MATHEMATICAL THEORY OF CONFOUNDING IN ASYMMETRICAL AND SYMMETRICAL FACTORIAL DESIGNS

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1. INTRODUCTION

The problem of confounding in the general symmetrical factorial design $s^n$, where $s$ is a prime positive integer or a power of prime and $m$ any positive integer, was solved by Bose and Kishen (1940) by representing each treatment combination by a finite point of the associated $m$-dimensional finite projective geometry $PG(m,s)$ constructed from the Galois field $GF(s)$ and using linear spaces or flats represented by linear equations in $m$ variables. This method is not applicable in the construction of confounded symmetrical factorial designs $s^n$, where $s$ is not a prime number or its power, nor in obtaining confounded designs in the general asymmetrical factorial experiment $s_1 \times s_2 \times \ldots \times s_m$, where $s_1, s_2, \ldots, s_m$ are not all equal. Special methods have, therefore, to be applied for the construction of such designs.
The problem of confounding in designs of the type $3^{n_1} \times 2^{n_2}$, where $n_1$, $n_2$ are any positive integers, and all cases reducible to it, has been completely solved by Yates (1937). Using methods similar to Yates's, Li (1944) has constructed confounded designs for the asymmetrical factorial experiments $4 \times 2^5$, $5 \times 2^5$, $4 \times 3 \times 2$, $4^2 \times 2$, $4 \times 3^2$, $4^2 \times 3$ and $4^2 \times 2$. Nair and Rao (1941, 1942) have developed a set of sufficient combinatorial conditions which lead to the construction of confounded designs of the general asymmetrical factorial experiment. Thompson and Dick (1951), starting from a basic $p \times q$ design in blocks of $q$ plots ($q < p$, $p$ being a prime number or a power of a prime), have obtained three-factor designs with the same block size, the number of levels being $p$, $q$ or factors of $q$. Kishen (1958) has given balanced designs of the type $q \times 2^2$ and $q \times p^2$.

The method of finite geometries has been recently extended by Kishen and Srivastava (1959) to the construction of balanced confounded asymmetrical factorial designs. This has been done by using curvilinear spaces or hypersurfaces and truncating the $EG(m, s)$ suitably. This method has been further developed in this paper and has been supplemented by more general methods using vectors in Galois fields. With the help of these methods, almost all confounded asymmetrical and symmetrical factorial designs having optimum properties have been constructed. The method of analysis of these designs has also been briefly discussed. The appropriateness of the large number of factorial designs that have now become available under experimental situations commonly encountered will be discussed in a separate communication.

2. HYPERSURFACES IN FINITE GEOMETRIES

2.1. Simple Hypersurfaces

A hypersurface in $EG(m, s)$ may be represented by the equation

$$
\phi(x_1, x_2, \ldots, x_m) = 0
$$

of which a particular case is given by the equation

$$
a_0 + a_1f_1(x_1) + a_2f_2(x_2) + \ldots + a_mf_m(x_m) = 0
$$

in which all the variables occur separately,

$$
a_0, a_1, a_2, \ldots, a_m \text{ being any elements of } GF(s)
$$

and

$$
f_i(x) = a_{i1}x + a_{i2}x^2 + \ldots + a_{is-1}x^{s-1}
$$
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where the \( a_i \)'s are also elements of \( EG(s) \).

When \( f_1(x) = x^k \), we get simple hypersurfaces of the type

\[
a_0 + a_1x_1^k + a_2x_2^k + \ldots + a_mx_m^k = 0
\]  

(4)

which we shall consider first.

The following theorem will be proved in this connection:

**Theorem 2.1.** If \( d \) is a divisor of \( s - 1 = p^n - 1 \), then \( x^d \) and \( x^{s-1-d} \) will give exactly \( (s-1)/d + 1 \) distinct values when \( x \) is varied from \( a_0 \) to \( a_{s-1} \) in \( GF(s) \).

Let \( d \) be a divisor of \( p^n - 1 \). It is known that the equation \( x^d = 1 \) will have exactly \( d \) roots. Let these roots be \( \mu_1, \mu_2, \ldots, \mu_d \). Since \( d < (s-1) \), there will be other elements not included in this set of \( \mu \)'s. Let \( a_t \) be such an element and let \( a_t \cdot d = \beta_1 \). Then the equation \( x^d = \beta_1 \) will be satisfied by \( \mu_1^t, \mu_2^t, \ldots, \mu_d^t \). Let \( a_s \) be another element not included in the two sets \( (\mu_i) \) and \( (\mu_1^t) \), and let \( \mu_s^d = \beta_2 \). Then the roots of \( x^d = \beta_2 \) will be \( \mu_s^{t_1}, \mu_s^{t_2}, \ldots, \mu_s^{t_d} \). If \( (s-1) = qa \), then obviously we will get the sets \( (\mu_i^t) \), \( (\mu_1^t) \), \( (\mu_s^{t_1}) \), and all these sets together will exhaust the \( (s-1) \) elements (excluding zero) of \( GF(s) \). The set \( (\mu_1^t, \mu_s^{t_1}) \) (where \( \beta_0 = 1 \)) will satisfy the equation \( x^d = \beta \), hence \( x^d \) will give \( q \) distinct values when \( x \) is varied from \( a_1 \) to \( a_{s-1} \), and these will be \( \beta_0, \beta_1, \ldots, \beta_{q-1} \). Including \( x = 0 \), we shall thus obtain \( (q+1) \) distinct values when \( x = a_0, a_1, \ldots, a_{s-1} \). Further, we know that all the elements of \( GF(s) \) satisfy the equation \( x^{s-1} = 1 \). Hence \( x^{s-1} = 0 \) when \( x = a_0 \) and equal to \( a_1 \) for all other values of \( x \). Since, for \( x = a_0, a_1, \ldots, a_{s-1} \), we get \( q \) distinct values for \( x^d \), we shall obtain \( q \) distinct values for \( 1/x^d \) and, consequently, also for \( x^{s-1-d} \). Hence the theorem.

### 2.2. Polynomials Yielding \( k \) Distinct Levels

The question now arises whether it is possible to get \( k \) distinct levels by taking instead of \( f_1(x) = x^k \) in equation (2), an appropriate polynomial in \( x \), say,

\[
y = f(x) = a_1x + a_2x^2 + \ldots + a_{s-1}x^{s-1}
\]  

(5)

where \( k \) is any number less than \( s \). This means that \( f(x) \) should be such that for \( x = a_0, a_1, \ldots, a_{s-1} \), \( f(x) \) provides only \( k \) distinct values, say, \( y_1, y_2, \ldots, y_k \). This result will be proved in the two theorems that follow.
Theorem 2.2.—The power-matrix

\[
S = \begin{bmatrix}
a_1 & a_1^2 & \ldots & a_1^t & \ldots & a_1^{s-1} \\
a_2 & a_2^2 & \ldots & a_2^t & \ldots & a_2^{s-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_r & a_r^2 & \ldots & a_r^t & \ldots & a_r^{s-1} \\
a_{s-1} & a_{s-1}^2 & \ldots & a_{s-1}^t & \ldots & a_{s-1}^{s-1}
\end{bmatrix} = [a_t^i] \quad (6)
\]

where \( a_t^i (r, t = 0, 1, \ldots, s - 1) \) are all non-zero elements of GF (s), is of rank \( s - 1 \) and its inverse is given by

\[
T = S^{-1} = a_{p-1}^{-1} \begin{bmatrix}
a_1^{s-2} & a_2^{s-2} & \ldots & a_r^{s-2} & \ldots & a_{s-1}^{s-2} \\
a_1^{s-3} & a_2^{s-3} & \ldots & a_r^{s-3} & \ldots & a_{s-1}^{s-3} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_1^{s-r-1} & a_2^{s-r-1} & \ldots & a_r^{s-r-1} & \ldots & a_{s-1}^{s-r-1} \\
a_1 & a_2 & \ldots & a_r & \ldots & a_{s-1}
\end{bmatrix} \quad (7)
\]

where \( s = p^n \ (n \geq 1) \).

Let us consider the product \( ST \). The element in the \( r \)-th row and \( t \)-th column of \( (ST) \), where \( r \neq t \), is given by the sum of products of elements in the \( r \)-th row of \( S \) and \( t \)-th column of \( T \), and equals

\[
a_{p-1}^{-1} [a_t a_t^{s-2} + a_t^2 a_t^{s-3} + \ldots + a_t^t a_t^{s-t-1} + \ldots + a_t^{s-1} a_t^0] = a_{p-1}^{-1} a_t^{s-1} \left[ \frac{a_t}{a_t} + \left( \frac{a_t}{a_t} \right)^2 + \ldots + \left( \frac{a_t}{a_t} \right)^t \right.
\]

\[
+ \ldots + \left( \frac{a_t}{a_t} \right)^{s-1} \right]
\]
Now, since \( a_r \) and \( a_t \) are both non-zero elements of \( GF(s) \), the quotient \( (a_r/a_t) \) exists. Let

\[
\frac{a_r}{a_t} = \omega \neq 1
\]

(8)
The above expression then reduces to

\[
a^{-1} \cdot \frac{a_r}{a_t} \cdot \frac{a_t}{a_t} \cdot [\omega + \omega^2 + \omega^3 + \ldots + \omega^{s-1}] = 0
\]
since

\[
\omega \neq a_1 \quad \text{and} \quad \omega^{s-1} = a_1 \quad \text{for all} \quad \omega \neq 0.
\]

Also, for \( r = t \) the product (8) becomes

\[
a^{-1} \cdot \frac{a_r}{a_t} \cdot \frac{a_t}{a_t} \cdot [a_1 + a_1 + \ldots + a_1 + \ldots (p - 1) \text{ times}] = a^{-1} \times a_1 \times a_{p-1} = a_1.
\]

Hence the product \( ST \) is a unit matrix. Obviously, the rank of both \( S \) and \( T \) is \((s - 1)\).

THEOREM 2.3.—

Let \( y \) and \( f(x) \) be defined as in (5), and let

\[
A = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{s-1}
\end{bmatrix}
\]

Here \( f(0) = 0 \).

Then there exist a set of matrices such that as \( x \) is varied from \( a_1 \) to \( a_{s-1} \), only \((k - 1)\) distinct values of \( y \) other than \( a_0 \) are obtained, so that including \( x = 0 \) we have \( k \) distinct levels. Further, there will be a subset of this set of matrices such that the \((k - 1)\) distinct values of \( y \) correspond to certain given values of \( x \), say \( x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}} \).

The proof is simple. Consider the product \( SA \) and let \( Y = SA \). Obviously, since \( S \) is of rank \((s - 1)\), \( Y \) exists and equals

\[
Y = \begin{bmatrix}
a_1 \times a_1 + a_2 \times a_1^2 + \ldots + a_{s-1} \times a_1^{s-1} \\
\cdots \\
a_1 \times a_r + a_2 \times a_r^2 + \ldots + a_{s-1} \times a_r^{s-1} \\
\cdots \\
a_1 \times a_{s-1} + a_2 \times a_{s-1}^2 + \ldots + a_{s-1} \times a_{s-1}^{s-1}
\end{bmatrix} = \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_{s-1}
\end{bmatrix}, \quad \text{say}
\]

(9)
where \( y_i = f(x_i) \). Now we want only \((k - 1)\) distinct elements in the last matrix in (9). This may be done in \( [(s-1)!/(k-1)! \times (s-k)!] \times k^{k-b} \) ways, corresponding to each of which there will exist a surface giving \( k \) distinct values. The polynomials yielding distinct fixed values \( y_{i_1}, y_{i_2}, \ldots, y_{i_{k-1}} \) against fixed levels of \( x = x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}} \) can be obtained in \((k)^{k-b}\) ways. The corresponding set of surfaces may be called the \( k \)th-level isomorphic set of surfaces, and the corresponding set of \( A \) matrices will be given by \( A = TY \), where the \((k - 1)\) elements in \( Y \) are fixed and the rest may vary from \( a_0 \) to \( a_{s-1} \).

