CIRCULAR DESIGNS—FURTHER RESULTS

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1. INTRODUCTION

A design obtained from a given incomplete block design by considering its blocks as varieties and varieties as blocks is said to be its dual. Bose and Nair [2] gave examples of partially balanced incomplete block (*PBIB*) designs obtained by dualizing some balanced incomplete block (*BIB*) designs. Youden [9] investigated these further. He called the dual of a *BIB* design by the suggestive name of Linked Block designs. Roy and Laha [7] have made exhaustive study, classification and enumeration of all linked block design with r, $k \leqslant 10$. Ramakrishnan [6] gave the structure and analysis of the dual of some *PBIB* designs with two associate classes. Recently, Saha and Mishra [8] dualized a series of two-associate *PBIB* designs and obtained a new series of three replicate three-associate *PBIB* designs.

The present authors [5] in an earlier communication have put forward a unified method of analysis of circular designs of Das [4]. In this paper, a study is made on the duals and association schemes of circular designs.

2. CIRCULAR DESIGN AND ITS ASSOCIATION SCHEME

2·1. For completeness we have the following definitions :—

Definition 2·1. Let there be n equal arcs on the circumference of a circle denoted in order by a_1, a_2,......, a_n. If we form bigger

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arcs say A_{ki} such that it is the sum of k consecutive small arcs starting with a_i, we shall have in all n such arcs for n different values of i. Now, if we identify a_i with a set (s_i) of m treatments such that the different sets (s_i) are mutually exclusive, then the contents of the n arcs A_{ki} will form an incomplete block design with n blocks, mn treatments, blocks size mk and k replications. Such a design is called a circular design (Das [4]). We shall refer to these designs by CD (mn, n, k, mk).

Definition 2·2. Given v treatments, 1, 2, ..., v, a relation satisfying the following conditions is said to be an Association Scheme with M classes:

(*a*) Any two treatments are either 1st, 2nd,...... or M-th associates, the relation of association being symmetrical, *i.e.*, if the treatment α is the i-th associate of the treatment β, then β is the i-th associate of α.

(*b*) Each treatment α has n_i i-th associates, the number n_i being independent of α.

(*c*) If any two treatments, α and β are i-th associates, then the number of treatments which are j-th associates of α and k-th associates of β is p_{jk}^{i} where p_{jk}^{i} is independent of the pair of i-th associates α and β.

The numbers v, n_i, p_{jk}^{i} $(i, j, k=1, 2, ..., M)$ are the parameters of the association scheme ([1], [3]).

With these definitions we now show that the circular designs for $v=mn$ treatments $(m \geqslant 2)$ are *PBIB* designs having the association scheme described below. We deal with two cases—(*i*) n odd, (*ii*) n even separately.

Case 1. n Odd

Let, without loss of generality, s_0, s_1, ...s_{n-1} represent the n disjoint sets of treatments corresponding to the n arcs on the circumference of a circle. Let θ_{ij} be a treatment belonging to s_i ; $i=0, 1, ..., n-1$; $j=1, 2, ..., m$. Any other treatment $\theta_{i'j'}$, $(\neq \theta_{ij})$ will be said to be q-th associate to θ_{ij}, if $i'=i\pm(q-1)$, mod n, $q=1, 2, ..., M$, where M denotes the number of associate classes and is given by $(n+1)/2$.

Clearly then, $n_1=m-1$; $n_j=2m, j=2, 3, ...M$. To prove the constancy of p^{r}_{tu} $(r, t, u=1, 2......, M)$, we proceed as follows.

Let,

(*i*) $\{s_i\}$ denote the set of m treatments belonging to the arc labelled i ; $i=0, 1, \ldots\ldots, n-1$:

(*ii*) $[\{s_i\} - \Theta]$ denote the set which consists of all the treatments of $\{s_i\}$ except one denoted by Θ ; the number of elements in $[\{s_i\} - \Theta]$ is, thus, $m-1$;

(*iii*) $A_q\ (\Theta \in s_i)$ and $B_q\ (\phi \in s_j)$ denote respectively the q-th ($q=1, 2, \ldots\ldots, M$) associates of the treatment Θ and $\phi(\neq \Theta)$ where Θ belongs to $\{s_i\}$ and ϕ to $\{s_j\}$; $i, j=0, 1, \ldots\ldots, n-1$;

(*iv*) $N[T_1, T_2]$ denote the number of elements common between the sets T_1 and T_2.

We, then, have :

$$A_1 = A_1\ (\Theta \in s_i) = [\{s_i\} - \Theta] ;$$

$$A_q = A_q\ (\Theta \in s_i) = [\{s_{i+q-1}\}, \{s_{i-q+1}\}] ;$$

$$B_1 = B_1\ (\phi \in s_j) = [\{s_j\} - \phi] ; \qquad \ldots(1)$$

$$B_q = B_q(\phi \in s_j) = [\{s_{j+q-1}\}, \{s_{j-q+1}\}] ;$$

$i, j=0,1,\ldots, n-1$; $q=2, 3, \ldots, M$; the suffixes of s's are reduced mod n.

