ON THE ORTHOGONAL MATES OF SOME F-SQUARES

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Introduction

Hedayat (1) and Hedayat and Seiden (2) generalised the concepts of latin squares and orthogonality of latin squares to F-squares and orthogonality of F-squares. Hedayat and Seiden (2) picked up certain F-squares and gave their orthogonal mates without giving any method of constructing them. This tempted the present authors to obtain a method of constructing orthogonal mates of a given type of F-squares w.r t. certain decompositions. It may be pointed out that the existence of p., m.o. F-squares is implied by the existence of p., m.o. latin squares of the same order. However, this method does not provide an answer to the problem taken by the authors *i.e.* of writing orthogonal mate of a given F-square. In the paper we give some results about the existence and construction of m.o. mates w.r.t. the decomposition $v = 1+n_1+n_2$ of the NN' matrices of *PBIB* designs with two associate classes, which are F-squares with the frequency vector $(1, n_1, n_2)$. Moreover, through our method we also establish that we may have more m.o. F-squares than obtainable from a set of m.o. latin squares of the same order (cf. Example 2.2).

It may be worthwhile to point out that an F-square is a useful experimental design having the property that treatment effects, row effects and column effects are mutually orthogonal. In applications, an F-square may be preferred to a latin square, when the number of treatments is smaller than the order of the square and we wish to take advantage of the available units to increase the precision of the estimates of at least some of the treatments. Similarly, if from some past experiment, we learn that differences between some of the treatments are negligible, we would switch over from a latin square to an F-square. The use of t, m.o. F-squares to eliminate $(t+1)$ sources of variation may be justified as in the case of latin squares. However, in this case, we have an added advantage that we may have more m.o. F-squares than the number of m.o. latin squares of the same order.

2. Orthogonal mates of NN' matrices of *PBIB* designs with two associate classes.

We first give a few definitions [cf. Hedayat and Seiden (2)].

Definition 1. A $v \times v$ matrix with $\Sigma=(c_1, c_2, \ldots, c_m)$ as the ordered set of distinct elements of A is called an F-square with frequency vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and is denoted by $F(v; \lambda_1, \lambda_2, \ldots, \lambda_m)$ if for every k, c_k appears precisely $\lambda_k (\lambda_k \geqslant 1)$ times in each row and each column of A.

Definition 2. Given $F_1(v; \lambda_1, \lambda_2, \ldots, \lambda_k)$ over a k-set $\Sigma=(a_1, a_2, \ldots, a_k)$ and $F_2(v; u_1, u_2, \ldots, u_t)$ over a t-set$\Omega=(b_1, b_2, \ldots, b_t)$, we say F_2 is orthogonal mate for F_1, if upon superimopositon of F_2 on F_1, a_i appears $\lambda_i \mu_j$ times with b_j.

Definition 3. An $F(v; \lambda_1, \lambda_2, \ldots \lambda_t)$ on a t-set Σ is said to be of degree r,w.r.t. the decomposition $v=u_1+u_2+\ldots+u_s$ if there exist $(r-1)$ F-squares $F_1, F_2, \ldots, F_{r-1}$ on a s-set $\Omega=(b_1, b_2, \ldots, b_s)$ with frequency vector $(u_1, u_2, \ldots, u_s)$ s.t. $(F, F_1, F_2, \ldots, F_{r-1})$ is a set of r.m.o. F-squares and r is the largest such integer.

The NN' matrix of a *PBIB* design with two associate classes is an $F(v; 1, n_1, n_2)$. It is a $u \times v$ matrix consisting of three symbols r, λ_1, and λ_2. In the i-th row vector corresponding to the i-th treatment say, θ_i $(i=1, 2, \ldots, v)$, there are n_1, λ_1's; n_2, λ_2's; and r in the i-th position, where v, r, λ_1, λ_2, n_1 and n_2 have their usual meaning.