3. ASYMMETRICAL CONFOUNDED DESIGNS

Let us now consider equation (4). If \( A_i \) is a factor which is included in equation (4) as \( x_i^{k_i} \), the contribution made by it in the equation can take only \( s_i = (s - 1)!/b_i + 1 \) distinct values, \( s_i \) being thus less than \( s \), assuming that \( b_i \) is a divisor of \((s - 1)\). Let the \( s_i \) distinct values of \( x_i^{k_i} \) correspond in order to the values of \( x_i \) equal to \( a_0, a_1, a_{i-1}, a_i, \ldots, a_{i_k} \). This means in effect that equation (4) will behave as if the \( j \)-th factor had only \( s_i \) distinct levels, namely,

\[
(0, 1, j_1, j_2, \ldots, j_{s_i-2})
\]

In the context of asymmetrical designs, this suggests that the levels of \( x_i \) other than those given by (10) be left out of consideration and the Euclidean Geometry \( EG(m, s) \) containing \( s^m \) points be so truncated that all the treatment combinations in which the above levels of \( A_i \) occur are cut out. Such a truncation may be done with respect to any number of factors, as required.

Consider now \( m \) factors \( A_1, A_2, \ldots, A_m \) at levels \( s_1, s_2, \ldots, s_m \) respectively, where \( s_i \) is a prime number and \( s_i < s_1 \) for all \( i > 1 \). As shown in Section (2.2), it is possible to have \( s_i \) equal to any number less than \( s_1 \) by taking a suitable polynomial of \( x \) in \( GF(s_1) \). Here for simplicity, we shall consider the case where the factors \( A_2, A_3, \ldots, A_m \) corresponds to \( x_2^{b_2}, x_3^{b_3}, \ldots, x_m^{b_m} \) respectively. Suppose now we desire to confound an \( m \)-factor interaction. We then take the pencil of hypersurfaces represented by

\[
x_1 + [a_{1r}x_2^{i_2} + a_{i_2}x_3^{i_3} + \ldots + a_{i_m}x_m^{i_m}] = \alpha_r,
\]

where \( r = 0, 1, \ldots, s_1 - 1 \), in the suitably truncated \( EG(m, s_1) \), it being presumed that \( x_1 \) varies from \( a_0 \) to \( a_{s-1} \) and \( x_i (i \neq 1) \) varies over \( a_0, a_1, a_2, a_3, a_4, \ldots, a_{s_i-2} \). We may now proceed to divide the \( s_1 \times s_2 \times \ldots \times s_m \) treatment combinations in \( s_1 \) blocks of \( s_2 \times s_3 \times \ldots \times s_m \) plots each with the help of the pencil (11). It can be shown that the pencil (11) will divide the treatment combinations symmetrically.
into \( s_1 \) sets. For, if \( x_{1t_1}^1, x_{2t_2}^1, \ldots, x_{jt_j}^1, \ldots, x_{mt_m}^1 \) is any combination of the levels of the factors \( A_1, A_2, \ldots, A_j, \ldots, A_m \), the expression within brackets on the L.H.S. of equation (11) will have a fixed value, say, \( A (t_1, \ldots, t_j, \ldots, t_m) \) in \( GF(s_1) \). If \( x_{1t_1}^1 = a_r \) is the value of \( x_1 \) such that \( x_{1t_1}^1 + A = a_t \) (\( t_1, \ldots, t_j, \ldots, t_m \) in \( GF(s_1) \)), then the treatment combination \( (x_{1t_1}^1, x_{2t_2}^1, \ldots, x_{mt_m}^1) \) will appear in the \( a \)-th block. Thus, all the combinations of the levels of \( x_2, x_3, \ldots, x_m \) will appear with different levels of \( x_1 \) in different blocks. Since \( x_1 \) can have \( s_1 \) values, we shall get \( s_1 \) blocks of equal size, each block containing all the \( s_2 \times s_3 \times \ldots \times s_m \) combinations of \( A_2, A_3, \ldots, A_m \).

It appears that in the replication provided by a pencil of hypersurfaces, the interactions confounded may belong to two types. The interaction corresponding to equation (11) which generates the replication is always partially confounded. This is, so to say, the deliberately confounded interaction. However, some of the interactions may get partially confounded automatically owing to the fact that the number of combinations of levels of factors to which they relate is not equal to, or a factor of, the block size. For example, in the \( 4 \times 2 \times 2 \) design in blocks of 4 plots, our pencil will partially confound the \( ABC \) interaction in a particular replication, and the \( AB \) and \( AC \) interactions will also be partially confounded since there are 8 combinations of levels of \( AB \) and \( AC \) and the block size is only 4. Thus, in the replication corresponding to equation (11), the main effect \( A \) and all the interactions in which it enters will be partially confounded if \( s_1 > s_i (i = 2, \ldots, m) \).

For obtaining a design balanced with respect to all main effects and interactions, we may have to take all the replications obtained by varying the \( a_{ij} (j = 2, 3, \ldots, m) \) over \( a_1, a_2, \ldots, a_{j-1} \). Varying only a particular \( a_{ij} (j \text{ fixed}) \) from \( a_1 \) to \( a_{j-1} \) will mean, in a sense, balance over a particular contrast of all factors other than \( A_j \).

When there are at least two factors at \( s_1 \) levels each, no main effect will be partially confounded. The interaction \( A_1A_2 \) will be partially confounded if only there is no third factor at \( s_1 \) levels, and so on. In the former case, varying \( l_2 \) in \( a_{ij} \) from 1 to \( (s_j - 1) \) we shall obtain the \( (s_1 - 1) \) replications required for balancing the \( A_1A_2 \) interaction with respect to the rest of the factors.

4. Illustrative Examples

4.1. The \( 3 \times 3 \times 2 \) Design in Blocks of 6 Plots

Let the three factors be \( A (0, 1, 2), B (0, 1, 2) \) and \( C (0, 1) \). In \( EG(3, 3) \), we truncate all the points with \( x_3 = 2 \), since \( x^2 = 0, 1, 1 \) for
\( x = 0, 1, 2 \) and \( x = 2 \) does not give a distinct value for \( x^2 \). The truncated geometry will have 18 points left corresponding to the 18 treatment combinations. The balanced design in two replications, confounding \( AB (J) \) and \( ABC (J) \), is generated by the pencils of hypersurfaces

\[
\begin{align*}
    x_1 + x_2 + x_3^2 &= 0, 1, 2 \\
    x_1 + x_2 + 2x_3^2 &= 0, 1, 2
\end{align*}
\]

(12)

For obtaining complete balance on \( AB \), we may take two more replications generated by the two pencils

\[
\begin{align*}
    x_1 + 2x_2 + x_3^2 &= 0, 1, 2 \\
    x_1 + 2x_2 + 2x_3^2 &= 0, 1, 2
\end{align*}
\]

(13)

which, by themselves, provide a balanced design partially confounding \( AB (I) \) and \( ABC (I) \).

The above can be easily generalized to obtain \( 3^{n-1}\times2 \) confounded designs in blocks of \( 3^{n-2}\times2 \) plots.

4.2. The \( s^xq \) Design in Blocks of \( sq \) Plots, Balanced in \( (s - 1) \) Replications; where \( s > q \)

Let the \( q \) levels be obtained by taking the polynomial \( f(x) \). Let the factors be \( A (0, 1, \ldots, s - 1) \), \( B (0, 1, \ldots, s - 1) \) and \( C (0, 1, 2, \ldots, q - 1) \). Let us consider the replication given by the pencil

\[
x_1 + a_i x_2 + a_2 f(x_3) = a_r (r = 0, 1, \ldots, s - 1; i_2, i_3 \text{ fixed})
\]

(14)

which will partially confound the \( AB \) and \( ABC \) interactions. The \( (s - 1) \) replications obtained by allowing \( i_3 \) to vary from 1 to \( (s - 1) \) will give a balanced set. For complete balancing with respect to \( AB \), \( i_3 \) also is to be varied from 1 to \( (s - 1) \), giving the full set of \( (s - 1)^2 \) replications.

Some of the more useful designs derived from this series are \( 3^2\times2 \), \( 4^2\times2 \), \( 5^2\times2 \), \( 7^2\times2 \), \( 8^2\times2 \), \( 4^3\times3 \) and \( 5^3\times3 \) in blocks of 6, 8, 10, 14, 16, 12 and 15 plots respectively. As an illustration, the design \( 5\times5\times3 \) in 15 plot blocks is given in Table I, where the \( c_i \)'s denote the level of the factor \( C \), and \( X_i \)'s denote the sets of \( AB \):

\[
X_0: a_0b_0, a_1b_0, a_2b_0, a_3b_0, a_4b_0
\]
\[
X_1: a_0b_1, a_1b_0, a_2b_1, a_3b_1, a_4b_1
\]

\[
X_2: a_0b_2, a_1b_1, a_2b_2, a_3b_2, a_4b_2
\]
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\[\begin{align*}
X_2: & \quad a_0b_2, a_1b_2, a_2b_0, a_3b_4, a_4b_3 \\
X_3: & \quad a_0b_3, a_1b_2, a_2b_1, a_3b_0, a_4b_1 \\
X_4: & \quad a_0b_4, a_1b_2, a_2b_3, a_3b_1, a_4b_0
\end{align*}\]

The relative loss of information in an \(s^2 \times q\) design, on each of the \((s - 1)\) confounded d.f. of \(AB\) is \((s - q)/q (s - 1)\) and on each of the \((s - 1)(q - 1)\) confounded d.f. of \(ABC\), it is \(s/(q (s - 1))\).

**Table I**

5x5x3 balanced design in 15 plot blocks involving four replications

<table>
<thead>
<tr>
<th>Replication I</th>
<th>Replication II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block No.</td>
<td>1 2 3 4 5 6 7 8 9 10</td>
</tr>
<tr>
<td>(X_0c_0)</td>
<td>(X_1c_0)</td>
</tr>
<tr>
<td>(X_4c_1)</td>
<td>(X_0c_1)</td>
</tr>
<tr>
<td>(X_3c_2)</td>
<td>(X_4c_2)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Replication III</th>
<th>Replication IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block No.</td>
<td>11 12 13 14 15 16 17 18 19 20</td>
</tr>
<tr>
<td>(X_0c_0)</td>
<td>(X_1c_0)</td>
</tr>
<tr>
<td>(X_2c_1)</td>
<td>(X_4c_1)</td>
</tr>
<tr>
<td>(X_4c_2)</td>
<td>(X_0c_2)</td>
</tr>
</tbody>
</table>

The above can be easily generalized to the corresponding \(s^{m-1} \times q^t\) designs in blocks of \(s^{m-1} \times q^t\) plots, e.g., \(3 \times 3 \times 2 \times 2\) design in 12 plot blocks and \(4 \times 4 \times 2 \times 2\) in 16 plot blocks.

4.3. \(s \times q_1 \times q_2\) Design in Blocks of \(q_1q_2\) Plots, \((s \geq q_1, q_2)\), Balanced in \((s - 1)^2\) Replications

In this design, the main effect \(A\) is confounded partially.

