^2 > \sigma_1^2$ (i.e. $\lambda_2 > 0$) in succession on the outcomes of which they based their final test H_0: $\sigma_4^2 = \sigma_3^2$ (i.e. $\lambda_4 = 0$) against H_1: $\sigma_4^2 > \sigma_3^2$ (i.e. $\lambda_4 > 0$), where σ_4^2 is true treatment variance. In this study the same preliminary tests are used to estimate σ_3^2.

Using the same pooling procedure as adopted by Ali and Srivastava, a sometimes pool estimator. V, for estimating σ_3^2 is proposed as follows:

$$
\begin{aligned}
V = \; & V_3 && \text{if} && V_3/V_1 \geq F(n_3, n_1; \alpha_1) \\
= \; & V_{13} && \text{if} && \text{(i) } V_3/V_1 < F(n_3, n_1; \alpha_1) \\
& && \text{and} && \text{(ii) } V_2/V_{13} \geq F(n_2, n_{13}; \alpha_2) \\
= \; & V_{123} && \text{if} && \text{(i) } V_3/V_1 < F(n_3, n_1; \alpha_1) \\
& && \text{and} && \text{(ii) } V_2/V_{13} < F(N_2, n_{13}; \alpha_2)
\end{aligned}
\tag{1.2}
$$

where $V_{13} = (n_1 V_1 + n_3 V_3)/(n_1 + n_3)$, $V_{123} = (n_1 V_1 + n_2 V_2 + n_3 V_3)/(n_1 + n_2 + n_3)$ are the different pooled mean squares with respective degrees of freedom $n_{13} = n_1 + n_3$, $n_{123} = n_1 + n_2 + n_3$ and $F(n_i, n_j; \alpha_k)$ is the upper 100 α_k % point of the central F-distribution with (n_i, n_j) degrees of freedom.

The motivation behind proposing V is that, it gives an estimator of σ_3^2 under the most general parametric situation, i.e., σ_3^2 and/or $\sigma_2^2 \geq \sigma_1^2$. The usual estimator V_3 corresponds to the only situation $\sigma_3^2 \neq \sigma_2^2 \neq \sigma_1^2$.

2. *Mean Value, Bias and Mean Square Error of Estimator V Along With Its Efficiency Relative to Never Pool Estimator* V_3

The mean value of estimator V, E(V), may be written as:

$$E(V) = E[V_3 \mid (V_3/V_1) \geq F(n_3, n_1; \alpha_1)] \Pr[(V_3/V_1) \geq F(n_3, n_1; \alpha_1)]$$

$$+ E[V_{13} \mid (V_3/V_1) < F(n_3, n_1; \alpha_1), (V_2/V_{13}) \geq F(n_2, n_{13}; \alpha_2)]$$

$$\Pr[(V_3/V_1) < F(n_3, n_1; \alpha 1), (V_2/V_{13}) \geq F(n_2, n_{13;} \alpha 2)]$$

$$+ E[V_{123} \mid (V_3/V_1) < F(n_3, n_1; \alpha_1), (V_2/V_{13}) < F(n_2, n_{13}; \alpha_2)]$$

$$\Pr[(V_3/V_1) < F(n_3, n_1; \alpha 1), (V_2/V_{13}) < F(n_2, n_{13;} \alpha 2)] \quad (2.1a)$$

or, say,

$$E(v) = E_1 P_1 + E_2 P_2 + E_3 P_3 \quad (2.1b)$$

where, $E_1 = E[V_3 \mid (V_3/V_1) \geq F(N_3, n_1; \alpha_1)]$,

$P_1 = \Pr[(V_3/V_1) \geq F(n_3, n_1; \alpha_1)]$

and E_2P_2, E_3P_3 are similarly defined.

For maintaining the continuity of presentation the derivations for E_1P_1, E_2P_2 and E_3P_3 have been relegated to the appendix. The expressions derived there are substituted in (2.1) to get the mean value, E(V). Then the bias is obtained by BIAS (V) = E (V) - σ_3^2.

The mean square error of the estimator V is defined as,

$$MSE(V) = E[V - \sigma_3^2]^2 = E(V^2) - 2\sigma_3^2 E(V) + (\sigma_3^2)^2 \quad (2.2)$$

In the r.h.s. of equation (2.2), the only unevaluated quantity is E (V^2), given σ_3^2. Therefore, to evaluate E (V^2) it can be expressed as in case of E(V) given by (2.1). Thus,

$$E(V^2) = E_{11} P_1 + E_{22} P_2 + E_{33} P_3 \quad (2.3)$$

where,

$E_{11} = E[V_3^2 \mid (V_3/V_1) \geq F(n_3, n_1; \alpha_1)]$,

$P_1 = \Pr[(V_3/V_1) \geq F(n_3, n_1; \alpha_1)]$

and $E_{22} P_2$, $E_{33} P_3$ are similarly defined.