Theorem 2.1. The NN' matrix of a *PBIB* design with two associate classes has at least one orthogonal mate as an F-square $F^*(V; 1, n_1, n_2)$ *i.e.* it is of degree at least two w.r.t. the decomposition $v=1+n_1+n_2$, if the elements of the set of treatments say $(\theta_1, \theta_2, \ldots, \theta_v)$ can be permuted as $\left(\overset{*}{\theta_1}, \overset{*}{\theta_2}, \ldots, \overset{*}{\theta_v}\right)$ such that out of the v pairs$\left(\theta_i, \overset{*}{\theta_i}\right)$$(i=1, 2, \ldots, v)$ so formed.

(*i*) for exactly one pair $\theta_j = \theta_j^*$;

(*ii*) exactly n_1 pairs are Ist associate pairs.

$F^*(v; 1, n_1, n_2)$ is obtained from $F(v; 1, n_1, n_2)$ by replacing the row vector corresponding to θ_i by the row vector corresponding to θ_i^* $(i=1, 2, \ldots, v)$.

Proof : F^* is obviously an F-square w.r.t. the decomposition $v=1+n_1+n_2$. Superimpose F^* on F and note that—

(*a*) the row vectors of F and F^* corresponding to the pair (θ_j, θ_j) provide pairs (r, r), (λ_1, λ_1) and (λ_2, λ_2); 1, n_1 and n_2 times respectively;

(*b*) the row vectors of F and F^* corresponding to each of the n_1 Ist associate pairs$\left(\theta_i,\ \overset{*}{\theta_i}\right)$, provide pairs (r, λ_1), (λ_1, λ_1) (λ_1, λ_2) (λ_2, λ_1) and (λ_2, λ_2) 1, p^1_{11}, p^1_{12}, p^1_{21}, and p^1_{22} times respectively;

(*c*) the row vectors of F and F^* corresponding to each of the remaining n_2, 2nd associate pairs $\left(\theta_i, \overset{*}{\theta_i}\right)$ provide pairs (r, λ_2), (λ_1, λ_1), (λ_1, λ_2), (λ_2, λ_1) and (λ_2, λ_2) 1, p^2_{11}, p^2_{12} p^2_{21} and p^2_{22} times respectively. Thus the total number of (r, r), (r, λ_1) and (r, λ_2) pairs are 1, n_1 and n_2 respectively. In view of the relations (2.1);

$$\begin{aligned} n_1+n_1\, p^1_{11} + n_2\, p^2_{11} &= n^2_1 \\ n_1\, p^1_{12} + n_2\, p^2_{12} &= n_1\, n_2 \\ n_1\, p^1_{21} + n_2\, p^2_{21} &= n_2\, n_1 \\ n_2 + n_1\, p^1_{22} + n_2\, p^2_{22} &= n^2_2 \end{aligned} \tag{2.1}$$

the total number of pairs of the type (λ_1, λ_1), (λ_1, λ_2), (λ_2, λ_1) and (λ_2, λ_2) are $n^2{}_1$, n_1n_2, n_2n_1 and $n^2{}_2$ respectively and this proves the theorem.

Remark 2.1. It can be easily verified that by interchanging λ_1 and λ_2 in F^* $(v;\, 1, n_1, n_2)$ one can obtain an orthogonal mate of $F(v; 1, n_1, n_2)$ as F^* $(v;\, 1, n_2, n_1)$, other permutations of r, λ_1 and λ_2 in F^* $(v;\, 1, n_1, n_2)$ will also provide orthogonal mate of $F(v;\, 1, n_1, n_2)$ with the appropriate decompositions.

The following is an immediate corollary of theorem 2.1 :

Corollary 2.1 If the v treatments of a *PBIB* design with two associate classes can be permuted in p different ways such that the v

pairs of treatments formed from the corresponding positions of any two of the permutations satisfy (*i*) and (*ii*) of Theorem 2.1 then these p permutations give rise to a set of p,m.o. F-squares *i.e.* the NN' matrix is of degree p w.r.t. the decomposition $v=1+n_1+n_2$.

Theorem 2.2. If for an association scheme of *PBIB* design with two associate classes, p^1_{11} and p^2_{22} are both positive, then the set of v treatments can always be permuted so as to provide v pairs satisfying the requirements (*i*) and (*ii*) of Theorem 2.1 and as a consequence $F^*(v;\ 1,\ n_1,\ n_2)$ an orthogonal mate of the NN' matrix of the *PBIB* design can be constructed.

