A NOTE ON DIFFERENCE EQUATIONS AND COMBINATORIAL IDENTITIES ARISING OUT OF COIN TOSSING PROBLEMS

By T. V. Narayana and S. G. Mohanty

McGill University, Montreal, Canada, and Indian Council of Agricultural Research, New Delhi

Summary

We solve a class of difference equations and desire some combinatorial identities arising from "returns to equilibrium" in coin tossing problems. We shall use the results and the notations introduced by the senior author in three previous papers which are referred to in what follows as (1), (2) and (3).

1. Difference Equations Related to Partition of an Integer

Consider sequences of trials made with a coin, limiting ourselves to those sequences S_N ($N = 1, 2, \ldots$) which satisfy the two following conditions:—

(i) A sequence S_N consists of $2N$ trials and the total number of heads and tails obtained in this sequence is equal, being N each.

(ii) If the number of heads and tails obtained in this sequence S_N were equal at the $(2k)$-th trial, $k = 0, 1, \ldots, N-1$, the $(2k+1)$-st trial of S_N is always a tail.

We represent a tail by 'O' and a head by 'X' in what follows. For $N = 1$, we consider thus the single sequence 'OX'. For each sequence we are interested in the three variables N, n, r, where N is the total number of tails (O's) in the sequence, n represents the number of heads in the run of X's at the end of the sequence and r represents the total number of changes from tail (O) to head (X) in the sequence.

For example, the sequence '$OOXOOXXX$' which satisfies (i) and (ii) will correspond to $N = 4$, $n = 3$, $r = 2$.

It is easy to see that given all the sequences S_{N_1} for some value of $N = N_1$ say, we obtain without repetition or omission all the sequences S_{N_1+1} *by placing a* 0 *either before any of the* X's *in the run of* X's *at the end of a* S_{N_1} *sequence or after the last* X *of a* S_{N_1} *sequence and adding an* X *at the end.* Let this procedure be called (P).

For example, 'OX' gives by (P)

$O\ O\ X\ X$
$O\ X\ O\ X$

which represent the two possible sequences S_2.

Let (N, n, r) be the number of such sequences, possible for given values of N, n, and r. Evidently, $(1, 1, 1) = 1$ and we could obtain, by recursion, using (P), the values of (N, n, r) for all N, n, r. For one of N, n, r non-integral or zero or negative $(N, n, r) = 0$. If

$$N < n + r - 1, \quad (N, n, r) = 0.$$

Finally we have the difference equation obtained from (P),

$$(N, n, r) = (N-1, n-1, r) + \sum_{\eta=n}^{N-r+1} (N-1, \eta, r-1).$$

2. Difference Equations Connected with k-Dominations

It is seen easily [*cf.* (2)] that a sequence S_N which contains r changes of sign corresponds to a "domination" of an r-partition of N by another and conversely. Thus, the procedure (P) gives us a constructive method for obtaining all the dominations [in the sense of (2)] amongst the partitions of an integer $N + 1$ *given* all the dominations amongst the partitions of N. We shall obtain a similar difference equation corresponding to the k-dominations [*cf.* (3)] of the partitions of the pair of integers $(1, k + 1)$; $(2, k + 2)$;... etc. Consider the sequence: $O\ \underline{X\ X \ldots X}$ ($k + 1$ X's), consisting of a 0 followed by $(k + 1)$ X's. We shall not for the moment, think of the O's and X's as heads and tails. This represents a k-domination of the 1-partition of $k + 1$ by the 1-partition, of 1. Applying (P) to this set-up we get $(k + 2)$-sequences;

$O\ O$ XX ... XX (k+2 X's); OXO XX ... XX (k+1 X's);
$O\ X\ XO$ XX ... XX (k X's); ... ...
O XX ... X OXX (k X's); O XX ... X OX
($k + 1$ X's).