Again the derived results from the appendix are used in (2.3) to get the expression for $E(V^2)$. Then MSE(V) is evaluated from (2.2) using the final expressing for $E(V^2)$ and E(V).

The relative efficiency of the estimator V with respect to the never pool estimator V_3 is given by R.E. $= \frac{\text{MSE }(V_3)}{\text{MSE }(V)} = \frac{2\,(\sigma_3^2)^2/n_3}{\text{MSE }(V)}$, since MSE $(V_3) = E\,(V_3^2) - (\sigma_3^2)^2$, $E\,(V_3^2) = (\sigma_3^2)^2 + \{2\,(\sigma_3^2)^2/n_3\}$.

3. *Discussion*

In order to examine whether the proposed estimation procedure conforms to the results of the corresponding test procedure the bias, mean square error and relative efficiency have been calculated for the same set of parameters considered by Ali [1], Ali and Srivastava [2] and presented in the Tables A.1 to A.5.

3.1 *Bias*

On perusal of Table A.1 through A.5 in the appendix it is found for any fixed λ_2 ($\lambda_2 \geq 0$), the bias of V decreases continuously with the increase in variance ratio θ_{31} $(= \sigma_3^2/\sigma_1^2)$ the chances of estimator V reducing to the unbiased estimator V_3 corresponding to never pool case become higher, and hence the bias is reduced. On the other hand, for a given value for variance ratio θ_{31} $(1.0 \leq \theta_{31} \leq 8.0)$ the bias increases consistently with λ_2. This is also evident from Table 1 and the expression (2.1a) for E(V), that the increase in λ_2 increases $E(V_2)$ resulting in the increase of E(V) and BIAS (V).

Now, for $\lambda_2 > 0$, and for all the values of θ_{31} under study, it is further noticed that the bias decreases substantially when any one of the error degrees of freedom (n_1, n_2 or n_3) is increased (compare Tables A.1 to A.5). On ther other hand, when $\lambda_2 = 0$, for all θ_{31}, the increase in any one of the doubtful error degrees of freedom (n_1 or n_2) increases the bias while the increase in n_3 decreases it because with $\lambda_2 = 0$ (i.e. $\sigma_2^2 = \sigma_1^2$, $\alpha_2 = 0 \Rightarrow F\,(N_2, n_{13};\alpha_2) \rightarrow \infty$), the chances of V reducing to V_3 or V_{123} increases according as n_3 or (n_1 and/or n_2) increases (see Table 1 and equations (1.2), (2.1).

3.2. *Mean Square Error and Relative Efficiency*

The entries for the mean square errors and relative efficiency have been presented in the columns four to six of Tables A.1 through A.5. Since the effect of mean square error of V manifests itself through its relative efficiency (RE) over V_3, the numerical discussion will be confined to the RE only.

It is observed that for $\lambda = 0$, efficiency of the estimator V is uniformly higher than that of the usual estimator V_3 for all values of θ_{31} However, this relative efficiency generally decreases with the increase in the variance ratio θ_{31} and both the estimator (V and V_3) become equally efficient for $\theta_{31} = 8.00$. This may be due to the fact that for populations with higher values of θ_{31}, the chances of the estimator V reducing to never pool estimator V_3 become high.

For $\lambda_2 > 0$ and for a particular case $\theta_{31} = 1.0$, $(\sigma_3^2 = \sigma_1^2)$, the proposed estimator V is less efficient than V_3 for a few combinations of parameters.

For $\lambda_2 > 0$ and for moderate values of θ_{31} $(1.0 < \theta_{31} < 3.0)$, the estimator V is in general more efficient that V_3 in situations where n_1 and/or n_2 are large. However, for small values of n_1 and/or n_2, V is less efficient for higher values of λ_2.

For $\lambda_2 > 0$, $\theta_{31} \geq 3.0$, the efficiency of the proposed estimator is uniformly higher than that of the usual estimator V_3 and thus V is again superior to V_3. This efficiency of V increases as λ_2 increases for a given θ_{31} in the range $\theta_{31} \geq 5.0$.

As regards the effect of degrees of freedom when n_1 and/or n_2 are increased the efficiency of V increases as against the usual estimator V_3 except when $\theta_{31} \geq 5.0$. This increase in n_1 and/or n_2 can be achieved by increasing the number of varieties and/or replications in planning stage of the experiments under the model situation considered. With the increase in n_3, the relative efficiency of V over V_3 is observed to be smaller for $\theta_{31} \geq 5.0$.

4. *Conclusions*

When the interaction of cuttings into varieties is not significant, the proposed estimator V is more efficient than the usual estimator V_3 for a true error up to almost eight times more than the first doubtful error. Even if the above interaction is significant and the true error is three times or more than the first doubtfull error, V is uniformly more efficient than V_3. For the true error five or more times, this efficiency is further increased as the significance of interaction becomes more powerful. More efficiency of V can also be ensured by keeping the degrees of freedom n_1 and/or n_2 large, say $\geq$ 10. Thus, it is concluded that the estimator V should be preferred to V_3 as in most regions it is superior. This conclusion is in conformity with the results of the corresponding test procedure discussed in Ali [1], Ali and Srivastava [2].