Many useful designs, e.g., \(s \times 3 \times 2\), \(s \times 4 \times 2\), \(s \times 2 \times 2\), \(s \times 4 \times 3\), \(s \times 5 \times 2\), etc. in blocks of plots 6, 8, 4, 12 and 10 may be derived from this general design. If \(s > q_1q_2\), balancing may be achieved in \((s - 1)\) replications only.
4.4. $4 \times 3 \times 2 \times 2$ Design in Blocks of 12 Plots

Consider $GF(2^4)$ with minimum function $x^2 = x + 1$ and elements $a_0, a_1 = 1, a_2 = x, a_3 = x^2 = x + 1$. Denote the factors by $A (0, 1, 2, 3), B (0, 1, 2), C (0, 1), D (0, 1)$. The functions $f(x)$ corresponding to $C$ and $D$ may be taken as $x_3^3$ and $x_4^3$ respectively. For $B$, let us choose $f(x) = a_0, a_1, a_2$ and $a_3$ respectively corresponding to $x = a_0, a_1, a_2$ and $a_3$. Then, from (9), we have

$$A = a_1^{-1} \begin{bmatrix} a_1 & a_2 & a_3^2 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \\ a_3 + a_2 a_3 \\ a_3 + a_2 a_3 \\ a_3 + a_2 a_3 \end{bmatrix} = \begin{bmatrix} a_3 \\ a_0 \\ a_1 + a_3 \\ a_1 + a_3 \\ a_1 + a_3 \end{bmatrix}$$

(15)

Hence $f(x_2) = a_3 x_2 + a_2 x_2^3$.

The pencil which confounds $ABCD$ may be represented by the equation

$$x_1 + (a_2 x_2 + a_2 x_2^3) + [x_3^3 + a_2 x_4^3] = a_r \quad (r = 0, 1, 2)$$

(16)

in the truncated geometry $EG (4, 4)$. The replication generated also partially confounds the interaction $AC, AD, ACD, ABC$ and $ABD$. A design in 3 replications providing balance over $AC$ and $AD$ is given in Table II.

**Table II**

$4 \times 3 \times 2 \times 2$ Balanced design in 12 plot blocks, involving three replications

<table>
<thead>
<tr>
<th>Replication</th>
<th>I</th>
<th>II</th>
<th>III</th>
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</thead>
<tbody>
<tr>
<td>Block No.</td>
<td></td>
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<tr>
<td>Level of $A, B$</td>
<td>00</td>
<td>01</td>
<td>02</td>
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<td></td>
<td>32</td>
<td>10</td>
<td>11</td>
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Level of $C, D$

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<th>Replication</th>
<th>I</th>
<th>II</th>
<th>III</th>
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<tbody>
<tr>
<td>Block No.</td>
<td>00</td>
<td>01</td>
<td>02</td>
</tr>
<tr>
<td>Level of $A, B$</td>
<td>10</td>
<td>00</td>
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<th>II</th>
<th>III</th>
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<tbody>
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<td>02</td>
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<tr>
<td>Level of $A, B$</td>
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It has to be noted that the coefficients of $x_3^8$ and $x_4^3$ have been kept different in (16). This has to be done as otherwise one degree of freedom belonging to the interaction $AB$ will also be totally confounded. The reason is that if their coefficients are not distinct, $x_3^8$ and $x_4^3$, when combined together, will not generate all the elements of $GF(2^3)$.

4.5. $s_1 \times s_2 \times s_3 \times \ldots \times s_m$ Design where $s_i$ are equal to, or are Powers of, a Prime Number $p$

As a special case of this design, let us consider the $4 \times 2 \times 2$ design in blocks of 4 plots. Denote the factors by $A(0, 1, 2, 3)$, $B(0, 1)$ and $C(0, 1)$. A suitable pencil confounding the $ABC$ interaction is represented by

$$x_1 + a_{i_k}(x_2^3 + a_2x_3^2) = a_r \quad (r = 0, 1, 2, 3; \ i_k \text{ fixed}) \quad (17)$$

This also partially confounds the $A'B$ and $AC$ interactions. A design in 3 replications is obtained by taking $i_k = 1, 2$ and 3 and is shown in Table III below, in which the confounded interactions are also given.

<table>
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<tr>
<th>Replication</th>
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<th>III</th>
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<tbody>
<tr>
<td>Block No.</td>
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<td>2</td>
<td>3</td>
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<td>01</td>
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<td>2</td>
<td>0</td>
<td>11</td>
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<td>3</td>
<td>0</td>
<td>20</td>
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<tr>
<td></td>
<td></td>
<td>4</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>Confounded</td>
<td></td>
<td>$A'C$, $A''B$</td>
<td>$A''C$, $A''B$</td>
<td>$A'B$, $A''C$</td>
</tr>
<tr>
<td>Interactions</td>
<td></td>
<td>$A''BC$</td>
<td>$A'BC$</td>
<td>$A''BC$</td>
</tr>
</tbody>
</table>
|             |  | Here $A' = (a_3 + a_2 - a_1 - a_0)$, $A'' = (a_3 - a_2 - a_1 + a_0)$ and $A''' = (a_3 - a_2 + a_1 - a_0)$. The loss of information on each of the $AB$, $AC$ and $ABC$ interactions is $1/3$. The above design can be immediately extended to $4 \times 2^n$ designs in blocks of $2^n$ plots, balanced in 3 replications, as above.
Designs of the type $9 \times 3 \times 3$ in blocks of 9 plots, $16 \times 4 \times 4$ in blocks of 16 plots, $8 \times 2 \times 2 \times 2$ in blocks of 8 plots, $8 \times 4 \times 2$ in blocks of 8 plots, etc., can also be constructed by similar methods. These are balanced in 8, 15, 7 and 7 replications respectively. These confounded designs are amenable to arrangement in quasi-Latin squares.

4.6. $s_1 \times s_2 \times s_3 \times \ldots \times s_m$ Design in Blocks of $s_1 \times s_2 \times s_3 \times \ldots \times s_m$ plots where $s_2$ is a factor of $s_1 \times s_3 \times s_4 \times \ldots \times s_m$ and is a Prime Number or a Power of a Prime, and $s_4^2 = s_i (i \neq 2)$

Consider $GF(s_2^2)$. By Theorem 2.2, we can obtain $s_4$ distinct levels from a suitable polynomial in $GF(s_4^2)$. Then a pencil of hypersurfaces will divide the total number of treatment combinations into $s_2^2$ blocks, each block containing $(1/s_2) (s_1 \times s_2 \times s_3 \times \ldots \times s_m)$ plots. We may then suitably combine sets of $s_4$ blocks out of these $s_2^2$ blocks to get $s_3$ new blocks, each containing $s_1 \times s_3 \times \ldots \times s_m$ plots.

4.7. $s^3 \times s_4 \times s_5 \ldots \times s_m$ Design in $s^2$ Blocks of $s \times s_4 \times s_5 \times \ldots \times s_m$ Plots each

Here 3 factors have been taken at $s$ levels so that the number of plots in each of the $s^2$ blocks may still remain a multiple of $s$ so as to keep all the main effects unconfounded. As in the symmetrical case of $(s^n, s^2)$, we have here to confound two pencils simultaneously.

As an illustration, let us take the $5^2 \times 2$ design in blocks of 10 plots each.

Let us take the pencils

\[
\begin{align*}
\sum x + 2x_2 + 2x_4 &= 0, 1, 2, 3, 4 \\
\text{and} \\
\sum x + x_3 + x_4 &= 0, 1, 2, 3, 4
\end{align*}
\]

in the truncated $EG (4, 5)$. Balance on any particular contrast belonging to the first three factors $A$, $B$, and $C$ can be achieved in 4 replications.

A generalization of the above procedure leads to the construction of balanced confounded designs of the type $s^{m-k} \times s_1 \times s_2 \times \ldots \times s_p$ in $s^2$ blocks of $s^{m-k} \times s_1 \times s_2 \times \ldots \times s_p$ plots each where $k < m_1$; balancing being achieved in $(s-1)$ replications only, if $k < (s-1)$.

4.8. Method of Cutting out from an $s^n$ Design

Suppose we have got an $s^n$ design in $s^2$ blocks of $s^{n-k}$ plots each, where $s$ is a prime power. Then it can be easily seen that we can
derive a design of the type $s^k \times s_{k+1} \times s_{k+2} \times \ldots \times s_m$ where $s_i \leq s$, from it by cutting out in all blocks all the treatment combinations which contain any out of the last $(s - s_i)$ levels of the factor $A_i$, $i$ varying from $k + 1$ to $m$. This method is essentially equivalent to cutting out points lying on the $(m - 1)$-flats $x_i = a_r$ [$r$ varying from $s_k$ to $(s - 1)$; $i = k + 1, \ldots, m$] from a set of pencils of linear $(m - 1)$-flats giving the confounded symmetrical design. The designs obtained by this method will, however, all correspond to simple confounding to be described in the next section, and are obviously a particular case of the designs obtainable from hypersurfaces. However, as would appear from the foregoing sections, the hypersurfaces provide a natural representation of all asymmetrical designs which are derivable by the above method of cutting.

5. Use of Galois Fields in Confounding in Factorial Designs

Let us now examine the role played by Galois fields and finite geometries in the construction of confounded factorial designs. In the case of confounding in symmetrical designs with $s^m$ treatment combinations, the number $s$ enters both as the level of each of the $m$ factors and is also used in the pencils in finite geometries in splitting up the $s^m$ treatment combinations symmetrically into $s$ parts. We have seen, however, that with truncated geometries, the levels of each factor may not be the same and still the use of $GF(s)$ leads us to $s$ symmetrical partitions. If the total number of treatment combinations is $v$ and we want $s$ blocks in which the treatments occur symmetrically, evidently $s$ should firstly be a factor of $v$, which means that at least one factor is to be at $s$ levels. As we have shown in Section 3, we can, in that case, put all the treatments $v$ into a sort of correspondence with the $s$ elements of $GF(s)$. The construction of a confounded factorial design necessarily involves the partitioning of treatments into $s$ parts, i.e., putting the $v$ objects into correspondence with the $s$ blocks. The $GF(s)$ is thus simply a mathematical device for effecting such a correspondence.

The above suggests that we may construct the $(s^m, s^k)$ design directly from $GF(s^k)$. This procedure should appear to be more natural than the ordinary one inasmuch as the blocks required correspond one-to-one to the elements

$$a_0, a_1, a_2, a_3, \ldots, a_{s^k-2}, a_{s^k-1}$$

of $GF(s^k)$.

It also appears that the use of $GF(s)$ to group the factors even when all of them are not at $s$ levels may also be possible and may lead us to
interesting designs since the groupings made by \( GF(s) \) are in a sense symmetrical.

5.1. General Theory for Symmetrical Case

For developing the above approach, the use of vectors in Galois fields will be made. We first define the basic terminology.

(a) Any set of \( n \) elements in \( GF(s) \) will be called an \( n \)-vector in \( GF(s) \).

(b) Corresponding to a factor \( A \) at \( k \) levels, the vector \((0, 1, 2, \ldots, k - 1)\) in the real field will be called the Level Vector of \( A \).

(c) Corresponding to the level vector of \( A \), there is an Associated Vector \((\beta_0, \beta_1, \beta_2, \ldots, \beta_{k-1})\) of \( A \), where the \( \beta \)'s are all elements of \( GF(s) \), not necessarily distinct.

(d) Any vector in \( GF(s) \) used to generate the required design will be called a Generator. With \( m \) factors, the generator will be an \( m \)-vector in \( GF(s) \).

(e) The sum \( S \) and product \( P \) of two vectors \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_m)\) in \( GF(s) \) will be \( S = (a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) \) and \( P = (a_1b_1, a_2b_2, \ldots, a_mb_m) \) while their product sum will be \( Q = a_1b_1 + a_2b_2 + \ldots + a_mb_m \).

(f) If the elements of the Associated Vector of a factor correspond one-to-one to the elements of the Level Vector, all the confounded interactions in which the factor enters may be said to be simply confounded where this is not the case the confounding is said to be non-simple.

