A FURTHER STUDY ON THE USE OF POPULATION GENERATION MATRIX IN DAIRY HERDS

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1. Introduction

Jain and Narain [3] discussed the method and use of population generation matrix in the study of growth of female population of dairy herds grouped in unequal age-intervals. Since the cost of operating any animal breeding programme and genetic advance occurring as a result of selection depend not only on the female herd strength but of males as well, it is necessary to have a clear idea of the pattern of growth in both male and female herd strengths at periodic intervals. In the present paper, joint generation matrix for the two-sexes has been presented and its properties studied. Further, explicit formulae for working out the elements of matrices corresponding to a particular breeding plan for progeny testing in dairy herds in terms of specific schedule of vital characteristics and culling rates have been derived.

2. An Outline of the Breeding Programme

Since the systematic plan of progeny testing outlined by Jain *et al.* [2] leads to the maximum possible genetic advance, the pattern of growth and structure of the breeding population have, therefore, been studied under the constraints of that plan. The relevant constraints are reproduced here :

(*i*) At any time u bulls are in use, of which v best proven bulls are used on superior cows for securing males for future breeding and the remaining w, which are sons of v best proven bulls of an earlier set, are used on other cows.

(*ii*) In addition to u bulls, v' ($\leqslant v$) tested bulls and w' ($\leqslant w$) untested young bulls are kept as standby.

(*iii*) Each set of young bulls is in use for one calving interval of m months.

(*iv*) The v best proven bulls selected among those under test are used for another m months for securing males.

(*v*) A representative set of equal number of females are allotted to each bull.

(*vi*) No male progeny is kept from the bulls under test.

(*vii*) Of the male calves born to the tested bulls, equal number, say a, from each (*i.e.* av per set) and coming from high yielding dams are retained and others disposed of at one year of age. The number of sons from each tested bull is so chosen that at the time of their use there are $(w+w')$ of them remaining after allowing for mortality and involuntary culling during the intervening period. The reason for retaining all the young male calves upto one year is obvious in unweaned herds and, at the same time, will ensure selection on current lactation performance.

(*viii*) All the young bulls under test after first use and one among the reserves to allow for mortality are to be maintained apart till the completion of their test. At this stage, $(v+v')$ best sires are retained, v of them for use on best yielding cows for securing a set of young males for further breeding and the remaining v' serving as reserves.

(*ix*) At the time of initiating a progeny testing programme, normally tested bulls will not be available. As such initially a slightly different programme than the one outlined in the foregoing is required to be followed till such time the scheme generates its own quota of desired type of young and tested bulls. Instead of starting with u bulls, w young bulls per batch are tested at intervals of 'm' months and v of them selected from each batch on the basis of their progeny tests. This initial plan can either be continued upto the time the testing of first set of bulls has been completed or till the sons of first set of bulls can be pressed for service. In the former case, d sets of w young bulls where md equals the testing period will come from outside at intervals of m months and thereafter the herd would be closed. Since the sons born to first set of selected proven v bulls mated to the best dams at the $(d+1)^{th}$ round of mating will become available for use only after another two rounds of mating (Section 3.1), the next three sets each of w young bulls would therefore be the sons of v best proven bulls of the first three sets respectively. This would necessitate retaining $a.w$ young male calves per set ('a' calves from each bull and coming from high yielding dams) from first three sets of bulls till the testing of their sires is completed and by this time they would be quite advanced in age. As per the alternative plan, instead of d sets only three sets of w young bulls will come from outside as by the time the females come up for 4th round of mating the sons of first set of bulls would become available for use. Further, since the sons of best proven bulls become available from $(d+4)^{th}$ round of mating, the next d rounds of mating (4 to $3+d$) would be covered by the sons of first d sets of w young bulls selected solely on the basis of their dams performance. In what follows the latter strategy has been adopted because of the cost and operational convenience. A pictorial representation of typical scheme under Indian condition is given in Fig. 1.

3. Grouping of Male Stock

3.1. Assumptions and Limitations in Approach

(*i*) The time is measured in units of m months *i.e.* one calving interval.

(*ii*) The females in any stage group at time t are uniformly distributed over the group interval. That is, if we conceive n_{it} females at time t in stage group i of m months duration as the aggregation of m 'sets' of n_{it}/m females each, then these sets of females are uniformly distributed over the 'm' discrete points in which this stage group is assumed to have been divided.

(*iii*) Individual variation in the duration of different stage groups is ignored.

(*iv*) Multiple births are excluded.