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REFERENCES

[1] Ali, M.A., 1979. "Hypothesis testing in a mixed model based on conditional specifications." Ph.D. thesis, Banaras, Hindu University. India.

[2] Ali. M.A. and Srivastava S.R., 1983. "On power function of a sometimes pool test procedure in a mixed model-I: A theoretical investigation." *Jour. Ind. Soc. Agri. Stat.* **35**, 80–90.

[3] Han, C.P., Rao, C.V. and Ravichandran, J., 1988. "Inference based on conditional specification: a second bibliography." *Communications in Statistics - Theory and Methods.* **17(6)**. 1945–1964.

[4] Patnaik, P.B., 1949. "The non-central χ^2 and F-distributions and their applications." *Biometrika*, **36**. 202-232.

[5] Steel, R.G.D. and Torrie J.H., 1980. "Principles and Procedures of Statistices." Mc-Graw Hill Book Company, Inc., New York.

APPENDIX

Joint density function

In order to find the mean value, obtain first the joint density function of V_i, $i = 1, 2, 3, 4$, namely

$$f(V_1, V_2, V_3) = A\ V_1^{1/2\, n_1 - 1}\ V_2^{1/2\, v_2 - 1}\ V_3^{1/2\, n_3 - 1} .\exp\left[-\frac{1}{2}\left\{\frac{n_1 V_1}{\sigma_1^2} + \frac{n_2 V_2}{(\sigma_1^2 c_2)} - \frac{n_3 V_3}{\sigma_3^2}\right\}\right] \quad \text{(A.1a)}$$

where

$$A = \frac{\left(\frac{n_1}{\sigma_1^2}\right)^{\frac{1}{2}n_1}\left(\frac{n_2}{\sigma_1^2 c_2}\right)^{\frac{1}{2}v_2}\left(\frac{n_3}{\sigma_3^2}\right)^{\frac{1}{2}n_3}}{2^{1/2(n_1+v_2+n_3)}\,\Gamma\left(\frac{1}{2}n_1\right)\Gamma\left(\frac{1}{2}v_2\right)\Gamma\left(\frac{1}{2}n_3\right)} \quad \text{(A.1b)}$$

after using the Patnaik's [4] approximation to non-central Chi-squares (for V_2), so that

$$V_2 = n_2 + \frac{4\lambda_2^2}{n_2 + 4\lambda_2},\quad c_2 = 1 + \frac{2\lambda_2}{n_2 + 2\lambda_2} \quad \text{(A.1c)}$$

where V_2's are to be taken as whole numbers.

Introducing the transformations:

$$u_1 = \frac{n_3 V_3}{\left(n_1 V_1 \theta_{31}\right)},\quad u_2 = \frac{n_2 V_2}{\left(n_2 V_1 c_2\right)},\quad u_3 = \frac{n_1 V_1}{(2\sigma_1^2)} \quad \text{(A.2)}$$

where $0 \le u_1 < \infty$, $0 \le u_2 < \infty$, $0 \le u_3 < \infty$; $\theta_{31} = \frac{\sigma_3^2}{\sigma_1^2}$, the joint density function can be rewritten as

$$f(u_1, u_2, u_3) = A_1\ u_1^{1/2\, n_3 - 1}\ u_2^{1/2 V_2 - 1}\ u_3^{1/2(n_1 + v_2 + n_3) - 1}\exp\left[-\frac{1}{2}\{2\, u_3\,(1 + u_1 + u_2)\}\right] \quad \text{(A.3a)}$$

$$\text{where,}\quad A_1 = \frac{1}{\Gamma(\frac{1}{2}n_1)\ \Gamma(\frac{1}{2}v_2)\ \Gamma(\frac{1}{2}n_3)} \quad \text{(A.3b)}$$

Derivation of E_1P_1, E_2P_2 and E_3P_3

To derive E_1P_1, express V_3 and $[(V_3/V_1) \ge F(n_3, n_1; \alpha_1)]$ in terms of u's, so that

$$E_1P_1 = E\,(2\,\sigma_3^2/n_3)\ u_1 u_3 \mid u_1 \ge a\ \ \text{pr}\ (u_1 \ge a)$$

$$= \int_{u_1=a}^{\infty}\int_{u_2=0}^{\infty}\int_{u_3=0}^{\infty} 2\,\frac{\sigma_3^2}{n_3}\,u_1 u_3\ f(u_1, u_2, u_3)\ du_3\, du_2\, du_1 \quad \text{(A. 4a)}$$

where $$a = \frac{n_3}{n_1 \ \theta_{31}} F(n_3, n_1; \alpha_1) \tag{A. 4b}$$

Then we apply the transformations,

$$z = u_3 \ (1 + u_1 + u_2) \quad \text{so that,} \quad dz = (1 + u_1 + u_2), \tag{A. 5}$$

