A MODEL FOR RANK ANALYSIS IN TRIAD COMPARISONS

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1. Introduction

The analysis of data involving ranking has received considerable attention in statistical and psychological methodologies. In psychology emphasis is given to the problem of scaling and in discipline of statistics, effort is made on testing and developing different models of analysis.

Pendergrass and Bradley (1960) have proposed a model for analysing rank in triple comparisons. Rai (1971) has developed a method for the analysis of data involving ranking in fractional triad comparisons. In the present paper, we shall formulate a model for rank analysis in triad comparisons, as an extension of the Bradley-Terry model for paired comparisons. A mathematical model involving treatment parameters has been proposed and test procedure has been developed. The method of estimation of treatment parameters and investigation regarding properties of the model have been discussed.

2. Mathematical Model

The model for triad comparisons has been obtained as an extension of Bradley-Terry model for paired comparisons. In paired comparisons, the existence of non-negative parameters $\pi_1,\ldots,\ \pi_t$ associated with t treatments $T_1,\ldots,\ T_t$ is postulated such that

$$\Sigma\ \pi_i = 1 \qquad (1)$$

and the probability that T_i is preferred over T_j is

$$P(T_i > T_j) = \pi_i/(\pi_i + \pi_j)\ ;\ i \neq j;\ i, j = 1,2,\ldots,\ t \qquad (2)$$

Probabilities associated with pairs of treatments are taken to be independent. When three items are compared together in triad comparisons the probability that $T_i > T_j > T_k$ is taken as

$$P\,(T_i > T_j > T_k) = \frac{\pi_i^2\,\pi_j}{(\pi_i + \pi_j)\,(\pi_i + \pi_k)\,(\pi_j + \pi_k)} \tag{3}$$

Here we retain the concept of non-negative parameters $\pi_1, \ldots, \pi_t$ associated with $T_1, \ldots, T_t$

and

$$\sum_{i=1}^{t} \pi_i = 1.$$

In a triad comparison consisting of

$$T_i;\; T_j \text{ and } T_k,$$

the six inequalities can be obtained:

$$T_i > T_j > T_k;\quad T_i > T_k > T_j;$$
$$T_j > T_i > T_k;\quad T_j > T_k > T_i;$$
$$T_k > T_i > T_j \text{ and } T_k > T_j > T_i$$

The probability for each case can be obtained from (3) and the sum of all the six probabilities is observed to be one.

We shall develop main results for experiments with n repetitions on all possible triplets formed by $T_1, \ldots, T_t$ objects. The total number of triplets formed out of all the t objects will be $\binom{t}{3}$. The members of each of $\binom{t}{3}$ triplplets will be ranked in order of acceptability. In a triplet the best treatment will be given rank 1, the second one rank 2 and the third will have rank 3.

In triplets having treatments

$$T_i,\; T_j \text{ and } T_k\; (i \neq j \neq k),$$

we have

$$P\,(T_i > T_j > T_k) = \pi_i^2\,\pi_j / \Delta_{ijk}$$

where

$$P\,(T_i > T_j > T_k)$$

represents the probability that treatment T_i is rated top, T_j central and T_k bottom and

$$\Delta_{ijk} = (\pi_i + \pi_j)\,(\pi_i + \pi_k)\,(\pi_j + \pi_k)$$

3. THE LIKELIHOOD FUNCTION

The likelihood function is obtained on the assumption of the probability independence for different triplets and for different replications. The rank of T_i, T_j and T_k in the pth comparisons will be denoted by $r_{ip,jk}$; $r_{jp,ik}$ and $r_{kp,ij}$ respectively where $p=1,\ldots,n$. The tied ranks are not permitted in the model. The probability of a specified ranking in the pth repetitions is given by

$$\pi_i^{3-r_{ip,jk}}\ \pi_j^{3-r_{jp,ik}}\ \pi_k^{3-r_{kp,ij}}/\Delta_{ijk} \tag{4}$$