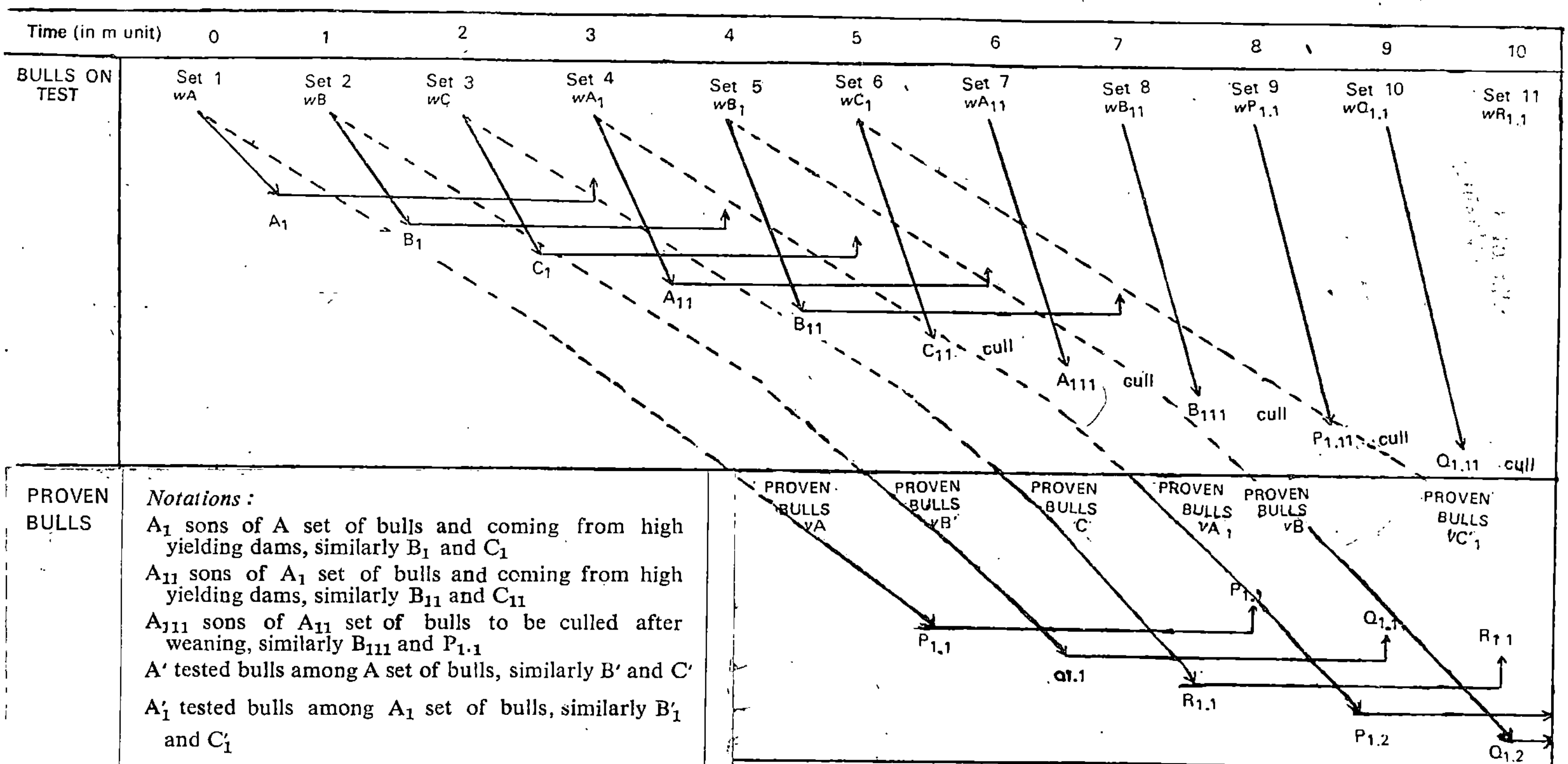
(*v*) Birth and deaths are independent.