Proof : Select any treatment say, θ. Let $A(\theta)=(\theta_1, \theta_2, \ldots, \theta_{n1})$ and $B(\theta)=(\phi_1, \phi_2, \ldots, \phi_{n2})$ denote the sets of Ist associates and 2nd associates of θ, respectively. Thus the set of v treatments can be written as $(\theta, \theta_1, \theta_2, \ldots, \theta_{n1}, \phi_1, \phi_2, \ldots, \phi_{n2})$. For each θ_j, consider the set $C(\theta_j)$, where

$$C(\theta_j)=A(\theta_j)\cap A(\theta),\ j=1, 2, \ldots, n_1.$$

For each θ_j, $C(\theta_j)\subset A(\theta)$ and has exactly p^1_{11} ($\neq 0$) elements.

Again, each element of $A(\theta)$ occurs exactly p^1_{11} times in the set $C=\cup_j C(\theta_j)$.

One can easily see that the necessary and sufficient condition Raghava Rao [3] for the existence of a system of distinct representatives (*S.D.R*) for the sets $C(\theta_1), C(\theta_2), \ldots, C(\theta_{n1})$ is satisfied and this *S.D.R.* is nothing but a permutation of $(\theta_1, \theta_2, \ldots, \theta_{n1})$. Let this *S.D.R.* be denoted by $\left(\overset{*}{\theta}_1, \overset{*}{\theta}_2, \ldots, \overset{*}{\theta}_{n1}\right)$. Again since $C(\theta_j)\subset A(\theta_j)$, it is clear that n_1 pairs $\left(\theta_i, \overset{*}{\theta}_i\right)$ are all first associate pairs.

Let for each ϕ_j, $D(\phi_j)=B(\phi_j)\cap B(\theta)$, $j=1, 2, \ldots, n_2$. Each $D(\phi_j)$ has p^2_{22} ($\neq 0$) elements. As before it follows that there exists a *S.D.R.* say $\left(\overset{*}{\phi}_1, \overset{*}{\phi}_2, \ldots, \overset{*}{\phi}_{n2}\right)$ for the sets $D(\phi_1), D(\phi_2), \ldots, D(\phi_{n2})$ and this is a permutation of $(\phi_1, \phi_2, \ldots, \phi_{n2})$. Again since $D(\phi_j)\subset B(\phi_j)$, the n_2 pairs $\left(\phi_j, \overset{*}{\phi}_j\right)$ $(j=1, 2, \ldots, n_2)$ are all 2nd associate pairs.

Thus there exists a permutation $\left(\theta, \theta^*_1, \theta^*_2, \ldots, \theta^*_{n1}, \phi^*_1, \phi^*_2, \ldots, \phi^*_{n2}\right)$ of $(\theta, \theta_1, \theta_2, \ldots, \theta_{n1}, \phi_1, \phi_2, \ldots, \phi_{n2})$ satisfying requirements of Theorem 2.1 and consequently $F^*(v; 1, n_1, n_2)$ can be constructed.

In the following theorems we shall give construction of an orthogonal mate $F^*(v; 1, n_1, n_2)$ for the NN' matrix of *G.D.*, Triangular, Latin Square, and cyclic type *PBIB* designs, even in cases where the Conditions of Theorm 2.2 do not hold.

Theorem 2.3. The NN' matrix of a group divisible (*G.D.*) *PBIB* design with parameters $v=mn, b, r, k, n_1=n-1, n_2=(m-1)n$, $\lambda_1, \lambda_2, p^i_{jk}$, $i, j, k=1, 2$ has at least one orthogonal mate $F^*(v; 1, n_1, n_2)$ except when $m=2$, and $n>2$.

Proof : We shall distinguish three cases.

Case (*i*). $m>2, n>2$, case (*ii*) $m=2, n=2$, cases (*iii*) $m>2, n=2$.

Case (*i*). p^1_{11} and p^2_{22} are both positive and therefore, the construction of $F^*(v; 1, n_1, n_2)$ an orthogonal mate of the NN' matrix follows from Theorems 2.1 and 2.2.