and $$y = \frac{1 + u_1}{1 + u_1 + u_2} \text{ so that, } u_2 = (1 + u_1)\left\{\left(\frac{1}{y}\right) - 1\right\}, du_2 = -(1 + u_1)\left(\frac{1}{y^2}\right) dy \tag{A.6}$$

in succession to integrate out u_3 and u_2 and then use the binomial expansion $(1 - y)^{1/2 \, v_2 - 1}$ to complete the integration w.r.t. u_2. Finally, transforming u_1 by

$$t = \frac{1}{1 + u_1} \quad \text{so that, } u_1 = \frac{1}{t} - 1, du_1 = -\frac{1}{t^2} dt \tag{A.7}$$

we obtain on simplification,

$$E_1 P_1 = A_2 \ B\left(\frac{1}{2}(n_1 + n_3) + 1, \frac{1}{2} v_2\right) Bx_1 \left(\frac{1}{2} n_1, \frac{1}{2} n_3 + 1\right) \tag{A.8a}$$

where,

$$A_2 = \frac{2\sigma_3^2}{n_3} \frac{\Gamma\left(\frac{1}{2}(n_1 + v_2 + n_3) + 1\right)}{\Gamma\left(\frac{1}{2} n_1\right) \Gamma\left(\frac{1}{2} v_2\right) \Gamma\left(\frac{1}{2} n_3\right)} \quad \text{and } x_1 = \frac{1}{1 + a} \tag{A.8b}$$

We can also show that (A.8a) reduces to,

$$E_1 P_1 = \sigma_3^2 \ I_{x_1}\left(\frac{1}{2} n_1, \frac{1}{2} n_3 + \right) \text{ where } I_x = \frac{B_x(p, q)}{B(p, q)} \tag{A.8c}$$

The expressions for $E_2 P_2$ and $E_3 P_3$ have been obtained in similar way.

$$E_2 P_2 = A_3 \sum_{i=0}^{\frac{1}{2} v_2 - 1} \frac{(-1)^i \binom{\frac{1}{2} v_2 - 1}{i}}{\frac{1}{2}(n_1 + n_3) + i + 1} \sum_{j=0}^{i} \frac{\binom{i}{j}}{(1 + b)^{\frac{1}{2} n_1 + i - j + 1} (1 + b\,\theta_{31})^{\frac{1}{2} n_3 + j}}$$

$$\left[B x_2\left(\frac{1}{2} n_3 + j, \frac{1}{2} n_1 + i - j + 1\right) + \theta_{31} \left[\frac{1 + b}{1 + b\,\theta_{31}}\right] B x_2 \left(\frac{1}{2} n_3 + j + 1, \frac{1}{2} n_1 + i - j\right)\right] \tag{A.9a}$$

and

$$E_3 P_3 = A_4 \ B\left(\frac{1}{2}(n_1 + n_3) + 1, \frac{1}{2} v_2\right) \left\{B x_3\left(\frac{1}{2} n_3, \frac{1}{2} n_1 + 1\right) + \theta_{31} \ B x_3\left(\frac{1}{2} n_3 + 1, \frac{1}{2} n_1\right)\right\}$$

$$- A_4 \sum_{i=0}^{\frac{1}{2}v_2-1} \frac{(-1)^i \binom{\frac{1}{2}v_2-1}{i}}{\frac{1}{2}(n_1+n_3)+i+1} \sum_{j=0}^{i} \frac{\binom{i}{j}}{(1+b)^{\frac{1}{2}n_1+i-j+1}(1+b\,\theta_{31})^{\frac{1}{2}n_3+j}}$$

$$\left[B\,x_2\left(\frac{1}{2}n_3+j,\frac{1}{2}n_1+i-j+1\right)+\theta_{31}\left[\frac{1+b}{1+b\,\theta_{31}}\right]B\,x_2\left(\frac{1}{2}n_3+j+1,\frac{1}{2}n_1+i-j\right)\right]$$

$$+ A_4\, c_2\, B\left(\frac{1}{2}(n_1+n_3),\frac{1}{2}v_2+1\right) B\,x_3\left(\frac{1}{2}n_3,\frac{1}{2}n_1\right)$$

$$- A_4\, c_2 \sum_{i=0}^{\frac{1}{2}v_2} \frac{(-1)^i \binom{\frac{1}{2}v_2}{i}}{\frac{1}{2}(n_1+n_3)+i} \sum_{j=0}^{i} \binom{i}{j} \frac{B\,x_2\left(\frac{1}{2}n_3+j,\frac{1}{2}n_1+i-j\right)}{(1+b)^{\frac{1}{2}n_1+i-j}(1+b\,\theta_{31})^{\frac{1}{2}n_3+j}}$$