(*vi*) The range of various parameters is restricted as :

$$12 \leqslant m \leqslant l+g \; ; \qquad o \leqslant k \leqslant m \; ;$$

$$9 \leqslant g \leqslant 10 \quad ; \quad \text{and} \quad q = 2m-g-12,$$

where l, g and m are the lactation length, gestation period and calving interval in months respectively. $24+k$ is the



Notations :

A_1 sons of A set of bulls and coming from high yielding dams, similarly B_1 and C_1

A_{11} sons of A_1 set of bulls and coming from high yielding dams, similarly B_{11} and C_{11}

A_{111} sons of A_{11} set of bulls to be culled after weaning, similarly B_{111} and $P_{1\cdot 1}$

A' tested bulls among A set of bulls, similarly B' and C'

A'_1 tested bulls among A_1 set of bulls, similarly B'_1 and C'_1

$P_{1\cdot 1}$ sons of A' and coming from high yielding dams, similarly $Q_{1\cdot 1}$ and $R_{1\cdot 1}$

$P_{1\cdot 2}$ sons of A'_1 and coming from high yielding dams, similary $Q_{1\cdot 2}$.

Fig. 1. Plan for testing period d=5 units of time.

age at first service of females and $12+q$ ($q \geqslant 0$) is the earliest age at which a bull is used for breeding. The implications of first three restrictions have been described in the earlier paper (Jain and Narain, 1974) and the last equality restriction is explained as follows. Since the male calves of a particular set of tested bulls will be born over one calving interval, their use will therefore have to be deferred till the youngest among these is $12+q$ months old. But just prior to taking off this set of bulls the youngest batch of sons from the bulls used in the previous set will be only $(m-g)$ months old which is very much less than $12+q$ and hence cannot be used for breeding. The fresh set of young bulls will be the sons of tested bulls of the set previous to the immediately preceding one, the youngest batch of their sons having attained the age of $2m-g$ months. Consequently, if these young bulls are not to remain idle after attaining maturity $2m$-g must be equal to $12+q$ or $q=2m-g-12$. Also if the best proven bulls are not to wait servicing any more after their final selection for securing males for future breeding, the testing period must be an exact multiple of m *i.e.* $[g+(24+k)+(g+l+m)]=0$ mod (m). In most situations this can be achieved by reducing this period by basing evaluation of sires on past records for some of the daughters (Searle, [5]; Van Vleck and Henderson, [6], [7]) and, if need be, by adjusting the value of l slightly on the lower side, say yield in 300 or 270 days. This situation has been assumed in the present study.

3.2. Grouping of Males

In the light of the constraints of the breeding plan and the foregoing considerations, the male stock is grouped in the following $4+d$ $(=r$, say) distinct stage groups :

Stage group	*Duration in months*
1. Youngstock between 0 to 12 months of age	12
2. Youngstock between the ages 12 to $12+q$ *i.e.* till they attain maturity	q
3. Waiting stage till the youngest batch of sons of a particular set of bulls attain maturity	m
4. Untested young males in use (including reserves)	m
5. to $5+j$ ($j=1, 2, \ldots, \overline{d-2}$) : Untested adult males maintained apart	m
$4+d$. Best proven bulls in use for securing males (including reserves)	m

The age differences among males in the first three stage groups is taken into consideration alongwith differential mortality rates in studying the movement from one stage group to another. However, this age difference is ignored from fourth stage group onwards since the reference is shifted to the time of use for breeding and since all in a set are used at the same point of time.

3.3. Grouping of Females

The female stock is assumed to be grouped in s distinct stage groups in the manner described by Jain and Narain [3].

4. The Deterministic Population Model

Let, $\underline{K_t}=\begin{bmatrix}\underline{m_t}\\ \underline{n_t}\end{bmatrix}$, be the column vector of $(r+s)$ elements at time t, with $\underline{m_t}$ and $\underline{n_t}$ as the respective vectors of male and female populations. The numbers in various stage groups at time $t+1$ can be obtained by the operation $H.\underline{K_t}$, where H is a $(r+s)\times(r+s)$ matrix embodying the regime of mortality, fertility and culling rates supposed to apply over the interval, and is of the form

$$H=\begin{bmatrix} T & B \\ O & F \end{bmatrix} \qquad \ldots(4.1)$$

In (4.1), T is $(r\times r)$ for males and by the row-by-column multiplication rule T will transfer the male stock into the succeeding stage groups; B is $(r\times s)$ also for males and the elements of which when multiplied by the column vector of females will give the number of males entering a particular stage-group through births; and F is $(s\times s)$ for females as defined in the earlier paper (Jain and Narain, [3]). For the type of grouping considered, the non-zero elements of T (c_{ij}) and B (b_{ij}) will be as follows as will become apparent from Section 6.4 :