With respect to a particular generator, the set of all treatment vectors, which we may call treatment space, may be divided into \( s \) parts, the \( j \)th part containing those treatment vectors the associated vectors corresponding to which give a product sum \( a_j \) when multiplied by the generator. Here \( a_j \) is the \((j + 1)\)-th element of \( GF(s) \). Since the usual arrangement of the \( s \) elements of \( GF(s) \) in the order \( a_0 = 0, a_1 = 1, a_2 = \theta, \ldots, a_{i-1} = \theta^{-1}, \ldots, a_{s-1} = \theta^{-s-1} \) presents difficulties in the addition of elements when \( n \neq 1 (s = p^n) \), it will be convenient to have a rearranged form for the elements of \( GF(s) \). In \( GF(s = p^n) \), where \( p \) is a prime number, the minimum function is of order \( n \) and of the form

\[ \theta^n = \mu_0 + \mu_1\theta + \mu_2\theta^2 + \ldots + \mu_{n-1}\theta^{n-1} \] (24)

where \( \mu_i \) are elements of \( GF(p) \). Hence any element of \( GF(p^n) \) may be represented in the form

\[ g_0 + g_2\theta + g_4\theta^2 + \ldots + g_{n-1}\theta^{n-1} \] (25)
where the \( g_i \) are elements of \( GF(p^r) \). Further, the elements of \( GF(p^r) \) will be so arranged that (25) is the \((1 + g_0 + g_1p + g_2p^2 + \ldots + g_{n-1}p^{n-1})\)-th element, where the \( g_i \)'s and \( p \) will be taken as belonging to the real field. Thus, in this rearranged form for the elements of \( GF(3^2) \), \( \theta + 2 \) will be the \((1 + 2 + 1 \times 3)\)-th, or the 6th element.

For simplicity, let us consider \((3^3, 3)\) design with blocks of 9 plots. Here we require 3 partitions. Hence we use \( GF(3) \). The level vectors are given by \( A (0, 1, 2) \), \( B (0, 1, 2) \) and \( C (0, 1, 2) \). Let us have simple confounding so that the effect vectors are \((0, 1, 2)\) or \((0, 2, 1)\). Now consider the different forms of generators. A generator like \((1, 0, 0)\) will divide the treatment space into 3 parts, the \( j \)-th part containing all the treatment vectors containing the level \( O_j \) of \( A \), which implies that the main effect is confounded. Considering the generator \((1, 1, 0)\), we find that the \( y \)-th part contains each level of \( C \) three times with each level of \( A \) or \( B \), which implies that only the factor \( C \) does not enter the confounded interactions. Similarly, it will be found that the generator \((1, 1, 1)\) corresponds to the \( ABC \) interaction and corresponds to the pencil \( x_1 + x_2 + x_3 = 0, 1, 2 \) in \( EG(3, 3) \) with the usual approach. Consider now \((3^3, 3^3)\) design, in which case we use \( GF(3^3) \) to get 9 blocks. A general element of \( GF(3^3) \) is \((r\theta + s)\), where \( r, s = 0, 1, 2 \). Let the effect vector be \((0, 1, 2)\) or \((0, 2, 1)\), as above. It will be found that factor or factors which correspond to a zero element in the generator are not confounded. Also, the generator should contain at least one element involving \( \theta \) and one element out of 0, 1 or 2; otherwise, since we are working with \( GF(3^4) \) but with factors at only 3 levels, the generator will not divide the treatment space into 9 equal parts. Now consider a generator of the type

\[
(z\theta + \lambda, \nu\theta, \mu)
\]  

(26)

This vector can be written as

\[
\theta (z, \nu, 0) + (\lambda, 0, \mu)
\]  

(27)

Suppose that the treatment vectors in a particular block have \((r\theta + s)\) as their product sum with (26). Then it is clear that they would give \( r \) and \( s \) respectively as products with the two component vectors of (27). If \( l_1, l_2, m_1, m_2 \) are any elements of \( GF(3^4) \), the same block would give \((l_1\theta + m_1s) \theta + (l_2\theta + m_2s)\) as product with the generator

\[
(l_1\theta + m_1\lambda)\theta + l_2\mu + m_2\lambda, l_1\nu\theta + l_2\nu, l_1\mu\theta + l_2\mu\lambda
\]  

(28)

and will be the block No. \([3 (l_1\theta + m_1s) + (l_2\theta + m_2s) + 1] \) of the same replicate, if the generator (28) is used. The two generators (26) and
(28) are equivalent. The close connection with the usual theory is evident, the two component vectors in (27) being the two confounded pencils represented by

\[ \xi x_1 + \nu x_2 = 0, 1, 2 \]
\[ \lambda x_1 + \mu x_3 = 0, 1, 2. \]

It is evident that in order that no main effect is confounded, all the elements in the generator (26) should be distinct.

Consider now a \( p^m \) design in \( p^k \) blocks of \( p^{m-k} \) plots each, where \( p \) is a prime number. Here we require \( p^k \) partitions and, therefore, use \( GF(p^k) \). The level vector corresponding to each factor is \((0, 1, 2, \ldots, p - 1)\). We have \((p - 1)\) distinct associated vectors for simple confounding. Let us use each one for one replication, getting \((p - 1)\) replications in all. Now suppose we want to confound the \( k \) independent interactions represented by the \( k \) equations

\[
\begin{align*}
&\alpha_{11}x_1 + \alpha_{21}x_2 + \ldots + \alpha_{m_1}x_{m_1} = a_{r_1} \\
&\alpha_{12}x_1 + \alpha_{22}x_2 + \ldots + \alpha_{m_2}x_{m_2} = a_{r_2} \\
&\hspace{1cm} \vdots \\
&\alpha_{1k}x_1 + \alpha_{2k}x_2 + \ldots + \alpha_{mk}x_{m_k} = a_{r_k} \\
&\hspace{1cm} \vdots \\
&\alpha_{1p}x_1 + \alpha_{2p}x_2 + \ldots + \alpha_{mp}x_{mp} = a_{r_{k+1}} \\
&\hspace{1cm} \vdots \\
&\alpha_{1p^k}x_1 + \alpha_{2p^k}x_2 + \ldots + \alpha_{mp^k}x_{mp^k} = a_{r_{2p^k}}
\end{align*}
\]

(29)

To the above corresponds the generator

\[
\begin{align*}
&\alpha_{11} + \alpha_{12} \theta + \alpha_{13} \theta^2 + \ldots + \alpha_{1k} \theta^{k-1}, \\
&\alpha_{21} + \alpha_{22} \theta + \alpha_{23} \theta^2 + \ldots + \alpha_{2k} \theta^{k-1}, \\
&\hspace{1cm} \vdots \\
&\alpha_{m_1} + \alpha_{m_2} \theta + \alpha_{m_3} \theta^2 + \ldots + \alpha_{mk} \theta^{k-1}.
\end{align*}
\]

or

\[
\left( \sum_{i=1}^{k} \alpha_{1i} \theta^{i-1}, \sum_{j=1}^{k} \alpha_{2j} \theta^{j-1}, \ldots, \sum_{j=1}^{k} \alpha_{mi} \theta^{j-1} \right)
\]

(30)

Then the generator

\[
\left( \sum_{j=0}^{k-1} \sum_{i=1}^{k} \gamma_{ij} a_{1j} \theta^{i}, \sum_{j=0}^{k-1} \sum_{i=1}^{k} \gamma_{ij} a_{2j} \theta^{i}, \ldots, \sum_{j=0}^{k-1} \sum_{i=1}^{k} \gamma_{ij} a_{mi} \theta^{i} \right)
\]

(31)
CONFOUNDING IN ASYMMETRICAL & SYMMETRICAL FACTORIAL DESIGNS

where $\gamma_{ij}'s$ are any elements of $GF(p)$, gives the same replication (for all $j$) as (31) under the condition that the $k$ vectors represented by

$$
\left( \sum_{i=1}^{k} \gamma_{ij}a_{1i}, \sum_{i=1}^{k} \gamma_{ij}a_{2i}, \ldots, \sum_{i=1}^{k} \gamma_{ij}a_{mi} \right),$$

$j$ varying from 0 to $(k - 1)$, are independent. This means that the vector space of rank $k$, with the basis given by $(a_{1j}, a_{2j}, \ldots, a_{mj})$, $j$ varying from 1 to $k$, is confounded, which corresponds to the principle of generalised interaction enunciated in this general case by Bose and Kishen.

It may be said that in a sense the generator (30) integrates the interactions confounded by the bundle of $k$ pencils corresponding to (29) just as, for example, the moment generating function of a distribution integrates its moments.

In the case of an $s^m$ design in $s^k$ blocks, we can proceed in the same manner as above, remembering that the $a_{ij}'s$ are now elements of $GF(s)$ and not of $GF(p)$.

5.2. Kishen’s Series of $q \times 2^s$ and $q \times p^s$ Designs, $q$ being any Integer and $p$ an Odd Prime Power

The two series of designs given by Kishen (1958) are typical examples of non-simple confounding defined earlier. In the $q \times 2^s$ series, the 3 factors are $A(0, 1, 2, \ldots, q - 1)$, $B(0, 1)$ and $C(0, 1)$. Since we want a design in $2q$ plot blocks, we use $GF(2)$. If $(1, 1, 1)$ is taken as the generator, $B(0, 1)$ and $C(0, 1)$ the associated vectors for $B$ and $C$, it can be easily seen that the $q$ replications required are obtained by taking the $q$ associated vectors for $A$ represented by the $q$ unit $q$-vectors in $GF(2)$, namely,

$$(1, 0, 0, \ldots, 0); \quad (0, 1, 0, \ldots, 0); \quad (0, 0, 1, 0, \ldots, 0); \quad \ldots$$

$$\ldots; \quad (0, 0, 0, \ldots, 0, 1)$$

For the $q \times p^s$ design, we use $GF(p)$, and the associated vectors of $B$ and $C$ are respectively

$$B(0, 1, 2, \ldots, p - 1) \quad \text{and} \quad C(0, 1, 2, \ldots, p - 1).$$

The set of associated vectors corresponding to $A$ are the $q$-vectors in $GF(p)$ represented by

$$(a_1, a_0, a_0, \ldots, a_0); \quad (a_0, a_1, a_0, \ldots, a_0); \quad \ldots; \quad (a_0, a_0, \ldots, a_0, a_2)$$

$$(a_0, a_0, a_0, \ldots, a_0); \quad (a_0, a_2, a_0, \ldots, a_0); \quad \ldots; \quad (a_0, a_0, \ldots, a_0, a_2)$$

$$(a_0, a_0, a_0, \ldots, a_0); \quad (a_0, a_{(p-1)/2}, a_0, \ldots, a_0); \quad \ldots; \quad (a_0, a_0, \ldots, a_0, a_{(p-1)/2}).$$

$$(a_0, a_0, a_0, \ldots, a_0); \quad (a_0, a_0, a_0, \ldots, a_0); \quad \ldots; \quad (a_0, a_0, \ldots, a_0, a_{(p-1)/2}).$$
It can be seen that if the vectors in (33) and each of the \((p - 1)/2\) sets of vectors in (34) are arranged in the form of a square, as under,

\[
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix},
\]

the element 1 of GF(2) falls once in each row and once in each column. Comparing it with the latin square

\[
\begin{bmatrix}
A_1 & A_2 & A_3 & \ldots & A_q \\
A_q & A_1 & A_2 & \ldots & A_{q-1} \\
A_{q-1} & A_q & A_1 & \ldots & A_{q-2} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
A_2 & A_3 & A_4 & \ldots & A_1
\end{bmatrix}
\]

we find that here the letter \(A\) has been replaced by 1 and the others by 0.

This makes possible a generalization of this approach for construction of \(q \times 2^a\) and \(q \times p^a\) designs with less loss of information on BC. For example, if \(A_1, A_2, \ldots, A_i\) in the second square are replaced by 1 and \(A_{i+1}, A_{i+2}, \ldots, A_q\) by 0, and the \(q\) rows of the resulting square are taken as associated vectors of \(A\), we shall get a \(q \times 2^a\) design in \(2q\) plot blocks, the loss of information on BC being \((q - 2i)^2/q^2\). A similar approach with the \(q \times p^a\) series can be made and can be utilized to construct the \(5 \times 3 \times 3\) design in 15 plot blocks.