(A.9b)

where,

$$A_3 = \frac{2\,\sigma_1^2}{(n_1+n_3)} \frac{\Gamma\left(\frac{1}{2}(n_1+v_2+n_3)+1\right)}{\Gamma\left(\frac{1}{2}n_1\right)\Gamma\left(\frac{1}{2}v_2\right)\Gamma\left(\frac{1}{2}n_3\right)},$$

$$A_4 = \frac{2\,\sigma_1^2}{(n_1+n_2+n_3)} \frac{\Gamma\left(\frac{1}{2}(n_1+v_2+n_3)+1\right)}{\Gamma\left(\frac{1}{2}n_1\right)\Gamma\left(\frac{1}{2}v_2\right)\Gamma\left(\frac{1}{2}n_3\right)},$$

and $x_2 = \dfrac{(1+b\theta_{31})\,a}{1+b\,(1+b\theta_{31})\,a}, \quad x_3 = \dfrac{a}{1+a}$

$$a = \left\{\frac{n_3}{(n_1\,\theta_{31})}\right\} F(n_3, n_1; \alpha_1),\quad b = \frac{n_2}{(c_2\,n_{13})} F(n_2, n_{13}; \alpha_2) \qquad \text{(A.9c)}$$

Derivation of E_{11} P_1, E_{22} P_2 and E_{33} P_3

Using similar procedures as in the evaluation of E_1 P_1, E_2 P_2 and E_3 P_3 one can also evaluate E_{11} P_1, E_{22} P_2 and E_{33} P_3. For the sake of brevity only the final expressions are given below:

$$E_{11}\,P_1 = A_5\; B\left(\frac{1}{2}(n_1+n_3)+2,\frac{1}{2}v_2\right)\; B\,x_1\left(\frac{1}{2}n_1,\frac{1}{2}n_3+2\right)$$

$$= \left\{\left(\sigma_3^2\right)^2 + \frac{2\,(\sigma_3^2)^2}{n_3}\right\} I\,x_1\left(\frac{1}{2}n_1,\frac{1}{2}n_3+2\right) \qquad \text{(A.10a)}$$

$$E_{22} P_2 = A_6 \sum_{i=0}^{\frac{1}{2} v_2 - 1} (-1)^i \frac{\binom{\frac{1}{2} v_2 - 1}{i}}{\frac{1}{2} (n_1 + n_3) + i + 2}$$

$$\sum_{j=0}^{i} \frac{\binom{i}{j}}{(1+b)^{\frac{1}{2} n_1 + i - j + 2} (1 + b\,\theta_{31})^{\frac{1}{2} n_3 + j}}$$

$$\Bigg[B\,x_2 \left(\frac{1}{2} n_3 + j, \frac{1}{2} n_1 + i - j + 2 \right) + 2\,\theta_{31} \left[\frac{1+b}{1+b\,\theta_{31}} \right]$$

$$B\,x_2 \left(\frac{1}{2} n_3 + j + 1, \frac{1}{2} n_1 + i - j + 1 \right)$$

$$+ \theta_{31}^2 \left[\frac{1+b}{1+b\,\theta_{31}} \right]^2 B\,x_2 \left(\frac{1}{2} n_3 + j + 2, \frac{1}{2} n_1 + i - j \right) \Bigg] \qquad \text{(A, 10b)}$$

and

$$E_{33} P_3 = A_7 \; B \left(\frac{1}{2} (n_1 + n_3) + 2, \frac{1}{2} v_2 \right) \Bigg\{ B\,x_3 \left(\frac{1}{2} n_3, \frac{1}{2} n_1 + 2 \right)$$

$$+ 2\,\theta_{31}\, B\,x_3 \left(\frac{1}{2} n_3 + 1, \frac{1}{2} n_1 + 1 \right) + \theta_{31}^2 \; B\,x_3 \left(\frac{1}{2} n_3 + 2, \frac{1}{2} n_1 \right) \Bigg\}$$

$$- A_7 \sum_{i=0}^{\frac{1}{2} v_2 - 1} (-1)^i \frac{\binom{\frac{1}{2} v_2 - 1}{i}}{\frac{1}{2} (n_1 + n_3) + i + 2} \sum_{j=0}^{i} \frac{\binom{i}{j}}{(1+b)^{\frac{1}{2} n_1 + i - j + 2} (1 + b\,\theta_{31})^{\frac{1}{2} n_3 + j}}$$

$$\Bigg[B\,x_2 \left(\frac{1}{2} n_3 + j, \frac{1}{2} n_1 + i - j + 2 \right) + 2\,\theta_{31} \left[\frac{1+b}{1+b\,\theta_{31}} \right]$$

$$B\,x_2 \left(\frac{1}{2} n_3 + j + 1, \frac{1}{2} n_1 + i - j + 1 \right)$$

$$+ \theta_{31}^2 \left[\frac{1+b}{1+b\,\theta_{31}} \right]^2 B\,x_2 \left(\frac{1}{2} n_3 + j + 2, \frac{1}{2} n_1 + i - j \right) \Bigg]$$