Because if T_i obtained the top rank T_j as second and T_k as third, then $r_{ip,jk}=1$, $r_{jp,ik}=2$ and $r_{kp,ij}=3$ and the expression (4) takes the form $\pi_i^2\,\pi_j/\Delta_{ijk}$. Similarly if T_j is ranked as first T_i as second and T_k as third then (4) becomes $\pi_j^2\,\pi_i/\Delta_{ijk}$ and so on. Multiplying the appropriate expression for all comparisons within a repetition and for all n replications, we obtain the likelihood function as given below :—

$$L=\frac{\prod_{i=1}^{t}\pi_i^{\frac{3n}{2}(t-1)(t-2)-\sum_{p=1}^{n}\sum_{j<k=1}^{t} r_{ip,jk}}}{\prod_{i<j<k}^{t}\Delta_{ijk}^{\,n}} \tag{5}$$

When the repetitions of the design is performed by groups with distinct parameters, the likelihood function will be product over the groups of functions of the form (4).

4. LIKELIHOOD RATIO TESTS AND ESTIMATION

We can apply the method of maximum likelihood to obtain the estimators $p_1\ldots,p_t$ of $\pi_1,\ldots,\pi_t$. The significance of equality of treatment effects can also be tested. Consider the hypothesis :

$$H_o:\qquad \pi_1=\pi_2=\ldots=\pi_t=\frac{1}{t}$$

against the alternative :

$$H_1:\qquad \pi_i\neq\pi_j\qquad \text{for some}\qquad i\neq j\,;\ i,j=1,\ldots,t.$$

The maximum likelihood estimators $p_1,\ldots,p_t$ of $\pi_1,\ldots,\pi_t$ are obtained by maximising log L with respect to $\pi_1,\ldots,\pi_t$ subject to the condition that $\sum_{i=1}^{t}\pi_i=1$. These values of the parameters maximise the likelihood function L. The resulting normal equations after minor simplifications are given by

$$\frac{\frac{3n}{2}(t-1)(t-2)-\Sigma\,\Sigma\, rip, jk}{p_i} = n \sum_{i<k} \frac{(p_j+p_k)(2p_i+p_j+p_k)}{D_{ijk}} \quad (6)$$

$$i=1,\ldots,t$$

where

$$D_{ijk}=(p_i+p_j)(p_i+p_k)(p_j+p_k)$$

This equation together with $\Sigma p_i=1$ yield the solutions for $p_1, \ldots, p_t$.

The normal equations given in (6) can be solved by iterative methods. The iteration proceeds as follows :—

Let $p_1^{(0)}, \ldots, p_t^{(0)}$ be first trial values for $p_1, \ldots, p_t$. Second trial values are obtained by putting the first trial values in the following equations:

$$\frac{C}{p_i^{(1)}}=n \sum_{j<k} \frac{\{p_j^{(0)}+p_k^{(0)}\}\{2pi^{(0)}+p_j^{(0)}+p_k^{(0)}\}}{D_{ijk}^{(0)}} \quad (7)$$

$$i=1, \ldots, t$$

where C is eliminated through the assumption that $\Sigma pi=1$ and $D_{ijk}^{(0)}$ is the value of D_{ijk} evaluated by using $p_1^{(0)}, \ldots, p_t^{(0)}$. The above procedure is continued until the process coverges to the required accuracy. The method is readily adoptable and the rapidity of convergence is good if the initial values are good. The values of p_i in the initial trial are taken in proportion to

$$\sum_{i=2}^{t} r_i : r_1+\sum_{i=3}^{t} r_i : \ldots\ldots : \sum_{i=1}^{t-1} r_i$$

where $r_1, r_2, \ldots, r_t$ are the sums of ranks for treatments $T_1, T_i, \ldots, T_t$ respectively over all repetitions. In many cases these values are good first approximations (Rai, 1971 and Sadasivan and Rai. 1973). Sometimes extreme sets of values of sums of ranks occur. This happens when $T_1, \ldots, T_t$ have a sub-set that always outranks the complementary sub-set. In case of extreme values of ranks where a particular treatment (say T_j) is always given the rank 1 in all the comparisons, the corresponding value of p_j is taken as 1. Similarly when a particular treatment is always rated as third in all the comparisons, the corresponding values of p for this treatment is taken as zero.