The first sequence represents the k-domination of the 1-partition of $(k + 2)$ by the 1-partition of 2 and the remaining $k + 1$ sequences represent all the possible k-dominations of the 2-partitions of $k + 2$ by the 2-partitions of 2, *viz.*, 1, 1. It is easily verified and was proved implicitly in (2), (3) by a geometrical interpretation that (P) applied to the above sequences, will yield all possible k-dominations of the partitions of $k + 3$ by those of 3. The procedure (P) could obtain by recursion all k-dominations of the partitions of $n + k$ by those of n, for all integral n. Section 1 corresponded to the case $k = 0$. The general difference equation for k-dominations using (P) is

$$(N, n, r)^k = (N-1, n-1, r)^k + \sum_{\eta=1}^{N-r+k+1} (N-1, \eta, r-1)^k, \quad (1)$$

where $(N, n, r)^k$ is defined analogously to (N, n, r). We note $(N, n, r)^k = 0$, if $N < n + r - k - 1$.

3. Lemma

The solution of the difference equation (1) is given by

$$\left.\begin{aligned} (N, n, 1) &= \begin{cases} 1 & \text{if } n = N + k \\ 0 & \text{otherwise} \end{cases} \\ (N, n, r)^k &= (N-1)_{(r-1)} (N+k-n-1)_{(r-2)} \\ &\quad - (N+k-1)_{(r-2)} (N-n-1)_{(r-1)} \end{aligned}\right\}. \quad (2)$$

Proof.—Let us define, using a notation similar to (3), Section 4 (*b*), the function $(a, b; t)_r^k$ which represents the number of k-dominations of those r-partitions of b which have their rth partition value equal to t exactly (*i.e.*, $t_r^{(2)} = t$) by the r-partitions of a.

It is evident from the geometrical interpretation or otherwise that $(a, b; t)^k_{r+1} = (a-1, b-t)_r^k + (a-2, b-t)_r^k + \ldots + (b-t-k, b-t)_r^k$.

By an induction on r, we can prove the result

$$(a, b; t)_r^k = (a-1)_{(r-1)} (b-t-1)_{(r-2)} - (a+k-1)_{(r-2)} (b-k-1-t)_{(r-1)} \quad (3)$$

for all given a, b, k, and t.

Setting $a = N$, $b = N + k$, $t = n$ in (3), we have the value of $(N, n, r)^k$ as shown above.

From (2), we have the results:

$$\sum_{r=1}^{N+k+1-n} (N, n, r)^k = (N, n, \Sigma_r)^k = \binom{2N+k-2-n}{N+k-n} - \binom{2N+k-2-n}{N-n-2}, \quad (4)$$

$$\sum_{n=1}^{N+k+1-r} (N, n, r)^k = (N, \Sigma_n, r)^k = (N-1)_{(r-1)} (N+k-1)_{(r-1)} - (N+k-1)_{(r-2)} (N-1)_{(r)}, \quad (5)$$

$$(N, \Sigma_n, \Sigma_r)^k = \frac{k+2}{N+k+1} \binom{2N+k-1}{N-1}, \quad (6)$$

and for all N, k

$$(N, 1, \Sigma_r)^k = \frac{k+2}{N+k}\binom{2N+k-3}{N-2} = (N, 2, \Sigma_r)^k$$

$$= (N-1, \Sigma_n \Sigma_r)^k. \qquad (7)$$

4. Relation to the Game: g_{k+2}

Consider the game g_{k+2} $(k > 0)$ [*cf.* (1)]. The first few sequences of g_{k+2} corresponding to the case of no O's and of exactly 1 0 are:

XX XX	$(k + 2\ X\text{'s})$
O XX XXX	$(k + 3\ X\text{'s})$
$X\,O$ XX ... XX	$(k + 2\ X\text{'s})$
$X\,X\,O$ XX ... XX	$(k + 1\ X\text{'s})$
⋮	
XX ... XX $O\,X\,X$	$(k + 1\ X\text{'s})$.

It will be noticed that a procedure very similar to (P) can be used to generate recursively the sequences of g_{k+2}. For any sequence of g_{k+2}, let

N' = (number of zeros in the sequence) + 1,

n' = (number of X's in the block terminating the sequence) $- 1$,

r' = $l + 1$, where l represents the number of XO's in the sequence.

If $(N', n', r')^k$ represents the number of sequences in g_{k+2} for given values of N', n', r' then $(N', n', r')^k$ satisfies the same difference equation (1). It has the same solution.