T : c_{31} and $c_{i+1,i}$ $(i=1, 2, \ldots\ldots, r-1)$

B : b_{ij} $(i=1$ and 2, and $j=4, 5, \ldots\ldots, s)$

Thus the composition of herd strength at time $t+1$ can be written as :

$$\begin{bmatrix}\underline{m_{t+1}}\\ \underline{n_{t+1}}\end{bmatrix}=\begin{bmatrix} T & B \\ O & F \end{bmatrix}\begin{bmatrix}\underline{m_t}\\ \underline{n_t}\end{bmatrix} \qquad \ldots(4.2)$$

This leads to the following relations for $\underline{m}_t$ and $\underline{n}_t$ on the assumption of time homogeneity of the process :

$$\left.\begin{array}{l} \underline{m}_t = T^t\,\underline{m}_0 + \sum_{j=1}^{t} T^{t-j}\,\bar{B}\,\underline{n}_{j-1} \\ \underline{n}_t = F^t\,\underline{n}_0 \end{array}\right] \qquad \ldots(4.3)$$

5. Asymptotic Behaviour of the Population

To determine the asymptotic behaviour of $\underline{m}_t$ and $\underline{n}_t$, we need the dominant latent root of matrix H. This matrix has characteristic equation

$$\begin{vmatrix} T-\omega I & B \\ 0 & F-\omega I \end{vmatrix} = 0 \qquad \ldots(5.1)$$

which is equivalent to $|T-\omega I|\;|F-\omega I| = 0$. Hence the matrix has as its latent roots all the roots of T together with all the latent roots of F. Hence the $(r+s)$ roots of H are r zeroes together with the s roots of F, the matrix that projects the female population. Since F is a square matrix with non-negative elements and is such that all the elements of F^t are positive for some integer t, it has a positive latent root ω_f of algebraic multiplicity one. This latent root is greater in absolute size than any other latent root of F. Therefore, the asymptotic behaviour of the population will depend upon ω_f. In the notation of the previous papers $\omega_f = \omega_1$.

The results concerning the asymptotic behaviour of the population as shown by Jain (1975) are as follows :

$$\underline{m}_t/\omega_f^t = (\underline{V}'\underline{n}_0))\,\{\omega_f\, I - T\}^{-1}\, B\underline{X} + O\,(1)$$

$$\underline{n}_t/\omega_f^t = (\underline{V}'\underline{n}_0)\,\underline{X} + O\,(1) \qquad \ldots(5.2)$$

where $\underline{X}$ and $\underline{V}'$ are the right and left hand latent vectors of F corresponding to ω_f and $\underline{V}'\,\underline{X} = 1$.

The total size of the population will be changing at a rate determined by ω_f. When ω_f is greater than 1, the population is increasing, when it is less than 1, the population is decreasing ; when $\omega_f = 1$, the population is stable in size.

In situations where lengths of some of the stage groupings exceed one unit interval of time, additional elements will occur along the main diagonal of matrix T in the rows corresponding to such stage groupings. The form of T would then be a triangular one with its latent roots as C_{ii}'s, $i=1, 2, \ldots, r$, some of these being zeroes. Let the largest root among these be denoted by ω_m which may be $\lesseqgtr \omega_f$. But since the number of males required for breeding is very much restricted as compared to females and as also the male births are tied to the female population, the matrix H is female dominant. This means that the dominant root of F is also the dominant root of H, and therefore $\omega_m < \omega_f$. The asymptotic behaviour of the population even when duration of some of the stage groupings exceed one unit of time would still be given by equations (5.2).

6. Derivation of the Elements of Matrices T and B for Males

6.1. Notations

To distinguish between the two sets of parameters for males and females, subscript 'm' is added to denote these for males. The parameters for females are as defined in the earlier paper.

- μ_{im} : monthly mortality rate in the i-th stage group, for $i=1$, 2 and 3 ;
- λ_{im} : the corresponding involuntary culling rate ;
- μ'_{im} : mortality rate over the entire duration of ith stage group, for $i=1$, 2 and 3 ;
- λ'_{im} : the corresponding involuntary culling rate :
- μ_m : monthly mortality rate in the stage groups 4 to r ;
- λ_m : the corresponding involuntary culling rate :
- μ'_m : annual adult mortality rate ;
- λ'_m : the corresponding involuntary culling rate ;
- f : infertility rate ;
- v : rate of abnormal calvings including abortion, still births and premature calvings ; and
- r : proportion of female births.

(r and v have earlier been used to denote respectively the number of male stage-groupings and the number of best proven bulls. Their double use will not cause confusion as their meaning will be clear from the context in which the symbols are used in the text).