$$+ A_7 \; (2\,c_2) \; B \left(\frac{1}{2} (n_1 + n_3) + 1, \frac{1}{2} v_2 + 1 \right) \Bigg\{ B\,x_3 \left(\frac{1}{2} n_3, \frac{1}{2} n_1 + 1 \right)$$

$$+ \theta_{31}\, B\,x_3 \left(\frac{1}{2} n_3 + 1, \frac{1}{2} n_1 \right) \Bigg\}$$

$$
-A_7\ (2\ c_2) \sum_{i=0}^{\frac{1}{2}v_2} \frac{(-1)^i \binom{\frac{1}{2}v^2}{i}}{\frac{1}{2}(n_1+n_3)+j+1}
$$

$$
\sum_{j=0}^{i} \frac{\binom{i}{j}}{(1+b)^{\frac{1}{2}n_1+i-j+1}\ (1+b\,\theta_{31})^{\frac{1}{2}n_3+j}}
$$

$$
\left[B\,x_2\left(\frac{1}{2}n_3+j,\ \frac{1}{2}n_1+i-j+1\right)+\theta_{31}\left[\frac{1+b}{1+b\,\theta_{31}}\right.\right.
$$

$$
\left.\left. B\,x_2\left(\frac{1}{2}n_3+j+1,\ \frac{1}{2}n_1+i-j\right)\right]\right]
$$

$$
+A_7\ (c_2^2)\ B\left(\frac{1}{2}(n_1+n_3),\ \frac{1}{2}v_2+2\right)\ B\,x_3\left(\frac{1}{2}n_3,\ \frac{1}{2}n_1\right)
$$

$$
-A_7\,(c_2^2) \sum_{i=0}^{\frac{1}{2}v_2+1} \frac{(-1)^i \binom{\frac{1}{2}v_2+1}{i}}{\frac{1}{2}(n_1+n_3)+i} \sum_{j=0}^{i} \binom{i}{j} \frac{B\,x_2\left(\frac{1}{2}n_3+j,\ \frac{1}{2}n_1+i-j\right)}{(1+b)^{\frac{1}{2}n_1+i-j}\ (1+b\,\theta_{31})^{\frac{1}{2}n_3+j}} \tag{A.10c}
$$

where,

$$
A_5 = \frac{4\,(\sigma_3^2)^2}{n_3^2}\ \frac{\Gamma\left(\frac{1}{2}(n_1+v_2+n_3)+2\right)}{\Gamma\left(\frac{1}{2}n_1\right)\Gamma\left(\frac{1}{2}v_2\right)\Gamma\left(\frac{1}{2}n_3\right)} \tag{A.10d}
$$

$$
A_6 = \frac{4\,(\sigma_1^2)^2}{(n_1+n_3)^2}\ \frac{\Gamma\left(\frac{1}{2}(n_1+v_2+n_3)+2\right)}{\Gamma\left(\frac{1}{2}n_1\right)\Gamma\left(\frac{1}{2}v_2\right)\Gamma\left(\frac{1}{2}n_3\right)} \tag{A.10e}
$$

$$
A_7 = \frac{4\,(\sigma_1^2)^2}{(n_1+n_2+n_3)^2}\ \frac{\Gamma\left(\frac{1}{2}(n_1+v_2+n_3)+2\right)}{\Gamma\left(\frac{1}{2}n_1\right)\Gamma\left(\frac{1}{2}v_2\right)\Gamma\left(\frac{1}{2}n_3\right)} \tag{A.10f}
$$

Tables A.1 to A.5. Bias and MSE of the sometimes pool estimator V involving two preliminary tests and its relative efficiency over never pool estimator V_3.

Table A.1

$n_1 = n_2 = n_3 = 2, \alpha_1 = \alpha_2 = \alpha_p = 0.50$					
θ_{31}	λ_2	BIAS (V)	MSE (V)	MSE (V_3)	e (V, V_3) %
1.0	0.0000	0.1669	0.9708	1.00	103.00
	2.4142	0.6908	1.7898	1.00	55.87
	4.4494	0.9915	2.5109	1.00	39.82
	6.4641	1.3273	3.9562	1.00	25.27
	8.4721	1.6620	5.8455	1.00	17.10
	10.4772	1.9961	8.1791	1.00	12.22
1.5	0.0000	0.1269	2.1076	2.25	106.75
	2.4142	0.5663	2.3491	2.25	95.77
	4.4494	0.7799	2.7712	2.25	81.19
	6.4641	1.0485	3.6947	2.25	60.89
	8.4721	1.3163	4.9740	2.25	45.23
	10.4772	1.5835	6.6094	2.25	34.04
2.0	0.0000	0.1014	3.7849	4.00	105.68
	2.4142	0.4747	3.6292	4.00	110.21
	4.4494	0.6425	3.8464	4.00	103.99
	6.4641	0.8663	4.4120	4.00	90.66
	8.4721	1.0895	5.2749	4.00	75.82
	10.4772	1.3122	6.4348	4.00	62.16
3.0	0.0000	0.0716	8.7020	9.00	103.42
	2.4142	0.3556	8.0367	9.00	111.98
	4.4494	0.4749	7.9896	9.00	112.64
	6.4641	0.6428	8.0966	9.00	111.15
	8.4721	0.8102	8.4278	9.00	106.78
	10.4772	0.9772	8.9824	9.00	100.19
5.0	0.0000	0.0444	24.6323	25.00	101.49
	2.4142	0.2350	23.4434	25.00	106.63
	4.4494	0.3120	23.1215	25.00	108.12
	6.4641	0.4239	22.7573	25.00	109.85
	8.4721	0.5355	22.5447	25.00	110.89
	10.4772	0.6468	22.4821	25.00	111.19
8.0	0.0000	0.0281	63.5956	64.00	100.63
	2.4142	0.1553	62.0497	64.00	103.14
	4.4494	0.2059	61.5383	64.00	104.00
	6.4641	0.2805	60.8529	64.00	105.17
	8.4721	0.3551	60.2712	64.00	106.18
	10.4772	0.4291	59.7918	64.00	107.03