5.3. Certain Factorial Designs Using b.i.b. Property

Let \(s\) be a prime power and \(a\) any integer. Let there be two factors \(A\) and \(B\), each at \(a \times s\) levels. We can divide the total number of \(a^2s^2\) treatment combinations into \(a \times s\) blocks of \(a \times s\) plots each in such a way that no main effect is confounded. Suppose these \(a \times s\) blocks confounding \((as - 1)\) d.f. belonging to interaction \(AB\) are \(X_1, X_2, \ldots, X_{as}\). Now, suppose, a balanced incomplete block design (b.i.b.d.) exists with \(v = as\), and block size \(k < v\). Then immediately we get a confounded factorial design \(as \times as\) in blocks of size \(kas\).
CONFOUNDING IN ASYMMETRICAL & SYMMETRICAL FACTORIAL DESIGNS

partially confounding \((as - 1) d.f.\) of interaction \(AB\). Preferably \(k\) should be small. In particular, we can always have \(k = 2\), the b.i.b.d. in this case being \(v = as, b = as (as - 1)/2, k = 2, r = (as - 1), \lambda = 1\), giving \(as \times as\) design in 2 as plot blocks.

Alternatively, the b.i.b.d., \(v = as, b = s(as - 1), k = a, r = (as - 1), \lambda = a - 1\), can also be considered if one exists.

When \(a = 2\) and \(s = 3\), we get the b.i.b.d. \(v = 6, b = 15, k = 2, r = 5\) and \(\lambda = 1\), which gives the \(6 \times 6\) design in 12 plot blocks shown below.

**Table IV**

*6 \times 6 Balanced design in 12-plot blocks*

<table>
<thead>
<tr>
<th>Replication</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block No.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
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<td></td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(X_1)</th>
<th>(X_3)</th>
<th>(X_5)</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(X_4)</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(X_3)</th>
<th>(X_1)</th>
<th>(X_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X_2)</td>
<td>(X_4)</td>
<td>(X_6)</td>
<td>(X_3)</td>
<td>(X_6)</td>
<td></td>
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<tr>
<td>(X_1)</td>
<td>(X_2)</td>
<td>(X_3)</td>
<td>(X_5)</td>
<td>(X_4)</td>
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</table>

Here the treatment combinations included in the set \(X_i\) are those which satisfy the equation

\[x_1 + x_2 = i \mod 6\]  

\((i = 0, 1, 2, 3, 4, 5)\).

It will be seen that this method gives designs in which the total loss of information due to confounding is less than in the designs obtained by using the \(a \times s\) sets \(X_1, X_2, \ldots, X_s\) as blocks.

The method is particularly useful in symmetrical or asymmetrical designs in which the number of levels of each factor is not large, and the block size can be increased without appreciable increase in error. For example, we can construct a \(5 \times 5\) factorial design in 10 plot blocks in 4 replications by taking a b.i.b.d. with \(v = 5, b = 10, k = 2, r = 4\) and \(\lambda = 1\). Similarly, we can construct a \(7 \times 4\) factorial design in 12 plot blocks in 3 replications by considering the b.i.b.d., \(v = 7, b = 7, k = 3, r = 3\) and \(\lambda = 1\), on the 7 sets obtained by using \(GF(7)\) along with the associated vectors \(A(0, 1, 2, \ldots, 6), B(0, 1, 2, 3)\) and the generator say \((p, q)\) where \(p, q\) are non-zero elements. It is noticeable that by this procedure the number of replications required for balancing is
only 3 as against 6 replications in the design derivable directly from the Galois field with block of 4 plots.

5.4. \((p_1, p_2)^m\) Designs in Blocks of \(p_1^{m-t_1} p_2^{m-t_2}\) Plots, \(p_1, p_2\) being any Prime Powers and \(r_1, r_2 \leq m\).

The procedure is simple. First, we form \(p_1^{r_1}\) blocks by considering a particular generator the elements of which belong to \(GF(p_1^{r_1})\). The associated vector for each factor may be such that it is divisible into \(p_2\) sets of elements, each set containing \(p_1\) distinct elements belonging to \(GF(p_1)\). At the next stage, we similarly consider \(GF(p_2^{r_2})\) for further dividing each block, for each of which we take the same generator. The associated vector for each factor in this case consists of \(p_2\) distinct elements belonging to \(GF(p_2)\), one element corresponding to all elements in one of the sets out of the \(p_2\) sets defined above for the earlier associated vector. The procedure will be illustrated by deriving a \(6 \times 6\) design in 6 plot blocks.

The associated vectors for \(A\) and \(B\) at the first stage may be taken as \((0, 1, 2, 0, 1, 2)\), and the generator as \((1, 1)\). We use \(GF(3)\) since \(6 = 3 \times 2\). At the second stage, we use \(GF(2)\) with the generator \((1, 1)\) and associated vector for both \(A\) and \(B\) as \((0, 0, 1, 1, 1, 1)\). This gives a set of 6 blocks for the first replication. To this we may add another replication obtained by taking \((1, 2)\) as the generator at the first stage. The two replications together provide a balanced design, which is given in Table V.

**Table V**

6×6 Design in 6-plot blocks

<table>
<thead>
<tr>
<th>Replication</th>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks</td>
<td>(X_0) (X_1) (X_2) (X_3) (X_4) (X_5)</td>
<td>(Y_0) (Y_1) (Y_2) (Y_3) (Y_4) (Y_5)</td>
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<td>51</td>
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</table>
In this design the single degree of freedom for $AB$ corresponding to the contrast $(a_6 + a_4 + a_3 - a_1 - a_0)(b_5 + b_4 + b_3 - b_2 - b_1 - b_0)$ is totally confounded in both the replications. Further, 8 more degrees of freedom belonging to $AB$ are partially confounded, on which the loss of information is $\frac{1}{2}$. The total loss of information is $1 + 8 \times \frac{1}{2} = 5$, which is equal to the number of degrees of freedom confounded in each replicate so that the design is a balanced arrangement.

Balancing for the case $m = 2$ would be achieved in $(p_1 - 1)(p_2 - 1)$ replications, which would be obtained by varying the second element of the first stage generator over $(p_1 - 1)$ non-null elements of $GF(p_1)$ and further for each of these cases by varying the second element of the second stage generator over the $(p_2 - 1)$ non-null elements of $GF(p_2)$. Balanced designs of the type $(p_1, p_2, \ldots, p_6)^m$ in blocks of $p_1^{m-r}, p_2^{m-r}, \ldots, p_6^{m-r}$ plots, where $r_1, r_2, \ldots, r_k \leq m$, can be constructed in a similar manner in $(p_1 - 1)(p_2 - 1) \ldots (p_k - 1)$ replications (provided $r_j \leq p_j - 1; j = 1, 2, \ldots, k$).

5.5. Balanced Asymmetrical Designs with Reduced Number of Replications

Firstly, let us consider three-factor designs of the type $s_1 \times s_2 \times s_3$ where both $s_1$ and $s_2$ are prime powers. Let $s_1 \geq s_2 \geq s_3$. From Section 4, we know that if $s_1 = s_2$, we can construct a design in blocks of $s_2 s_3$ plots balanced in $(s_1 - 1)$ replications. In case $s_1 \neq s_2$, the method given there provides a balanced design in $(s_1 - 1)^2$ replications. We may, therefore, use a modified method in such a case.

Consider, first, $GF(s_2)$. Let the associated vectors corresponding to $A_2$ and $A_3$ be $(a_0, a_1, a_2, \ldots, a_{s_2-1})$ and $(a_0, a_1, \ldots, a_{s_3-1})$ respectively. Taking a generator, say, $(a_1, a_2)$ in $GF(s_2)$, we can form one replication of $s_2$ sets of treatment combinations of the factors $A_2$ and $A_3$, each set containing $s_3$ treatment combinations. The $j$-th set will obviously contain those combinations of levels of $A_2$ and $A_3$, the elements of the associated vectors corresponding to which give a sum product $a_{j-1}$ when multiplied by the generator. Let us denote these sets by $X_{s_2}$, $X_{s_3}$, $X_{s_2}$, $X_{s_3}$, $X_{s_2}$, $X_{s_3}$, $X_{s_2}$, $X_{s_3}$, respectively.

Proceeding, as above, we can similarly construct a design in $s_2$ plot blocks in $(s_1 - 1)$ replications by considering the two factors $A_1$ and $A_3$ only with associated vectors $(a_0, a_1, \ldots, a_{s_3-1})$ and $(a_0, a_1, \ldots, a_{s_3-1})$ in $GF(s_3)$ and taking the $(s_1 - 1)$ generators represented respectively by $(a_1, a_2)$, where $r$ varies from 1 to $s_2 - 1$. This is simply an $s_1 \times s_2$ design; and to extend it to the $s_1 \times s_2 \times s_3$ design, we may now replace
the \( j \)-th level of the factor \( A_2 \) in this design by the set \( X_j \) defined above containing \( s_a \) combinations of levels of \( A_2 \) and \( A_3 \).

As an illustration of this procedure, consider the \( 5 \times 3 \times 2 \) design. The sets \( X_j \) obtained would be

\[
\begin{align*}
X_0 & : b_0 c_0, b_2 c_1 \\
X_1 & : b_1 c_0, b_0 c_1 \\
X_2 & : b_2 c_0, b_1 c_1
\end{align*}
\]

These sets, when combined with the 5 levels of \( A_1 \) with the help of \( GF(5) \), will give the following \( 5 \times 3 \times 2 \) design:

**Table VI**

\( 5 \times 3 \times 2 \) Design in 6-plot blocks

<table>
<thead>
<tr>
<th>Block No.</th>
<th>Replication I</th>
<th>Replication II</th>
<th>Replication III</th>
<th>Replication IV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 2 3 4 5</td>
<td>6 7 8 9 10</td>
<td>11 12 13 14 15</td>
<td>16 17 18 19 20</td>
</tr>
<tr>
<td>000</td>
<td>100 200 300 400</td>
<td>000 300 100 400</td>
<td>000 200 300 400</td>
<td>000 400 300 200</td>
</tr>
<tr>
<td>021</td>
<td>021 221 321 421</td>
<td>021 321 121 421</td>
<td>021 221 321 421</td>
<td>021 421 321 221</td>
</tr>
<tr>
<td>311</td>
<td>311 011 111 211</td>
<td>311 211 011 111</td>
<td>311 011 111 211</td>
<td>311 111 011 211</td>
</tr>
<tr>
<td>320</td>
<td>320 420 020 120</td>
<td>320 420 020 120</td>
<td>320 420 020 120</td>
<td>320 120 020 420</td>
</tr>
<tr>
<td>410</td>
<td>410 010 110 210</td>
<td>410 010 110 210</td>
<td>410 010 110 210</td>
<td>410 110 010 210</td>
</tr>
<tr>
<td>401</td>
<td>401 001 101 201</td>
<td>401 001 101 201</td>
<td>401 001 101 201</td>
<td>401 101 001 201</td>
</tr>
</tbody>
</table>
In the above design, the main effects and interactions confounded are $A$, $AB$ (4 d.f.) and $ABC$. It is noticeable that $AC$ is not confounded although the number of combination of levels of $A$ and $C$ is 10 and the block size is 6. The loss of information on each of 4 d.f. of $A$ is $1/6$; that on 4 d.f. of $AB$ is 0 and on each of the remaining 4 d.f. of $AB$ is $5/24$; and, finally, on each of 4 d.f. of $ABC$, the loss is $5/24$ and on each of the remaining 4 d.f. of $ABC$, it is $10/24$. The total loss of information is, therefore, 4, so that the design is balanced.