Also, from (1) we obtain

$$N_{11} = N[A_1, B_1] = (m-2)\delta_{ij} ;$$

$$N_{1q} = N[A_1, B_q] = N_{q1} = N[A_q, B_1]$$

$$= (m-1)[\delta_{d_1 d_2} + \delta_{d_1 d_3}] ; \qquad \ldots (2)$$

$$N_{qq} = N[A_q, B_q] = m[\delta_{d_4 d_6} + \delta_{d_4 d_7} + \delta_{d_5 d_6} + \delta_{d_5 d_7}] ;$$

where, $\delta_{ab}=1$, if $a=b$, and $\delta_{ab}=0$ if $a \neq b$;

$$d_1 = i-j,\ d_2 = q-1,\ d_3 = 1-q,\ d_4 = i+q-1 ;\ d_5 = i-q+1,$$

$$d_6 = j+q-1,\ d_7 = j-q+1,$$

all the d's being reduced modulo n.

Now, evidently, for a pair of treatments $\Theta \in \{s_i\}$ and $\phi \in \{s_j\}$, which are mutually r-th associates, we can write :

$p^r_{tu}\ (\Theta, \phi) =$ [Number of treatments which are simultaneously t-th associates of Θ and u-th associates of ϕ]

$=N_{tu}=N_{ut}=N[A_t, B_u]$, where i and j in (1) and (2) are related by either (*i*) $(i-j)=(r-1)$, or (*ii*) $(i-j)=-(r-1)$, mod n ; $r, t, u,=1, 2, \ldots\ldots, M$.

It is clear from (2) that, the value of N_{tu} in (2) with $(i-j)=r-1$ is same as that with $(i-j)=-(r-1)$, mod n, which proves that p^r_{tu} remains the same for all the pairs of treatments Θ and ϕ which are mutually r-th associates. Thus, condition (*c*) of Definition 2.2 is satisfied. Hence the assertion about the Association Scheme of circular designs with odd n. It may be of interest to note that the parameters p^r_{tu} of this association scheme are readily obtainable for all values of $r, t, u=1\ 2, \ldots\ldots, M$ from the expressions given in (2).

Case II. *n*-Even

Following similar lines as in Case I, we can arrive at the results of this case. The points of differences only are mentionde below. The definition of the association relation remains the same in this case.

Here,

$$M=(n+2)/2\ ;$$

$$n_1=m-1\ ;\ (m\geqq 2)$$

$$n_q=2m\ ;\ q=2, 3,\ldots, M-1\ ;$$

$$n_M=m.$$

A_q's, B_q's and N_{tu}'s are as follows :

$$A_1=A_1\ (\Theta\ \varepsilon\ s_i)=[\{s_i\}-\Theta]\ ;$$

$$A_q=A_q(\Theta\ \varepsilon\ s_i)=[\{s_{i+q-1}\}, \{s_{i-q+1}\}]\ ;$$

$$q=2, 3,\ldots,M-1\ ;$$

$$A_M=A_M(\Theta\,\varepsilon s_i)=\{s_{i+n/3}\}\ ; \qquad \ldots(3)$$

B_q's are obtained from A_q's by replacing Θ by ϕ and i by j.

N_{11}, N_{1q} and N_{qq} are same as in (2) except that here q ranges from 2 to $M-1$.

$$N_{MI}=N_{MI}=(m-1)\ \delta_{id_8}, \text{ where } d_8=j+n/2, \text{ mod } n; \qquad ..\ (4)$$

$$N_{Mq}=N_{qM}=m[\delta_{d_4\,d_8}+\delta_{d_5 d_8}].$$

$$N_{MM}=m.\ \delta_{ij}.$$

A little reflection shows that the cases $(i-j)=(r-1)$ and $(i-j) = -(r-1)$, mod n yield identical sets of N_{tu}'s in (4) also. Hence the assertion about the Association Scheme of circular designs with even n.

2.2 The association scheme of circular designs having $m=1$ can be built up on exactly the same lines as those presented above for designs with $m \geqslant 2$, the only exception being that, here the first associate classes in both cases of odd and even n are dropped out, and the t-th class is now renamed as $(t-1)$-th class, $t=2, 3, \ldots M$. Clearly then, for the circular designs having $m=1$, $M=(n-1)/2$ or $(n/2)$ according as n is odd or even.

3. Dual of a circular design

Let D be a circular design for n treatments (putting $m=1$) in n blocks, k replications and block size k which is obtainable through the initial block $(0, 1, 2, \ldots, k-1)$, mod n.

Since k consecutive arcs form a block in a CD (mn, n, k, mk) it is easy to see that the operation of dualizing CD (mn, n, k, mk) results in a design D^* which consists of D repeated $(m-1)$ times *i.e.*, m replications of D. Evidently then, the association scheme of D^* which is the dual of a CD (mn, n, k, mk) is the same as that of D described in section 2.2. Thus, the duals of circular designs are also *PBIB* designs of $(n-1)|2$ or $n/2$ associate classes according as n is odd or even.

SUMMARY

Duals of circular design of Das [4] are studied in this paper. Circular designs for mn treatments in n blocks of mk plots and k replications are shown to be *PBIB* designs of $(n+1)/2$ or $(n+2)/2$ associate classes according as n is odd or even. The duals of circular designs are also *PBIB* designs of $(n-1)/2$ or $n/2$ associate classes according as n is odd or even.

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