Now the estimates of $\pi_1, \ldots, \pi_t$ are obtained under the hypothesis H_1. The likelihood function L given by (5) is used to obtain the likelihood ratio λ and Z which is given by

$$Z=-2 \log_e \lambda$$

Therefore

$$Z=2n \ \binom{t}{3}\log_e^{8}+2 \sum_{i=1}^{t} a_i \log_e p_i - 2n \sum_{i<j<k} \log_e D_{ijk} \qquad (8)$$

where

$$a_i = \frac{3n}{2}(t-1)\ (t-2)-\sum_{p=1}^{n}\sum_{j<k}^{t} rip,\ jk$$

For large n, Z may be taken to have the Chi-square distribution with $(t-1)$ degrees of freedom under the null hypothesis H_o.

Small sample tables for the distribution of Z given H_0 may be developed but these will be extremely laborious and voluminous. The procedure for developing such tables are similar to one given by Rai (1971) and Sadasivan and Rai (1973)

5. Some Generalisations on Estimation

For paired comparisons, Bradley and Terry proposed a general model in which the treatments might be grouped so that

$$\pi_i=\pi(b);\ b=1, \ldots, m \text{ and } i=S_{b-1}+1, \ldots, S_b$$

Where

$$S_o=1,\ S_m=t$$

and

$$\sum_{b=1}^{m} (S_b-S_{b-1})\ \pi(b)=1$$

This technique of grouping may also be done for triad comparisons. This simply involves substitution of $\pi(b)$ in (5) in the place of π_i at appropriate places and maximisation subject to the new restraint mentioned above. The maximum likelihood estimators $p(b)$ of $\pi(b)$ may be obtained.

An other generalisation is also possible in triple comparisons. Here we consider t distinct treatments but use n_{ijk} observations on the triplet T_i, T_j, T_k; $i \neq j \neq k$, $i, j, k = 1, \ldots, t$

In this case the normal equations given by (6) takes the following form.

$$\frac{a_i}{p_i} \sum_{\substack{j,\, k \neq i \\ j<k}}^{t} \frac{n_{ijk}\,(p_j+p_k)\,(2p_i+p_j+p_k)}{D_{ijk}} \tag{9}$$

for

$$i=1, \ldots, t$$

These equations together with $\Sigma\ p_i = 1$ give the solutions for p_i.

6. Combination of Results

Sometimes the ranking experiments may be completed in groups of repetitions by various judges at different times or under different circumstances. The experiment may be considered as one with groups of repetitions, the uth of which has n_u repetitions. Here $n = \sum_{u=1}^{g} n_u$. The difference between the treatment parameters represents a group $\times$ treatment interaction. For detecting such interaction let us consider,

$$H_o : \pi_{iu} = 1/t$$

for all i and u

and

$$H_a : \pi_{iu} \neq 1/t$$

for some i and u

Then

$$Z_c = -2 \log_e \lambda_c = \sum_{u=1}^{g} Z_u \tag{10}$$

where λ_c is the likelihood ratio, and Z_u is the value of Z given by (8) computed for the uth group. Asymptotically with the n_u, Z_c has the x^2 distribution with $g\ (t-1)$ degrees of freedom under H_o. The likelihood ratio test of interaction depends on $Z_c - Z$ where Z_c is definded in (10) and Z in (8) based on pooling the totality of the repetitions. For large values of n_u, $Z_c - Z$ has the Chi-square distribution with $(g-1)(t-1)$ degrees of freedom. The procedures of computations are clear. For obtaining the value of Z_u, $p_{1u}, \ldots, p_{tu}$ are obtained as estimates of $\pi_{1u}, \ldots, \pi_{tu}$ through consideration of only the uth group. The value of Z is computed from the values of $p_1, \ldots, p_t$ which are the estimates of $\pi_1, \ldots, \pi_t$ on the assumption that all groups of repetitions may be pooled in to a single group.