Table A.2

$n_1 = n_2 = 2,\ n_3 = 4,\ \alpha_1 = \alpha_2 = \alpha_p = 0.50$					
θ_{31}	λ_2	BIAS (V)	MSE (V)	MSE (v_3)	e (V, V_3) %
1.0	0.0000	0.0710	0.4679	0.500	106.85
	2.4142	0.5708	0.8948	0.500	55.87
	6.4641	0.9544	2.1668	0.500	23.07
	8.4721	1.2054	3.2296	0.500	15.48
	10.4772	1.4560	4.5422	0.500	11.00
1.5	0.0000	0.0350	1.0776	1.125	104.39
	2.4142	0.4625	1.0600	1.125	106.12
	4.4494	0.5118	1.3959	1.125	80.58
	6.4641	0.7019	1.8866	1.125	59.62
	8.4721	0.8929	2.5632	1.125	43.89
	10.4772	1.0835	3.4298	1.125	32.80
2.0	0.0000	0.0197	1.9370	2.000	103.25
	2.4142	0.3531	1.6448	2.000	121.58
	4.4494	0.3864	1.9132	2.000	104.53
	6.4641	0.5371	2.1807	2.000	91.71
	8.4721	0.6873	2.5984	2.00	76.97
	10.4772	0.8372	3.1654	2.000	63.18
3.0	0.0000	0.0038	4.4417	4.500	101.31
	2.4142	0.2125	3.9449	4.500	114.06
	4.4494	0.2426	4.0358	4.500	111.49
	6.4641	0.3428	4.0480	4.500	111.16
	8.4721	0.4427	4.1606	4.500	108.15
	10.4772	0.5423	4.3721	4.500	102.92
5.0	0.0000	-0.0047	12.4861	12.500	100.11
	2.4142	0.1043	11.8373	12.500	105.59
	4.4494	0.1204	11.8267	12.500	105.69
	6.4641	0.1737	11.6419	12.500	107.37
	8.4721	0.2271	11.5115	12.500	108.58
	10.4772	0.2802	11.4319	12.500	109.34
8.0	0.0000	-0.0053	32.0324	32.000	99.89
	2.4142	0.0492	31.3977	32.000	101.91
	6.4641	0.0844	31.0991	32.000	102.89
	8.4721	0.1116	30.8812	32.000	103.62
	10.4772	0.1386	30.6840	32.000	104.28

Table A.3

$n_1 = 10, n_2 = n_3 = 2, \bar{\alpha}_1 = \alpha_2 = \alpha_p = 0.50$					
θ_{31}	λ_2	BIAS (V)	MSE (V)	MSE (V_3)	e(V, V_3) %
1.0	0.0000	0.2954	0.8427	1.00	118.66
	2.4142	0.5620	0.9393	1.00	106.46
	4.4494	0.6414	1.1156	1.00	89.63
	6.4641	0.7853	1.3810	1.00	72.41
	8.4721	0.9288	1.7271	1.00	57.90
	10.4772	1.0710	2.1506	1.00	46.50
1.5	0.0000	0.2196	1.9229	2.25	117.00
	2.4142	0.4201	1.8028	2.25	124.80
	4.4494	0.4795	1.8780	2.25	119.81
	6.4641	0.5880	1.9733	2.25	114.02
	8.4721	0.6962	2.1305	2.25	105.61
	10.4772	0.8034	2.3439	2.25	95.99
2.0	0.0000	0.1745	3.5697	4.00	112.05
	2.4142	0.3348	3.3167	4.00	120.60
	4.4494	0.3822	3.3306	4.00	120.10
	6.4641	0.4690	3.3218	4.00	120.41
	8.4721	0.5556	3.3635	4.00	118.92
	10.4772	0.6415	3.4468	4.00	116.05
3.0	0.0000	0.1235	8.4514	9.00	106.49
	2.4142	0.2378	8.0450	9.00	111.87
	4.4494	0.2714	7.9887	9.00	112.66
	6.4641	0.3333	7.8599	9.00	114.50
	8.4721	0.3953	7.7696	9.00	115.84
	10.4772	0.4568	7.6982	9.00	116.91
5.0	0.0000	0.0779	24.3440	25.00	102.69
	2.4142	0.1504	23.7974	25.00	105.05
	4.4494	0.1716	23.6776	25.00	105.58
	6.4641	0.2109	23.4393	25.00	106.66
	8.4721	0.2507	23.2319	25.00	107.61
	10.4772	0.2902	23.0011	25.00	108.69
8.0	0.0000	0.0501	63.2779	64.00	101.14
	2.4142	0.0969	62.6445	64.00	102.16
	4.4494	0.1106	62.4858	64.00	102.42
	6.4641	0.1360	62.1792	64.00	102.93
	8.4721	0.1623	61.9078	64.00	103.38
	10.4772	0.1886	61.5314	64.00	104.01