6.2. Lemma

In addition to the results of five lemmas of the previous paper [3] the following result will also be used :

The average number of male calves born every month during $(t,\ t+1)$ of m months after allowing for mortality, infertility, abnormal calvings and involuntary cullings in different stage groups is given by

$$\bar{O}_{mt}=p_b\left[\frac{n_{4t}}{l+g}+\sum_{i=5}^{s}\frac{n_{it}}{m}\right]$$

where $$p_b=\left(1-\frac{m}{2}\mu\right)\left(1-\frac{m}{2}\lambda\right)\left(1-f\right)\left(1-v\right)\left(1-r\right),$$

and n_{it} is the number of females in the i-th stage group at time t.

Proof: On the same lines as for Lemma 5 of the previous paper (Jain and Narain, [3]).

6.3. Proportion of Males Retained among those completing one year of Age

In the light of considerations (*viii*) and (*ix*) of Section 2 we define two paramaters $p^{(1)}$ and $p^{(2)}$ corresponding to the two schemes of selection in respect of the sons born to first d sets of bulls and in respect of those born to bulls used thereafter.

The breeding plan envisages more or less constant number of females coming up for mating every calving interval, say N_f^* ,

where $$N_f^*=\left(\frac{m}{l+g}\right)n_4^*+\sum_{i=5}^{s}n_i^*$$

n_i^*'s are the number of females in different stage groups when the population has attained stability in size and can be determined from the relation

$$n_i^*=\left(\frac{N_o}{\Sigma l_i}\right)l_i \qquad (i=1,\ 2,\ \ldots\ s)$$

where l_i' are the elements of the right hand eigen vector of the equation

$$\underline{F}\underline{l}=\underline{l}$$

and $\Sigma n_{io}=N_o$ is the initial population number.

$\overset{*}{N_f}$ females on an average would give rise to N_m males attaining the age of one year,

where $$N_m = p_b \left(1-\mu'_{1m}\right)\left(1-\lambda_{1m}\right) N_f^*$$

Since during the initial phase of the programme the number of sons to be retained from each batch of first d sets of w bulls are to be so chosen that at the time of their use there are $(w+w')$ of them, we therefore define $p^{(1)}$ as

$$p^{(1)} = \left(\frac{w+w'}{N_m}\right)\left[\left(1-\mu'_{2m}\right)\left(1-\lambda'_{2m}\right)\left(1-\frac{m}{2}\mu_{3m}\right)\left(1-\frac{m}{2}\lambda_{3m}\right)\right]^{-1}$$

After the completion of the initial phase of the programme, the selection of sons is restricted among those born to tested bulls only. Since of the N_m males attaining the age of one year only $\frac{v}{u} N_m$ would be sons from tested bulls. We now define $p^{(2)}$ as satisfying the following relation:

$$\frac{v}{u} N_m \, p^{(2)} = v.a$$

where a, the number of sons to be retained from each tested bull, will be obtained by solving the following equation for c:

$$(w+w') = c.v$$

and then taking the integral value rounded on the upper side of

$$\frac{c}{\left(1-\mu'_{2m}\right)\left(1-\lambda'_{2m}\right)\left(1-\frac{m}{2}\mu_{3m}\right)\left(1-\frac{m}{2}\lambda_{3m}\right)}.$$

6.4. Expressions for the Elements of Matrices T and B for Males

The elements c_{ij}'s and b_{ij}'s of matrices T and B are derived from the consideration that the number of males in stage group i at time $t+1$ can be obtained deterministically by taking a linear function of the number of males and females in different stage groups at time t. That is,

$$m_{i,t+1} = \sum_{j=1}^{r} m_{it} \, c_{ij} + \sum_{=1}^{s} n_{jt} \, b_{ij}$$

where m_{it} $(i=1, 2, \ldots, r)$

and n_{jt} $(j=1, 2, \ldots, s)$

are the number of males and females in their respective stage groups i and j at time t, and their coefficients represent the biological dependence of ith stage at time $t+1$ upon the jth stage at time t.

Since the regular progeny testing of bulls based on daughters' performance commences only after the selection of sons of first d sets of bulls (based on dam's performance) is completed *i.e.*, after $d+1$ units of time from the start of the scheme, therefore, for $t \leqslant d+1$, such of the elements of the generation matrix which depend upon selection scheme in respect of sons born to first d sets of bulls say the first phase, will be different from the ones which correspond to the regular scheme of retaining sons from tested bulls only. In order to differentiate the two sets of elements, those pertaining to the first phase have been superscripted with 1. Even during the first phase not all elements will remain constant as the sons from the bulls brought from outside will start getting generated after g months from the start of the scheme. The elements affected as a result thereof have been superscripted with o.