It was proved [*cf.* (1)] that the games g_{k+2} are equivalent to the " problème du scrutin " or returns to equilibrium in coin tossing. The results (4), (5), (6) and (7) can be used to obtain more information about the g-games or the problème du scrutin, paying due attention to the slight differences in the definition of N, n, r and N', n', r'.

5. Relation to the Game G_{k+2}: $(k > 0)$ [*cf.* (3)]

We remark finally that every sequence of g_{k+2} can be rearranged into a sequence of G_{k+2} and conversely. In fact, the two games are identical as *sequences*, except that the probabilities are more complicated in the case of G_{k+2}.

Consider a sequence of G_{k+2} containing $2m + k + 2$ observations. (*i.e.*, m O's and $m + k + 2$ X's) belonging to the series S_u. The base sequences of S_u are of length $k + 2u + 2t$ $(t = 1, 2, \ldots, u + 1)$ and the number of base sequences of S_u of length $k + 2u + 2t$ is by Theorem 2, [*cf.* (3)]

$$u_{(t-1)} (k + u - 1)_{(t-1)} - (k + u - 1)_{(t-2)} u_{(t)}.$$

As every sequence of G_{k+2} is generated from a base sequence [*cf.* (3)], the total number of sequences of G_{k+2} in series S_u having $2m + k + 2$ observations is the number of ways of putting $m - u + 1 - t$ balls in $k + 2u + 1$ boxes $(t = 1, 2, \ldots, u + 1)$ of a corresponding base sequence of length $k + 2u + 2t$ $(t = 1, 2, \ldots, u + 1)$. Hence the total number of G_{k+2} sequences in S_u having $2m + k + 2$ observations is

$$\sum_{t=1}^{u+1} \binom{m + k + u + 1 - t}{2u + k} \{u_{(t-1)} (k + u - 1)_{(t-1)} - (k + u - 1)_{(t-2)} u_{(t)}\}.$$

However, the number of g_{k+2} sequences with m 0's and l XO's in it, is from Section 4 (note definitions of N', n', r')

$$(m+1, \Sigma_n, l+1)^k = m_{(l)} (m+k)_{(l)} - (m+k)_{(l-1)} m_{(l+1)}.$$

But every g_{k+2} sequence can be deformed into a G_{k+2} sequence and conversely. In changing a g_{k+2} sequence of $2m + k + 2$ observations containing l XO's exactly into a G_{k+2} sequence, we note that of the m-0's in g_{k+2}, l will fall on the bottom line [or l 0's are obtained with coin 2 *cf.* (1) (3)]. Hence this g_{k+2} sequence of $2m + k + 2$ observations containing exactly l XO's when transformed into a G_{k+2} sequence will belong to S_{m-l}. Setting $m - 1 = u$ or $l = m - u$,

$$(m + 1, \Sigma_n, m - u + l)^k = m_{(u)} (m + k)_{(k+u)} - (m + k)_{(k+u+1)} m_{(u-1)}$$

represents the number of g_{k+2}-series, falling in S_u when transformed to a G_{k+2}-series. Equating these two for the number of G_{k+2}-series having $2m + k + 2$ observations in S_u, we have the identity,

$$m_{(u)} (m + k)_{(k+u)} - (m + k)_{(k+u+1)} m_{(u-1)} = \sum_{t=1}^{u+1} \binom{m + k + u + 1 - t}{2u + k} \{u_{(t-1)} (k + u - 1)_{(t-1)} - (k+u-1)_{(t-2)} u_{(t)}\} \text{ for } k \geqslant 0, u \geqslant 1 \text{ and } m \geqslant 1.$$

The case $k = 0$ is of special interest. With a change of notation, we have the identity,

$$\frac{{}^nC_r\, {}^nC_{r+1}}{n} = \frac{1}{r} \left({}^rC_1\, {}^rC_0\, {}^{n+r-1}C_{2r} + {}^rC_2\, {}^rC_1\, {}^{n+r-2}C_{2r} + \ldots + {}^rC_r\, {}^rC_{r-1}\, {}^nC_{2r}\right).$$

This identity explains why in the game G_2, the table of basic patterns for $n = 1, 2, 3, \ldots$, is identical with the table of G_2-sequences, falling in series $S_0, S_1, S_2, \ldots$. These tables were prepared by the junior author at the Indian Council of Agricultural Research.

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