Designs of the type $7 \times 3 \times 2$ (in 6-plot blocks involving 6 replications), $8 \times 3 \times 2$ (in 6-plot blocks involving 7 replications), $5 \times 4 \times 3$ (in 12-plot blocks involving 4 replications), $7 \times 4 \times 3$ (in 12-plot blocks involving 6 replications), $7 \times 5 \times 3$ (in 15-plot blocks involving 6 replications), etc., can be easily constructed by the above method.

For the construction of a four-factor design, say, $5 \times 3 \times 2 \times 2$ in 12-plot blocks, we can proceed exactly as in the above manner, combining first $A_1$ and $A_2$ and making two sets with two treatment combinations in each set; combining these two sets with $A_2$ in $GF(3)$, making 3 new sets of 4 treatment combinations each; and finally making five sets of 12 treatment combinations each by combining the 3 sets formed in the last case with the factor $A_1$, using $GF(5)$. The design so obtained will be balanced in 4 replications. Obviously, the design can be represented by the plan given above for the $5 \times 3 \times 2$ design with the modification that in place of the two levels of $C$, namely, 0 and 1, we have now to put respectively two sets of levels of $C$ and $D$, namely, $c_0d_0$, $c_1d_1$ and $c_0d_2$, $c_1d_2$. It will be found that in the design so generated, the main effect $A$ will be confounded and also the interactions $AB$ and $ABCD$.

The above procedure can be easily and usefully generalized for the construction of the general asymmetrical factorial design $s_1 \times s_2 \times \ldots \times s_m$ (where $s_1 \geq s_2 \geq s_3 \geq \ldots \geq s_m$, and $s_1$, $s_2$, $s_3$, $s_m$ are all prime powers) in blocks of $s_2 \times \ldots \times s_m$ plots. Further, having obtained such a design, we can reduce the block size one step further to $s_3 \times \ldots \times s_m$ by splitting all the $s_2 \times \ldots \times s_m$ treatment combinations in a block into $s_2$ sets of $s_3 \times s_4 \times \ldots \times s_m$ treatment combinations each by the use of $GF(s_3)$ over the factors $A_2A_3 \ldots A_m$. It has to be remembered that for doing this the same generator is to be used for all the blocks.

As an illustration, the $5 \times 3 \times 3 \times 2$ design in blocks of 6 plots will be presented.
In order to avoid complete confounding over \( AB \), we first combine \( B \) and \( C \), and make the one plot sets \( Z_{ij} \) given by

\[
Z_{ij} = b_c i
\]

where

\[
\begin{align*}
  r + t &= i \mod 3 \\
  r + 2t &= j \mod 3
\end{align*}
\]

Next, we combine \( Z_{ij} \) with \( D \), and get the sets \( X_i \)

\[
\begin{align*}
  X_0 &= Z_0 d_0, Z_0 d_1 \\
  X_1 &= Z_1 d_0, Z_0 d_1 \\
  X_2 &= Z_2 d_0, Z_1 d_1
\end{align*}
\]

where \( Z_i \) consist of \( Z_{ij} \) for all \( j \). We then get a \( 5 \times 3 \times 3 \times 2 \) design in \( 3 \times 3 \times 2 \) plot blocks by combining the \( X \)'s with the factor \( A \). To get a design in blocks of 6 plots, we simply put the treatments with the same \( j \) in \( Z_j \) in the same block, and those with separate \( j \)'s in separate blocks. This will give a design in 4 replications. However, in order to have a balanced design, we shall have to use four more replications obtained by separating \( Z_i \)'s with respect to \( i \) in the same way as was done with \( j \). The total number of replications required for balancing in this case is \( (5 - 1)(3 - 1) = 8 \), and is, in general, \((s_1 - 1)(s_2 - 1) \) since we use two Galois fields, each once, and make \( s_1 s_2 \) blocks per replication. The \( 5 \times 3 \times 3 \times 2 \) design is given in Table VII, where the loss of information is also shown. The total loss of information is seen to be 14, or one less than the number of blocks per replication, so that the design is balanced. This design can be easily generalized to \( s \times 3 \times 3 \times 2 \) where \( s \) is a prime power.

### Table VII

**5x3x3x2 Design in 6-plot blocks**

<table>
<thead>
<tr>
<th>Combinations of ( B, C, D )</th>
<th>Level of ( A )</th>
<th>Combinations of ( B, C, D ) for four exactly similar replications</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Replication I</td>
<td>Replication II</td>
</tr>
<tr>
<td>000 Block No. 000</td>
<td>1 4 7 10 13</td>
<td>16 19 22 25 28</td>
</tr>
<tr>
<td>111</td>
<td>0 1 2 3 4</td>
<td>0 3 1 4 2</td>
</tr>
<tr>
<td>220</td>
<td>4 0 1 2 3</td>
<td>2 0 3 1 4</td>
</tr>
<tr>
<td>001</td>
<td>3 4 0 1 2</td>
<td>4 2 0 3 1</td>
</tr>
<tr>
<td>110</td>
<td>221</td>
<td>111</td>
</tr>
</tbody>
</table>
Table VII (Contd.)

<table>
<thead>
<tr>
<th>Combinations of $B, C, D$</th>
<th>Level of $A$</th>
<th>Combinations $B, C, D$ of for four exactly similar replications</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Replication I</td>
<td>Replication II</td>
</tr>
<tr>
<td>Block No. 210</td>
<td>2 5 8 11 14</td>
<td>17 20 23 26 29</td>
</tr>
<tr>
<td>Block No. 021</td>
<td>0 1 2 3 4 0</td>
<td>0 3 1 4 2 0</td>
</tr>
<tr>
<td>Block No. 100</td>
<td>4 0 1 2 3 2</td>
<td>0 3 1 4 3 0</td>
</tr>
<tr>
<td>Block No. 211</td>
<td>3 4 0 1 2 4</td>
<td>2 0 3 1 4 3</td>
</tr>
<tr>
<td>Block No. 020</td>
<td>3 6 9 12 15</td>
<td>18 21 24 27 30</td>
</tr>
<tr>
<td>Block No. 101</td>
<td>0 1 2 3 4 0</td>
<td>0 3 1 4 3 0</td>
</tr>
<tr>
<td>Block No. 120</td>
<td>4 0 1 2 3 2</td>
<td>0 3 1 4 3 0</td>
</tr>
<tr>
<td>Block No. 201</td>
<td>3 4 0 1 2 4</td>
<td>2 0 3 1 4 3</td>
</tr>
</tbody>
</table>

Total number of replications = 8

Loss of information

<table>
<thead>
<tr>
<th>Relative loss per d.f.</th>
<th>Total loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>1/6</td>
</tr>
<tr>
<td>$AB$</td>
<td>5/48</td>
</tr>
<tr>
<td>$AC$</td>
<td>5/48</td>
</tr>
<tr>
<td>$BC (I)$</td>
<td>1</td>
</tr>
<tr>
<td>$BC (J)$</td>
<td>0</td>
</tr>
<tr>
<td>$ABC (I)$</td>
<td>8/48</td>
</tr>
<tr>
<td>$ABC (J)$</td>
<td>5/48</td>
</tr>
<tr>
<td>$ABD$</td>
<td>15/48</td>
</tr>
<tr>
<td>$ACD$</td>
<td>15/48</td>
</tr>
<tr>
<td>$ABC (I) D$</td>
<td>15/48</td>
</tr>
<tr>
<td>$ABC (J) D$</td>
<td>0</td>
</tr>
<tr>
<td>$BCD$</td>
<td></td>
</tr>
<tr>
<td>$CD$</td>
<td></td>
</tr>
<tr>
<td>$AD$</td>
<td></td>
</tr>
<tr>
<td>$BD$</td>
<td></td>
</tr>
<tr>
<td>$B$</td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td></td>
</tr>
</tbody>
</table>

Total Loss = 14 = No. of blocks per replication - 1.
6. SOME FURTHER BALANCED DESIGNS

6.1. Derivation of One Balanced Design from Another

Suppose there already exists a balanced design $s_1 \times s_2 \times \ldots \times s_m$ in blocks of $k$ plots ($k$ may be $s_2 \times s_3 \times \ldots \times s_m$), and we wish to derive the design for the $a_1 s_1 \times a_2 s_2 \times \ldots \times a_m s_m$ factorial experiment from it, where $a_1, a_2, \ldots, a_m$ are any positive integers. Also, suppose that in the construction of the given design $s_1 \times s_2 \times \ldots \times s_m$ we had used for the factor $A_i$, an associated vector

$$(Z_1, Z_2, \ldots, Z_{si})$$

where the $Z_j$'s are any elements, not necessarily distinct, of the Galois field used for the purpose of generating the blocks. Then, for the construction of the $a_1 s_1 \times \ldots \times a_m s_m$ design, we may simply take for the $A_i$ an associated vector of the form

$$(Z_1, Z_2, \ldots, Z_{si}; Z_1, Z_2, \ldots, Z_{si}; \ldots; Z_1, Z_2, \ldots, Z_{si})$$

each $Z_j$ being repeated $a_i$ times in this vector. Such associated vectors for $A_i$ should be used in the new design corresponding to all associated vectors which were used for the factor $A_i$ when the given design $s_1 \times s_2 \times \ldots \times s_m$ was constructed. The block size in the derived design will be $a_1 \times a_2 \times \ldots \times a_m \times k$. The block size can be further reduced to any extent by repeated use of suitable Galois fields. For this purpose factorisation of $a$'s into prime powers may also be done. This procedure of derivation of designs with non-prime levels gives a number of useful designs.

As an illustration, we derive the $6 \times 2 \times 2$ design from the $3 \times 2 \times 2$ design. The associated vectors that we use for the $3 \times 2 \times 2$ design are $(0, 1)$ for $B$ and $C$ and $(0, 0, 1); (0, 1, 0); (1, 0, 0)$ for $A$ to be used respectively for the three replications in which balancing is achieved. All these vectors are in $GF(2)$. For the $6 \times 2 \times 2$ designs, we use the same vectors for $B$ and $C$ and for $A$ we use $(0, 0, 1; 0, 0, 1); (0, 1, 0; 0, 1, 0)$ and $(1, 0, 0; 1, 0, 0)$ respectively for the three replications. This gives the $6 \times 2 \times 2$ design shown in Table VIII, in which $X_0$ and $X_1$ denote respectively the sets $(b_6 c_0, b_1 c_1)$ and $(b_1 c_0, b_0 c_1)$. In this design, the total loss of information on $BC$ is $1/9$ and that on the two confounded d.f. of $ABC$ is $8/9$, so that the total loss is unity and the design is balanced.

As already mentioned, the above procedure can be used for construction of all designs irrespective of the number and type of Galois
fields utilized for getting the block size $k$. Thus, for a $3 \times 2^3$ design in 6-plot blocks, we use $GF(2)$ twice with the associated vectors of $A$ as given above for the $3 \times 2^3$ design. From this design we can, therefore, immediately derive the $6 \times 2^3$ design simply by using the associated vector for $A$ as given above for the $6 \times 2^3$ design.