(*a*) *Elements of the first row*

(*i*) Considering males born to scheme bulls only, $m_{1,1}$ represents the number of males born during the last g to m months whereas for $t>o$, $m_{1,t+1}$ represents the number born during the last $m-12$ to m months and alive in stage group 1 at time $t+1$, we have using the lemmas,

$$m_{1,1}=(m-g)\,\overline{O}_{mo}\left(1-\frac{m-g}{2}\,\mu_{1m}\right)\left(1-\frac{m-g}{2}\,\lambda_{1m}\right)$$

$$=\sum_{i=1}^{r} m_{io}\;c_{1i}^{(o)}+\sum_{i=1}^{s} n_{io}\;b_{1i}^{(o)}$$

with $\qquad c_{1i}^{(o)}=o \qquad$ for all i,

$$b_{1i}^{(o)}=o \qquad \text{for all } i\leqslant 3,$$

$$b_{14}^{(o)}=\frac{m-g}{l+g}\left(1-\frac{m-g}{2}\,\mu_{1m}\right)\left(1-\frac{m-g}{2}\,\lambda_{1m}\right)p_b,$$

and $\quad b_{1i}^{(o)}=\dfrac{m-g}{m}\left(1-\dfrac{m-g}{2}\,\mu_{1m}\right)\left(1-\dfrac{m-g}{2}\,\lambda_{1m}\right)p_b$ for all $i\geqslant 5$.

and for $t>o$

$$m_{1,t+1}=12\bar{O}_{mt}\,(1-6\,\mu_{1m})(1-6\,\lambda_{1m})$$

$$=\sum_{i=1}^{r} m_{it}\,c_{1i}+\sum_{i=1}^{s} n_{it}\,b_{1i}$$

with $c_{1i}=o$ for all i,

$b_{1i}=$for all $i\leqslant 3$,

$$b_{14}=\frac{12}{l+g}\,(1-6\,\mu_{1m})\,(1-6\,\lambda_{1m})\,p_b,$$

and $$b_{1i}=\frac{12}{m}(1-6\mu_{1m})\,(1-6\,\lambda_{1m})\,p_b \text{ for all } i\geqslant 5.$$

(*b*) *Elements of the second row*

(*i*) For $t\leqslant d$, the numbers in group 2 will comprise the sons of bulls under test after allowing for mortality and voluntary and involuntary cullings during $(t, t+1)$ plus the males of last $q+12-m$ sets in stage group 1 which were also the sons of untested bulls and alive at time t. Using the lemmas, we have

$m_{2,1}=0$ (as the specified sons from scheme bulls will not get generated by this time)

$$=\sum_{i} m_{i0}\;c_{2i}^{(o)}+\sum_{i}\;n_{i0}\,b_{2i}^{(o)}$$

As such $c_{2i}^{(o)}=b_{2i}^{(o)}=o$ for all i.

For $1\leqslant t\leqslant d$

$$m_{1,t+1}=p^{(1)}\Bigg[(m-12)\,\bar{O}_{mt}\,\left(1-\mu'_{1m}\right)\left(1-\lambda'_{1m}\right)\left(1-\frac{m-12}{2}\,\mu_{2m}\right)\left(1-\frac{m-12}{2}\,\lambda_{2m}\right)$$

$$+\left(\frac{q+12-m}{12}\right)m_{1t}\,\left(1-\frac{m-q+12}{2}\,\mu_{1m}\right)\left(1-\frac{m-q+12}{2}\,\lambda_{1m}\right)\times\left(1-\frac{m+q-12}{2}\,\mu_{2m}\right)\left(1-\frac{m+q-12}{2}\,\lambda_{2m}\right)\Bigg]$$

$$=\sum_{i} m_{it}\;c_{2i}^{(1)}+\sum_{i} n_{it}\;b_{2i}^{(1)}$$

$$\text{with } c_{21}^{(1)} = p^{(1)}\left(\frac{q+12-m}{12}\right)\left(1-\frac{m-q+12}{2}\mu_{1m}\right)$$

$$\left(1-\frac{m-q+12}{2}\lambda_{1m}\right)\times\left(1-\frac{m+q-12}{2}\mu_{2m}\right)\left(1-\frac{m+q-12}{2}\lambda_{2m}\right),$$