7. Appropriateness of the Model

In statistical methodology it is essential that means be available to test the appropriateness of the model on which the method is based. In triple comparisons we postulate the existence of positive parameters $\pi_{ijk}, \ldots, \pi_{kji}$ Six in number of each triplet corresponding to the probabilities of occurrence of the six possible rankings of T_i, T_j and T_k. Here π_{ijk} indicates the probability that T_i, T_j and T_k receive ranks 1, 2 and 3 respectively in a triplet.

The sum of six parameters corresponding to each triplet is unity and their maximum likelihood estimators $\frac{f_{ijk}}{n}, \ldots, \frac{f_{kji}}{n}$ for the n comparisons of this triplet where f_{ijk} is the number of times of ranking 1, 2 and 3 for T_i, T_j and T_k respectively occurs in n triplets.

The model for triple comparison implies that

$$H_o : \pi_{ijk} = \pi_i^2 \, \pi_j / \triangle_{ijk} \, ;$$

$$i \neq j \neq k \, ; \; i, j, k = 1, \ldots, t$$

against the alternative

$$H_a : \pi_{ijk} \neq \pi_i^2 \, \pi_j / \triangle_{ijk}$$

for some

$$i, j, k.$$

The general likelihood function for triple comparisons is given by

$$L(\pi_{ijk}) = \underset{i<j<k}{\pi} \; \pi_{ijk}^{\,fijk} \tag{11}$$

Under H_o, this likelihood function reduces to the likelihood function given in (5). The likelihood ratio statistic for testing H_o against the alternative H_a is

$$-2 \log_e \lambda = 2\Big[\sum_{i<j<k} f_{ijk} \log f_{ijk} - n \tbinom{t}{3} \log n + n \sum_{i<j<k} \log D_{ijk} - \sum_i a_i \log p_i \Big] \quad (12)$$

This test constitutes the test of the model for triple comparisons and for large n, $-2 \log \lambda$ has Chi-square distribution with $[5\binom{t}{3} - (t-1)]$ degrees of freedom.

Let us define f'_{ijk} as the expected frequency corresponding to the observed frequency f_{ijk}, then the estimates of the expected frequencies is given by

$$f'_{ijk} = np_i^2 \, p_j / D_{ijk} \quad (13)$$

The likelihood ratio statistic for testing H_o in terms of observed and expected frequencies is given by

$$-2 \log \lambda = 2 \sum_{i<j<k} f_{ijk} \log [f_{ijk}/f'_{ijk}] \quad (14)$$

Now in equation (14) take

$$f_{ijk}/f'_{ijk} = 1 + e_{ijk}$$

where e_{ijk} may have either positive or negative values.

Then

$$-2 \log \lambda = 2 \sum_{i<j<k} f'_{ijk} (1 + e_{ijk}) \log (1 + e_{ijk})$$

Expanding the logarithmic series in powers of e_{ijk} and ignoring the higher power of e_{ijk}, we have

$$-2 \log \lambda \approx 2 \sum f'_{ijk} (1 + e_{ijk}) (e_{ijk} - e^2_{ijk}/2) \quad (15)$$

We notice that

$$\sum f'_{ijk} \, e_{ijk} = 0$$

and (15) takes the form

$$-2 \log \lambda \approx \sum f'_{ijk} \, e^2_{ijk}$$

After putting the value of e_{ijk} we have the final result in the following form

$$-2 \log \lambda \approx \sum (f_{ijk} - f'_{ijk})^2 / f'_{ijk} \quad (16)$$

Thus the statistic $-2 \log \lambda$ is transformed to the usual x^2 test of goodness of fit.