A similar procedure is adopted in those cases where two different Galois fields are to be used. For example, consider the $6 \times 6 \times 2$ design in 12-plot blocks. First, we divide the 72 treatment combinations into two sets of 36 each by using $GF(2)$ together with (i) a generator of the form $(1, 1, 1)$, (ii) the associated vector $(0, 0, 0, 1, 1, 1)$ for both $A$ and $B$ and $(0, 1)$ for $C$. At the second stage, we use $GF(3)$ and take $(0, 1, 2, 0, 1, 2)$ as the associated vectors of $A$ and $B$ and $(0, 1)$ as the associated vector of $C$ along with two generators $(1, 1, 1)$ and $(1, 2, 1)$ for getting two different replications, which will provide the balanced design given in Table IX. In this Table, $c_0$ and $c_1$ denote the two levels of $C$ and $X_i$ and $Y_i$ ($i = 0, 1, \ldots, 5$) the sets of combinations of levels of $A$ and $B$ as given in the plan for the $6 \times 6$ design. It can be easily seen that in this design, the interactions $AB$ and $ABC$ are confounded, the total loss of information being respectively 2 and 3.
The $s^3 \times t$ design in $s^t \times t$ plot blocks, where $s$ is non-prime, can be constructed by the methods of this Section as a particular case. Some of the other useful asymmetrical designs which can be constructed in an optimum manner by use of the above methods are $6 \times 4$, $6 \times 4 \times 3$, $6 \times 4 \times 2$, $6 \times 4 \times 4$, $4 \times 3 \times 2$, $6 \times 3 \times 2$, $3 \times 3 \times 3 \times 4$, $6 \times 3 \times 3 \times 2$, etc.

6.2. Further Use of b.i.b.d. Property in Obtaining Confounded Designs

Almost all types of factorial designs arising in practice can be constructed in an optimum manner by appropriately using the methods discussed so far. Still another method, which may be of use in certain situations and which serves to indicate the connection between balancing in asymmetrical factorial experiments and balanced incomplete block designs, will now be discussed.

Let us consider the construction of a $7 \times 2 \times 2$ design. By the use of hypersurfaces or associated vectors, we can construct a design in 4-plot blocks with 6 replications. In this design, the main effect $A$ is confounded, which is not desirable. The design belonging to Kishen's series $q^2 \times 2^q$ discussed in Section (5.2), taking $q = 7$, is to be preferred in this case, as only the interactions $BC$ and $ABC$ are partially confounded in this design. However, the loss of information on $BC$ in this case is $25/49$. An alternative approach for obtaining an optimum design in this case is by use of the b.i.b.d. property and will now be discussed.

Each replication will be divided into two blocks of $2q$ plots each. Let $X_0$ and $X_1$ denote the sets $(b_0c_0, b_1c_1)$ and $(b_1c_0, b_0c_1)$ of treatment combinations respectively. Consider a single replication. Block No. 1 of this replication will contain the treatment combinations $[a']X_0$ and $[a'']X_1$, where $a'$ and $a''$ represent two exhaustive groups for the levels
of $A$ and Block No. 2 will have the complement of this. If $a'_1$ and $a'_2$ are any two levels in $a'$, and $a''_1$, $a''_2$ belong to $a''$, then obviously $(a'_1 - a'_2)(X_1 - X_0)\), $(a''_1 - a''_2)(X_1 - X_0)$ and $[(a'_1 - a'_2) \pm (a''_1 - a''_2)] (X_1 - X_0)$, all of which belong to interaction $ABC$, will not be confounded. However, contrasts like $(a_1 - a_2)(X_1 - X_0)$ will be confounded. The question is how to determine $a'$ and $a''$ so that a balanced design is obtained.

A solution to this problem may be found by considering the possibility of selecting the set $a'$ in the different replications in such a manner that the levels of $A$ included in $a'$ form a balanced incomplete block design. Since, in that case, every pair of levels of $A$ will occur on equal number of times with $X_0$ or $X_1$, every contrast of the type $(a_i \pm a_j) (X_1 - X_0)\) (i \neq j = 1, 2, \ldots, q)$ will be partially confounded to the same extent. Thus, both $BC$ and $ABC$ will be estimable, and the design will be a balanced arrangement.

The loss of information on $BC$ depends on the number of levels in the sets $a'$ and $a''$. For $7 \times 2 \times 2$ design, we can take 3 levels for $a'$ and 4 for $a''$, and obtain the design given in Table X.

### Table X

$7 \times 2 \times 2$ Design in 14-plot blocks

<table>
<thead>
<tr>
<th>$A$</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>...</td>
<td>$X_0X_1$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>...</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>...</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>...</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>...</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>...</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
</tr>
<tr>
<td>$a_6$</td>
<td>...</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_0X_1$</td>
<td>$X_1X_0$</td>
<td>$X_1X_0$</td>
</tr>
</tbody>
</table>

In the above design, loss of information on $BC$ is $1/49$ and on each of the six d.f. of $ABC$ is $8/49$, so that the total loss of information is 1. The design is, therefore, balanced.
We shall now proceed to the general case and consider a \( q \times p^2 \)
design, where \( p \) is an odd prime power and \( q \) any integer. (It can be
easily seen that the method holds when \( p \) is of the form \( 2^n \)). Consider
a generator of the type, say, \( (a_1, a_1, a_2) \), where \( a_1 \) is an element of \( GF(p) \).
Let \( k_i (i = 1, 2, \ldots, p) \) be any integers such that \( 0 \leq k_i \leq q - 1, \)
and \( \sum_{i=1}^{p} k_i = q \), so that these divide the total number of levels of \( A \) in
\( p \) groups, of which one or more groups may have no elements in them.
Also, suppose that for \( i = 1, 2, \ldots, p \), balanced incomplete block
designs exist with parameters \( v = q, k = k_i \) and suitable values of \( b, \)
\( r_i \) and \( \lambda_i \) and can be superimposed on one another so as to give \( b \) blocks
of \( q \) plots each. Then we take an associated vector of the form
\( (a_0, a_1, \ldots, a_{p-1}) \) for \( B \) and \( C \), and \( b \) different associated vectors for \( A \),
one vector corresponding to each replication, such that each of the
elements in the \( j \)-th group (containing \( k_j \) elements), is \( a_{ij} \), an element of
\( GF(p) \). A balanced design in \( b \) replicates will then be obtained by
using the generator with each of the \( b \) associated vectors for \( A \) and the
common associated vector for \( B \) and \( C \).

As an illustration, consider the \( 7 \times 3 \times 3 \) design in 21-plot blocks.
We shall consider the three sets \( X_0, X_1, X_2 \) of treatment combinations
for the interaction \( BC \) defined below:

\[
X_0 : b_0 c_0, b_2 c_1, b_1 c_2 \\
X_1 : b_1 c_0, b_0 c_1, b_2 c_2 \\
X_2 : b_2 c_0, b_1 c_1, b_0 c_2
\]

Here \( q = 7 \). Also, if we take \( k_1 = 3, k_2 = 4 \) and \( k_3 = 0 \), we shall
find that two b.i.b. designs with \( v = 7 \) exist, which are superimposable,
as shown in Table XI.

**Table XI**

*Two b.i.b. designs with \( v = 7 \)*

<table>
<thead>
<tr>
<th>B.I.B.D. No.</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block No.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>
We can, therefore, form the 7 different associated vectors for \( A \) corresponding to the 7 replications in the design. Thus, for the 5th replication, the associated vector for \( A \) will be \((a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6})\) when the \( a_{ij} \) are elements of \( GF(3) \). Here \( a_{i_1} = 0 \) and \( a_{i_6} = 1 \). By using the generator \((1, 1, 1)\), we shall then obtain the design given in Table XII.

**Table XII**

\( 7 \times 3 \times 3 \) Design in 21-plot blocks

\[
\begin{array}{cccccccc}
  a_0 & \ldots & X_0X_1X_2 & X_0X_1X_2 & X_0X_1X_1 & X_2X_0X_1 & X_2X_0X_1 & X_2X_0X_1 \\
  a_1 & \ldots & X_0X_1X_2 & X_2X_0X_1 & X_2X_0X_1 & X_0X_1X_1 & X_0X_1X_1 & X_0X_1X_1 \\
  a_2 & \ldots & X_0X_1X_2 & X_2X_0X_1 & X_2X_0X_1 & X_0X_1X_1 & X_0X_1X_1 & X_0X_1X_1 \\
  a_3 & \ldots & X_2X_0X_1 & X_0X_1X_1 & X_2X_0X_1 & X_0X_1X_1 & X_0X_1X_1 & X_0X_1X_1 \\
  a_4 & \ldots & X_2X_0X_1 & X_0X_1X_1 & X_2X_0X_1 & X_0X_1X_1 & X_0X_1X_1 & X_0X_1X_1 \\
  a_5 & \ldots & X_2X_0X_1 & X_0X_1X_1 & X_2X_0X_1 & X_0X_1X_1 & X_0X_1X_1 & X_0X_1X_1 \\
  a_6 & \ldots & X_2X_0X_1 & X_0X_1X_1 & X_2X_0X_1 & X_0X_1X_1 & X_0X_1X_1 & X_0X_1X_1 \\
\end{array}
\]

It will be seen that in the above design, the interactions \( BC \) and \( ABC \) are confounded.

The above procedure can be used to obtain other asymmetrical designs also, for example, the \( 7 \times 3 \times 2 \) design in 14-plot blocks.

### 7. Analysis of Balanced Designs

Two general methods of analysing partially confounded designs, which are balanced, will now be briefly discussed to enable the interested reader to work out formulae for analysis in the case of any specific design.

#### 7.1. The \( Q_j \) Method

In any general design where

\[
\begin{align*}
N_i &= \text{number of plots in the } i\text{th block} \quad (i = 1, \ldots, b), \\
N_j &= \text{number of replications of the } j\text{-th treatment} \quad (j = 1, \ldots, t), \\
n_{ij} &= \text{number of times the } j\text{-th treatment occurs in the } i\text{th block},
\end{align*}
\]
and where \( t_j \) denotes the effect of the \( j \)-th treatment, it is known (Kempthorne, 1952) that the normal equations for estimating the \( t_j \) can be written in the form

\[
\left( N_i - \sum_i n_{ij}^2 \right) t_j - \sum_{k \neq j} \left( \sum_i n_{ij} h_{ik} \right) t_k = Q_j, \quad j = 1, 2, \ldots, t
\]

where

\[Q_j = Y_j - \sum_i n_{ij} \frac{Y_i}{N_i}.
\]

Since the block size and number of replicates of a treatment for all the designs discussed in this paper are constant, we shall take \( N_i = k \) and \( N_j = r \). Thus, the normal equations in (38) reduce to

\[
r \left( 1 - \frac{1}{k} \right) t_j - \frac{1}{k} \text{(sum of } t's \text{ that occur in a block with } t_j) = Q_j
\]

The quantities \( Q_j \) are the well-known adjusted yields for a treatment, and may be calculated in an easy and straightforward way for all designs, as indicated in the Table below:

<table>
<thead>
<tr>
<th>Treatment No.</th>
<th>Total yield from all replicates</th>
<th>Blocks in which the ( j )-th treatment occurs</th>
<th>Total yield of all blocks in which ( j )-th treatment occurs</th>
<th>( Q_j = \frac{Q}{S/k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Y_1, B_{11}, B_{21}, \ldots, B_{1r} )</td>
<td>( S_1 )</td>
<td>( S_1/k )</td>
<td>( Q_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( Y_2, B_{21}, B_{22}, \ldots, B_{2r} )</td>
<td>( S_2 )</td>
<td>( S_2/k )</td>
<td>( Q_2 )</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>( j )</td>
<td>( Y_j, B_{j1}, B_{j2}, \ldots, B_{jr} )</td>
<td>( S_j )</td>
<td>( S_j/k )</td>
<td>( Q_j )</td>
</tr>
<tr>
<td>( t )</td>
<td>( Y_t, B_{t1}, B_{t2}, \ldots, B_{tr} )</td>
<td>( S_t )</td>
<td>( S_t/k )</td>
<td>( Q_t )</td>
</tr>
</tbody>
</table>

The blocks \( B_{ij} \) are not all distinct for different \( i \) and \( j \).
Now, if $\Sigma \lambda_j f_j$ is a treatment contrast which we wish to estimate, it can be estimated by $1/I_r \Sigma \lambda_j Q_j$, where $I$ is the relative information on the contrast. For obtaining the relative information, we first calculate the expectation of $\Sigma \lambda_j Q_j$, where the expected value of $Q_j$ is as shown on the left-hand side of equation (40). Then, if $C_j$ is the coefficient of $t_j$ in this expected value; we shall find that if $\Sigma \lambda_j t_j$ is estimable from the full design, the value of $C_j/\lambda_j$ will be a constant $C$ for all $j$. Then, the relative information required is given by

$$I = \frac{C}{r}$$

(41)

where, as above, $r$ is the total number of replications. Having calculated $I$, the sum of squares corresponding to this contrast will be given by

$$\frac{1}{I_r} \times \frac{1}{\Sigma \lambda_j^2} (\Sigma \lambda_j Q_j)^2$$

(42)

where, for this purpose, $\lambda_j$'s should be $+1$, $-1$ or zero.