$$c_{2i}^{(1)} = o \text{ for all } i \geqslant 2$$

$$b_{2i}^{(1)} = o \text{ for all } i \leqslant 3$$

$$b_{24}^{(1)} = p^{(1)}\frac{m-12}{l+g}\,p_b\left(1-\mu'_{1m}\right)\left(1-\lambda'_{1m}\right)\left(1-\frac{m-12}{2}\mu_{2m}\right)\left(1-\frac{m-12}{2}\lambda_{2m}\right),$$

$$b_{2i}^{(1)} = p^{(1)}\frac{m-12}{m}\,p_b\left(1-\mu'_{1m}\right)\left(1-\lambda'_{1m}\right)\left(1-\frac{m-12}{2}\mu_{2m}\right)\left(1-\frac{m-12}{2}\lambda_{2m}\right) \text{ for all } i \geqslant 5$$

(*ii*) For $t>d$, the number in group 2 at time $t+1$ will comprise sons of tested bulls after allowing for mortality and culling during $(t,\ t+1)$ among those born during first $m-12$ months plus the males of last $q+12-m$ sets in stage group 1 which were the sons of tested bulls and alive at time t. Using the lemmas, we have

$$m_{2,t+1} = \sum_i m_{it}\,c_{2i} + \sum_i n_{it}\,b_{2i}$$

$$\text{with } c_{21} = p^{(2)}\frac{v}{u}\,\frac{q+12-m}{12}\left(1-\frac{m-q+12}{2}\mu_{1m}\right)\left(1-\frac{m-q+12}{2}\lambda_{1m}\right)$$

$$\times\left(1-\frac{m+q-12}{2}\mu_{2m}\right)\left(1-\frac{m+q-12}{2}\lambda_{2m}\right)$$

$$c_{2i} = o \text{ for all } i \geqslant 2$$

$$b_{2i} = o \text{ for all } i \leqslant 3,$$

$$b_{24} = p^{(2)}\frac{v}{u}\left(\frac{m-12}{l+g}\right)p_b\left(1-\mu'_{1m}\right)\left(1-\lambda_{1m}\right)$$

$$\left(1-\frac{m-12}{2}\mu_{2m}\right)\left(1-\frac{m-12}{2}\lambda_{2m}\right), \text{ and}$$

$$b_{2i} = p^{(2)}\frac{v}{u}\left(\frac{m-12}{m}\right)p_b\left(1-\mu'_{1m}\right)\left(1-\lambda'_{1m}\right)$$

$$\times\left(1-\frac{m-12}{2}\mu_{2m}\right)\left(1-\frac{m-12}{2}\lambda_{2m}\right) \text{ for all } i \geqslant 5.$$

(c) *Elements of the third row*

(*i*) The numbers in stage group 3 at time $t+1$ for $t \leqslant d+1$ will comprise sons of bulls of an earlier set previous to the immediately preceding one among the first $m-q$ sets of sons in stage group 1 and all the q sets in stage group 2 at time t, we have

$$m_{3,t+1} = \sum_i m_{it}\, c^{(1)}_{3i} + \sum_i n_{it}\, b^{(1)}_{3i}$$

with $$c^{(1)}_{31} = p^{(1)} \left(\frac{m-q}{12}\right)\left(1-\mu'_{2m}\right)\left(1-\lambda'_{2m}\right)\left(1-\frac{m-q}{2}\mu_{1m}\right)$$

$$\times\left(1-\frac{m-q}{2}\lambda_{1m}\right)\left(1-\frac{m-q}{2}\mu_{3m}\right)\left(1-\frac{m-q}{2}\lambda_{3m}\right)$$

$$c^{(1)}_{32} = \left(1-\frac{q}{2}\mu_{2m}\right)\left(1-\frac{q}{2}\lambda_{2m}\right)\left(1-\frac{2m-q}{2}\mu_{3m}\right)$$

$$\times\left(1-\frac{2m-q}{2}\lambda_{3m}\right)$$

$c^{(1)}_{3i} = o$ for all $i \geqslant 3$, and $b^{(1)}_{3i} = o$ for all i.

Since first three sets of $w+w'$ bulls each at intervals of one calving interval are to come from outside (consideration *ix* Section 2), we set $m_{3,t} = w+w'$ in the vector of male population at $t=o$, 1 and 2 before obtaining the composition of the population at time $t+1$. This amounts to assuming that the incoming bulls at time o, 1 and 2 are assembled in stage group 3 just prior to their commissioning for service.