Case (*ii*). Let the association scheme be

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

where treatment in the same row are Ist associates and treatments in different rows are 2nd associates. Consider the sets (a, b, c, d) and (a, c, d, b), the four pairs (a, a); (b, c); (c, d) and (d, d) satisfy the requirements (*i*) and (*ii*) of Theorem 2.1 and so an orthogonal mate $F^*(v; 1, n_1, n_2)$ of the NN' matrix can be constructed.

Case (*iii*). Let the association scheme be

$$\begin{matrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \\ \cdots & \cdots \\ \theta_{m-11} & \theta_{m-12} \\ \theta_{m1} & \theta_{m2} \end{matrix}$$

where treatments in the same row are Ist associates and treatments in different rows are 2nd associates. Write the set of treatments as θ_{11}, $\theta_{21}, \theta_{31}, \ldots, \theta_{m1}, \theta_{m2}, \theta_{m-12}, \ldots, \theta_{22}, \theta_{12})$ and permute the treatments as

$\theta_{11}, \theta_{31}, \theta_{41}, \ldots, \theta_{m2}, \theta_{m-1\cdot 2}, \theta_{m2-2}, \ldots, \theta_{12}, \theta_{21})$ then the v pairs $(\theta_{11}, \theta_{11})$, $(\theta_{21}, \theta_{31})$, $(\theta_{31}, \theta_{41}), \ldots, (\theta_{m1}, \theta_{m2})$, $(\theta_{m2}, \theta_{m-1\cdot 2})$, $(\theta_{m-12}, \theta_{m22}), \ldots, (\theta_{22}, \theta_{12})$ and $(\theta_{12}, \theta_{21})$ satisfy the requirements of Theorem 2 1 and cons-equently F^* $(v; 1, n_1, n_2)$ can be constructed.

Remark 2.2. For particular values of m and n, it is possible to construct $p(p>2)$ *m.o.* F-square of the type $F(v; 1, n_1, n_2)$ as illustrated by the following examples.

Example 2.1 For $m=2$, $n=2$, the four pairs of treatments formed by taking any two of the following three permutations (a, b, c, d), (a, c, d, b) and (a, d, b, c) satisfy the conditions of Corollary 2.1 and hence give rise to 3 m.o. F-squares. This establishes that NN' is of degree 3 w.r.t. the decomposition $4=1+1+2$.

Example 2.2. For $m=3$, $n=2$ the six pairs of treatments formed by taking any two of the following five permutations (a, b, c, d, e, f), (a, e, d, f, c, b), (a, f, e, c, b, d), (a, d, b, e, f, c) and (a, c, f, b, d, e) satisfy the conditions of corollary 2.1 and hence give rise to 5. m.o. F-squares of the type $F(6; 1, 1, 4)$. This establishes that NN' is of degree 5 w.r.t. the decomposition $6=1+1+4$.

Theorem 2.4. The NN' matrixof a triangular *PBIB* design with parameters $v=\dfrac{n(n-1)}{2}$, $b, r, k, n_1=2n-4, n_2=\dfrac{(n-2)(n-3)}{2}$, $\lambda_i, p^i_{jk}, i, j, k=1, 2$ has at least one orthogonal mate $F^*(v; 1, n_1, n_2)$.

Proof : Here we shall distinguish two cases, namely

Case (i) when $n>5$. Case (ii) when $n=4$ or $n=5$.

Case (i) p^1_{11} and p^2_{22} are both positive and hence the construction of $F^*(v; 1, n_1, n_2)$ follows from Theorem 2.1 and Theorem 2.2.

Case (ii) Let the association schemes for $n=4$ and $n=5$ be the following :

$\times$	a	b	c		$\times$	a	b	c	d
a	$\times$	d	e		a	$\times$	e	f	g
b	d	$\times$	f		b	e	$\times$	h	i
c	e	f	$\times$		c	f	h	$\times$	j
					d	g	i	j	$\times$

For $n=4$, consider the permutations (a, b, c, d, e, f) and (a, c, d, e, f, b), the six pairs (a, a), (b, c), (c, d), (d, e), (e, f) and (f, b) satisfy the requirements (i) and (ii) of Theorem 2.1.