As an illustration, consider the analysis of the $4 \times 3^2$ design, the plan of which is given in Table XIII.

**Table XIII**

<table>
<thead>
<tr>
<th>Plan of $4 \times 3^2$ Design in 12-plot blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level of $B$ and $C$</td>
</tr>
<tr>
<td>Block No. 1 2 3 4 5 6</td>
</tr>
</tbody>
</table>
| Level of $A$  
  $a_0$  1 1 1 1 1 1  
  $a_1$  1 1 1 1 1 1  
  $a_2$  1 1 1 1 1 1  
  $a_3$  1 1 1 1 1 1 |
| The $I$'s denote as usual the well-known $I$ sets, corresponding to the interaction $BC(I)$. It will be found that the relative loss of information on the interaction $BC(I)$ is $\frac{1}{4}$ and that on $(a_3 + a_2 - a_1 - a_0) BC(I)$ components of $ABC$ interaction is $\frac{1}{3}$. Two independent comparisons belonging to $BC(I)$ are $L_1 = (a_3 + a_2 - a_1 + a_0)(I_2 - I_6)$ and $L_2 =$ |
(a_1 + a_2 + a_3 + a_0) (I_2 - 2I_1 + I_0). The estimates of \( L_1 \) and \( L_2 \) will be found to be 
\[ \hat{L}_1 = \frac{4}{3} (Q_{1_2} - Q_{1_3}), \quad \hat{L}_2 = \frac{4}{3} (Q_{1_3} - 2Q_{1_1} + Q_{1_0}), \]
where \( Q_{1_0} \) is the sum of the \( Q_i \)'s for all the nine treatment combinations contained in the set \( a_i I_0 \) \((i = 0, 1, 2, 3; \quad I_0 = b_0 c_0, b_2 c_1, b_1 c_2)\), with similar definitions for \( Q_{1_1} \) and \( Q_{1_2} \). Similarly, if \( L_3 = (a_3 + a_2 - a_1 - a_0) (I_2 - I_0) \) and \( L_4 = (a_3 + a_2 - a_1 - a_0) (I_3 - 2I_1 + I_0) \), we shall have
\[ \hat{L}_3 = 4 (Qa_3 I_2 + Qa_3 I_3 - Qa_1 I_0 - Qa_0 I_0 - Qa_3 I_0 - Qa_0 I_0 + Qa_1 I_0 + Qa_0 I_0), \]
with a similar expression for \( \hat{L}_4 \). The sum of squares for BC (1) will be
\[ \frac{1}{9} (Q_{1_1} - Q_{1_3})^2 + \frac{1}{27} (Q_{1_3} - 2Q_{1_1} + Q_{1_0})^2. \]

7.2. Yates's Method

This method, which has been suggested by Yates (1937), has been used by him and Li (1944) for the analysis of balanced asymmetrical factorial designs. The method can be utilized for the analysis of the balanced designs discussed in this paper. However, it is particularly appropriate for the analysis of Kishen's series of designs presented in Section (5.2). For illustration, we shall now give briefly the method of analysis of the \( q \times 2^2 \) design in blocks of 2q plots, the plan for which is shown in Table XIV.

**Table XIV**

*Plan of \( q \times 2^2 \) Design*

<table>
<thead>
<tr>
<th>B_{11}</th>
<th>B_{12}</th>
<th>B_{21}</th>
<th>B_{22}</th>
<th>B_{31}</th>
<th>B_{32}</th>
<th>\ldots</th>
<th>B_{q1}</th>
<th>B_{q2}</th>
</tr>
</thead>
</table>

<table>
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<tr>
<th>Level of A</th>
<th>X_0</th>
<th>X_1</th>
<th>X_0</th>
<th>X_1</th>
<th>X_0</th>
<th>\ldots</th>
<th>X_1</th>
<th>X_0</th>
</tr>
</thead>
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<td>X_0</td>
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<td>X_1</td>
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<td>X_0</td>
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<tr>
<td>X_1</td>
<td>X_0</td>
<td>X_1</td>
<td>X_0</td>
<td>X_1</td>
<td>X_0</td>
<td>\ldots</td>
<td>X_1</td>
<td>X_0</td>
</tr>
</tbody>
</table>

Here \( X_0 = b_0 c_0 + b_1 c_1, \quad X_1 = b_0 c_1 + b_1 c_0. \)

The constants to be fitted by the method of least squares are chosen according to the following scheme:
Blocks: \(-b_1, b_1; -b_2, b_2; \ldots; -b_q, b_q\).

Interaction \(BC\): \(X_0, X_1; f, -f\).

Interaction \(ABC\): \(X_0a_j, X_1a_j; i_j, -i_j(j = 0, 1, 2, \ldots, q - 1)\),

\[\sum_{j=0}^{q-1} i_j = 0.\]

Denoting the block totals in the \(j\)-th replication by \(B_{j1}, B_{j2}(j = 1, 2, \ldots, q)\), we take, following Yates,

\[g_j = B_{j1} - B_{j2} \quad (j = 1, 2, \ldots, q).\]

We also use \([BC]\) to denote, as Yates has done, the ordinary total for this interaction. Similarly, we use \([BC.a_j]\) \((j = 0, 1, 2, \ldots, q - 1)\) for the total for the interaction in the presence of \(a_j\), the contrast between these \(q\) totals giving the interaction \(ABC\).

The normal equations for determining the above constants then come out as under:

\[4q^2f + 4(q - 2)\left(\sum_{i=1}^{q} b_i\right) = [BC]\]
\[4qi_1 + 4qf + 4(-b_1 + b_2 + \ldots + bq) = [BC.a_0]\]
\[4qi_2 + 4qf + 4(b_1 - b_2 + \ldots + bq) = [BC.a_1]\]
\[\vdots\]
\[4qi_{q-1} + 4qf + 4(b_1 + b_2 + \ldots + b_q) = [BC.a_{q-1}]\]
\[4qb_1 + 4(q - 2)f + 4(-i_1 + i_2 + \ldots + i_q) = -g_1\]
\[4qb_2 + 4(q - 2)f + 4(i_1 - i_2 + \ldots + i_q) = -g_2\]
\[\vdots\]
\[4qb_q + 4(q - 2)f + 4(i_1 + i_2 + \ldots - i_q) = -g_q\]

By solving the above equations, we obtain

\[16q(q - 1)f = q[BC] + (q - 2)\sum_{i=1}^{q} g_i,\]

Taking

\[q[BC] + (q - 2)\sum_{i=1}^{q} g_i = qQ,\]

we have

\[f = \frac{Q}{16(q - 1)}.\]
The estimate of $BC$ in units of the yield of a single plot is given by
\[
2f = \frac{(qQ)}{8q(q-1)}
\]

The error variance of $BC$ is given by
\[
V(2f) = \frac{\sigma^2}{4(q-1)}
\]

In an unconfounded experiment, the estimate of $BC$ would be $(1/2q^2) [BC]$ and its error variance would be $\sigma^2/q^2$.

The relative information is, therefore, given by the ratio
\[
\frac{1}{q^2} \int \frac{1}{4(q-1)} = \frac{4(q-1)}{q^2},
\]
so that the relative loss of information on $BC$ is given by
\[
L(BC) = \frac{(q-2)^2}{q^2}.
\]

The sum of squares for $BC$ is
\[
\frac{Q^2}{16(q-1)} = \frac{(qQ)^2}{16q^2(q-1)}
\]
as compared with $(1/4q^2) [BC]^2$ in an unconfounded experiment.

The estimate of $ABC$ is obtained in a similar manner by solution of the normal equations given above. Thus, for estimating $i_j (j = 0, 1, 2, \ldots, q - 1)$, we get
\[
4(q^2 - 4)i_j + 16(q-1)f = qR_j,
\]
where
\[
qR_j = q[BC.a_i] + a_1 + a_2 + \ldots + a_j - a_{j+1} + a_{j+2} + \ldots + a_q.
\]

We thus obtain
\[
i_j = \frac{q}{4(q^2 - 4)(R_j - \bar{R})} (j = 0, 1, 2, \ldots, q - 1)
\]

where
\[
\bar{R} = \frac{\sum_{j=0}^{q-1} R_j}{q}.
\]
The estimate of the interaction $ABC$, in units of a single plot yield, is given by:

$$2l_j (j = 0, 1, 2, \ldots, q - 1),$$

so that we may write

$$ABC = \frac{1}{2 (q^2 - 4)} \text{dev} (R_0, R_1, \ldots, R_{q-1})$$

and

$$= \frac{1}{2 (q^2 - 4)} \text{dev} (qR_0, qR_1, \ldots, qR_{q-1})$$

as compared to $(1/2q) [BC, a_0]$, etc., in an unconfounded experiment.

The error variance applicable to each of these quantities is

$$V (2l_i) = \frac{4q^2 \sigma^2}{4 (q^2 - 4)} = \frac{q}{q^2 - 4} \sigma^2$$

as compared to $\sigma^2/q^2$ when there is no confounding.

The relative information is, therefore,

$$\frac{q^2 - 4}{q^2},$$

so that the loss of information on each degree of freedom of $ABC$ is given by

$$L (ABC) = \frac{4}{q^2}.$$  

Hence the total loss of information on both the interactions is

$$\frac{(q - 2)^2 + 4 (q - 1)}{q^2} = 1,$$

so that the design is balanced. The sum of squares for the interaction $ABC$ is given by

$$\frac{q}{4 (q^2 - 4)} \text{dev}^2 (R_0, R_1, \ldots, R_{q-1})$$

and

$$= \frac{1}{4q (q^2 - 4)} \text{dev}^2 (qR_0, qR_1, \ldots, qR_{q-1}).$$

8. Summary

The method of finite geometries developed earlier by Bose and Kishen for solving the problem of confounding in the general symmetrical factorial design has been extended to the construction of balanced confounded asymmetrical factorial designs which were not so far amenable to this approach. This has been achieved by using
curvilinear spaces or hypersurfaces and truncating the $EG (m, s)$ suitably. A more general method, using vectors in Galois fields, has also been introduced and a unified theory for the construction of both symmetrical and asymmetrical factorial designs developed. It has been shown that, with the help of this theory, symmetrical confounded factorial designs $s^n$, where $s$ is not a prime number or a prime power, as also almost all types of asymmetrical factorial designs can be constructed in an optimum manner. Methods of deriving symmetrical and asymmetrical factorial designs, using the b.i.b. property, have also been given, besides methods of reducing the number of replications required for balancing in asymmetrical designs and of deriving balanced designs of the type $a_1s_1 \times a_2s_2 \times \ldots \times a_ms_m$ from a given $s_1 \times s_2 \times \ldots \times s_m$ design. Finally, two methods of analysis of balanced partially confounded designs have been briefly discussed to enable the interested reader to work out formulae for analysis of any specific design.

9. References


7. ——— "A general class of quasifactorial designs leading to confounded designs for factorial experiments," Ibid., 1942, 7, 457-58.