(*ii*) For $t > d+1$, the numbers in this stage group will comprise sons of tested bulls of an earlier set previous to the immediately preceding one among the first $m-q$ sets of sons in stage group 1 and all the q sets in stage group 2 at time t, *i.e.*

$$m_{3,t+1} = \sum_i m_{it}\, c_{3i} + \sum_i n_{it}\, b_{3i}$$

with $$c_{31} = p^{(2)} \frac{v}{u}\left(\frac{m-q}{12}\right)\left(1-\mu'_{2m}\right)\left(1-\mu'_{2m}\right)$$

$$\left(1-\frac{m-q}{2}\mu_{1m}\right)\left(1-\frac{m-q}{2}\lambda_{1m}\right)$$

$$\times\left(1-\frac{m-q}{2}\mu_{3m}\right)\left(1-\frac{m-q}{2}\lambda_{3m}\right)$$

$$c_{32}=\left(1-\frac{q}{2}\mu_{2m}\right)\left(1-\frac{q}{2}\lambda_{2m}\right)$$

$$\left(1-\frac{2m-q}{2}\mu_{3m}\right)\left(1-\frac{2m-q}{2}\lambda_{3m}\right)$$

$$c_{3i}=0 \text{ for all } i\geqslant 3,$$

and $$b_{3i}=0 \text{ for all } i.$$

(*b*) *Elements of the fourth row*

The numbers in stage group 4 at time $t+1$ will comprise males of stage group 3 at time t after allowing for mortality and culling during $(t, t+1)$, *i.e.*

$$m_{4,t+1}=\sum_i m_{it}\, c_{4i} + \Sigma\, n_{it}\, b_{4i}$$

with $$c_{4i}=0 \text{ for all } i\neq 3,$$

$$c_{43}=\left(1-\frac{m}{12}\mu'_m\right)\left(1-\frac{m}{12}\lambda'_m\right),$$

and $$b_{4i}=0 \text{ for all } i.$$

(*e*) *Elements of the fifth-row*

From the considerations (*viii*) of Section 2, the numbers in stage group 5 will comprise males after allowing for mortality and voluntary and involuntary cullings during $(t, t+1)$ among the males in stage group 4 at time t, *i.e.*

$$m_{5,t+1}=\sum_i m_{it}\, c_{5i}+\sum_i n_{it}\, b_{5i}$$

with $$c_{5i}=0 \text{ for all } i\neq 4,$$

$$c_{54}=\frac{(w+1)}{w+w'}\left(1-\frac{m}{12}\mu'_m\right)\left(1-\frac{m}{12}\lambda'_m\right),$$

and $$b_{5i}=0 \text{ for all } i.$$

(*f*) *Elements of the* $(5+j)$*th row for* $i=1, 2, \ldots \overline{d-2}$

The numbers in this stage group will comprise males of stage group $(4+j)$ at time t after allowing for mortality and culling during $(t, t+1)$, *i.e.*

$$m_{5+j,t+1} = \sum_i m_{it}\, c_{5+j,i} + \sum_i n_{it}\, b_{5+j,i}$$

with $c_{5+j,i}=0$ for all $i \neq 4+j$,

$$c_{5+j,\,4+j}=\left(1-\frac{m}{12}\,\mu'_m\right)\left(1-\frac{m}{12}\lambda'_m\right),$$

and $b_{5+j,i}=0$ for all i.

(g) *Elements of the r-th row*

The numbers in the last stage group at time $t+1$ will comprise males of stage group $r-1$ at time t after allowing for mortality and voluntary and involuntary cullings during $(t, t+1)$, *i.e.*

$$m_{r,t+1}=\sum_i m_{it}\,c_{ri}+\sum_i n_{it}\,b_{ri}$$

with $c_{ri}=0$ for all $i \neq r-1$,

$$c_{r,r-1}=\frac{v+v'}{w}\left(1-\frac{m}{12}\,\mu'_m\right)\left(1-\frac{m}{12}\,\lambda'_m\right),$$

and $b_{ri}=0$ for all i.

In the light of the foregoing values of c_{ij}'s and b_{ij}'s for $t>d+1$ *i.e.* after the selection of sons of the last batch of the first d sets of bulls has been affected, the matrices T and B corresponding to male stock take the forms as indicated in the earlier section.

SUMMARY

The method of population generation matrix used earlier (Jain and Narain, 1974) for studying the growth of female population of dairy herds grouped in unequal stage-groups has been extended to include both the sexes. Explicit formulae for working out the elements of matrices corresponding to a systematic breeding programme of progeny testing with a specific schedule of vital characteristics and culling rates have been derived.

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