Table A.4

$n_1 = n_2 = 10,\ n_3 = 2,\ \alpha_1 = \alpha_2 = \alpha_p = 0.50$					
θ_{31}	λ_2	BIAS (V)	MSE (V)	MSE (V_3)	e (V, V_3) %
1.0	0.0000	0.3237	0.7970	1.00	125.47
	3.4494	0.4801	0.8725	1.00	114.61
	5.7416	0.5812	0.9903	1.00	100.97
1.5	0.0000	0.2379	1.8703	2.25	120.30
	3.4494	0.3558	1.8098	2.25	124.32
	5.7416	0.4303	1.8163	2.25	123.87
2.0	0.0000	0.1879	3.5162	4.00	113.76
	3.4494	0.2823	3.3713	4.00	118.65
	5.7416	0.3402	3.3059	4.00	120.99
3.0	0.0000	0.1322	8.4001	9.00	107.14
	3.4494	0.1998	8.1534	9.00	110.38
	5.7416	0.2377	7.9930	9.00	112.60
5.0	0.0000	0.0834	24.2997	25.0	102.88
	3.4494	0.1267	23.9421	25.0	104.42
	5.7416	0.1441	23.6474	25.0	105.72
8.0	0.0000	0.0542	63.2463	64.00	101.19
	3.4494	0.0829	62.7730	64.00	101.95
	5.7416	0.0838	62.2739	64.00	102.77

Table A.5

$n_1 = 16, n_2 = n_3 = 4, \alpha_1 = \alpha_2 = \alpha_p = 0.50$					
θ_{31}	λ_2	BIAS (V)	MSE (V)	MSE (V_3)	e (V, V_3) %
1.0	0.0000	0.2780	0.3792	0.50	131.82
	2.7320	0.3528	0.4316	0.50	115.83
	4.8284	0.4408	0.5016	0.50	99.67
	6.8730	0.5271	0.5973	0.50	83.71
	8.8990	0.6082	0.7064	0.50	70.78
	10.9161	0.6620	1.2213	0.50	40.94
1.5	0.0000	0.1693	0.8949	1.12	125.70
	2.7320	0.2180	0.8830	1.12	127.40
	4.8284	0.2747	0.8751	1.12	128.55
	6.8730	0.3307	0.8855	1.12	127.04
	8.8990	0.3819	0.9071	1.12	124.01
	10.9161	0.3985	1.2386	1.12	90.82
2.0	0.0000	0.1130	1.7383	2.00	115.05
	2.7320	0.1467	1.6975	2.00	117.82
	4.8284	0.1857	1.6545	2.00	120.88
	6.8730	0.2248	1.6259	2.00	123.00
	8.8990	0.2588	1.6076	2.00	124.41
	10.9161	1.2498	1.8713	2.00	106.87
3.0	0.0000	0.0601	4.2431	4.50	106.05
	2.7320	0.0788	4.1839	4.50	107.55
	4.8284	0.1003	4.1181	4.50	109.27
	6.8730	0.1230	4.0636	4.50	110.74
	8.8990	0.1394	4.0196	4.50	111.95
	10.9161	0.0942	4.3158	4.50	104.27
5.0	0.0000	0.0252	12.2926	12.50	101.69
	2.7320	0.0333	12.2350	12.50	102.17
	4.8284	0.0427	12.1690	12.50	102.72
	6.8730	0.0548	12.1161	12.50	103.17
	8.8994	0.0575	12.0796	12.50	103.48
	10.9161	-0.0406	12.7564	12.50	97.99
8.0	0.0000	0.0108	31.8488	32.00	100.47
	2.7320	0.0143	31.8028	32.00	100.62
	4.8284	0.0187	31.7471	32.00	100.80
	6.8730	0.0276	31.7162	32.00	100.89
	8.8990	0.0218	31.7198	32.00	100.88
	10.9161.	-0.1443	33.4538	32.00	95.65