8. An Illustrative Example

In order to illustrate some of the procedures developed, we include a numerical example with $t=4$ and $n=40$. The data are given below in table-1.

Table No. 1

Frequencies of rankings with $t=4$ and $n=40$

$f_{123}=8$	$f_{124}=10$	$f_{134}=8$	$f_{234}=6$
$f_{132}=12$	$f_{142}=10$	$f_{143}=10$	$f_{243}=6$
$f_{213}=6$	$f_{214}=8$	$f_{341}=8$	$f_{342}=8$
$f_{231}=4$	$f_{241}=4$	$f_{314}=6$	$f_{324}=6$
$f_{312}=5$	$f_{412}=4$	$f_{413}=4$	$f_{423}=8$
$f_{321}=5$	$f_{421}=4$	$f_{431}=4$	$f_{432}=6$

From the above table we obtain the following preference matrix:

Table No. 2

Preference matrix and sum of ranks

Treatment Nos.	*Number of times ranked as* First	Second	Third	*Sums of ranks* Σr_i	a_i
1.	58	33	29	211	149
2.	34	41	45	251	109
3.	38	40	42	244	116
4.	30	46	44	254	106

We now obtain the value of p_1, p_2, p_3 and p_4. Successive approximations of these values along with the value of Z are presented below in table No. 3.

Table No. 3

Successive approximations to $p_1 \ldots, p_4$ and corresponding value of Z

Approximations	p_1	p_2	p_3	p_4	Z
1.	.261	.246	.248	.245	3.28
2.	.255	.249	.250	.246	4.41
3.	.254	.249	.250	.247	4.43

The successive approximations show lhe convergence of the estimates of $p_1, ..., p_4$ and of Z values. The final Z taken as x^2 with 3 degrees of freedom indicates that treatment main effects do not differ significantly from each other. We cannot illustrate the test of interaction as the data provided were not grouped.

The values of expected frequencies are obtained by using (13) and the goodness of fit test may be applied for testing the appropriateness of the model. The use of form (16) yield the value of $-2 \log \lambda = 8.3$ and this is taken as the value of x^2 with 17 degrees of freedom. The above valueindicates that the proposed model is quite appropriate for these data.

9. Discussion and Summary

A method of analysis of experiments involving ranking in triple comparisons is discussed which permits tests of hypotheses of general class and the estimation of treatment ratings or preferences. We assume, in the null hypothesis, that the treatment ratings are equal where as the alternative hypothesis does not make any assumption regarding the equality of treatment preference. The likelihood ratio test has been developed for testing the main effects. A test of interaction has also been obtained when the ranking experiments are completed in different groups or by different judges. A test has also been proposed for testing the appropriateness of the model of the triple comparisons. Some of the procedures developed in this paper, have been illustrated through numerical examples.

REFERENCES

1. Bradley, R.A. (1954) : Incomplete block rank analysis : on the arppropriateness of the model for a method of paired comparisons. Biometrics, **10**, 375-390.

2. Bradley, R.A. and Terry, M.E. (1952) : The rank analysis of incomplete Block Designs-I. The method of paired comparisons. Biometrika 39, 324-345.

3. Pendergrass, R.C. and Bradley, R.A. (1960) : Ranking in triple comparisons. Contribution to Probability and Statistics. Edited by I. Olkin, et al., Stanford University Press, Stanford, California. 331-351.

4. Rai, S.C. (1971) : Ranking in fractional triad comparisons. Jl. Ind. Soc. Agr. Stat. 23, 52-61.

5. Sadasivan, G. and Rai S.C. (1973) : A Bradley-Terry model for standard comparison Pairs. Sankhya, Vol. 35, 25-34.