For $n=5$ consider the permutations $(a, b, c, d, e, f, g, h, i, j)$ and $(a, c, d, e, f, g, h, i, j, b)$; the pairs (a, a), (b, c), (c, d), (e, f), (f, g), (g, h), (h, i), (i, j), (j, b) satisfy requirements (i) and (ii) of Theorem 2.1.

Hence it follows that for $n=4$ and $n=5$, an orthogonal mate $F^*(v; 1, n_1, n_2)$ of the NN' matrix can be constructed.

Remark 2.3. For particular values of n, we can construct $p(p>2)$ m.o. F-squares of the type $F(v; 1, n_1, n_2)$ as illustrated by the following example.

Example 2.3. For $n=5$, the pairs of treatments formed by taking any two of the following permutations $(a, b, c, d, e, f, g, h, i, j)$, $(a, f, d, c, i, b, h, j, e, g)$ and $(a, d, b, f, h, j, e, g, c, i)$ satisfy the conditions of Corollary 2.1 and hence give rise to 3. m.o. F-squares of the type $F(10; 1, 6, 0)$. This establishes that NN' is of degree at least three w.r. t. the decomposition $10=1+6+3$.

Theorem 2.5. The NN' matrix of a Latin-square type *PBIB* design with i constraints $(i>2)$ with parameters, $v=n^2$, b, r, k, $n_1=i(n-1)$, $n_2=(n-1)(n-i+1)$, λi, p^{i}_{jk}, $i, j, k=1, 2$, has at least one orthogonal mate $F^*(v; 1, n_1, n_2)$.

Proof. We distinguish two cases, namely

Case (i) when $n>2$, and case (ii) when $n=2$.

In this case p^{1}_{11} and p^{2}_{22} are both positive and therefore the construction of $F^*(v; 1, n_1, n_2)$ follows from Theorem 2.1 and Theorem 2.2.

Case (ii) Let the association scheme be

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

where treatments in the same row or same column are Ist associates otherwise they are 2nd associates. Consider the permutations (a, b, c, d) and (a, c, d, b), the pairs (a, a), (b, c), (c, d) and (d, b) satisfy (i) and (ii) of Theorem 2.1 and consequently $F^*(v; 1, n_1, n_2)$ can be constructed.

Theorem 2.6. The NN' matrix of a cyclic *PBIB* design with parameters $v+1$, b, r, k, $n_1=n_2=2t$, λ_1, λ_2 and

$$P_1=\begin{bmatrix} t-1 & t \\ t & t \end{bmatrix}, \quad P_2=\begin{bmatrix} t & t \\ t & t-1 \end{bmatrix}$$

has at least one orthogonal mate $F^*(v; 1, n_1, n_2)$.

Proof. We distinguish two cases namely case (*i*) $t>1$, case (*ii*) $t=1$.

Case (*i*) p^1_{11} and p^2_{22} are both positive and therefore the construction of $F^*(v; 1, n_1, n_2)$ follows from Theorems 2.1 and 2.2.

Case (*ii*) Let the five treatments be denoted by 1, 2, 3, 4, and 5.

Let $d_1=1$ and $d_2=4$, then the set of Ist asssociates of the i-th symbol are $(i+d_1, i+d_2)$ mod 5, $(i=1, 2, 3, 4, 5)$. Consider the permutations (1, 2, 3, 4, 5) and (1, 3, 5, 2, 4), the pairs (1, 1), (2, 3), (3, 5), (4, 2) and (5, 4) satisfy (*i*) and (*ii*) of Theorem 2.1 and hence F^* (5; 1, 2, 2) can be constructed.

Remark 2.4. For particular values of $v=4t+1$ it is possible to construct $p(p>2)$ m.o. F-squares of the type $F(4t+1; 1, 2t, 2t)$ as illustrated by the following example.

Example 2.4. For $v=5$, $n_1=2=n_2$, the five pairs of treatments formed by considering any two of the following for permutations (1, 2, 3, 4, 5), (1, 3, 5, 2, 4), (1, 5, 4, 3, 2) and (1, 4, 2, 5, 3) satisfy the conditions of Corollary 2.1 and hence give rise to 4 m.o. F-squares the type $F(5; 1, 2, 2)$. This establishes that the NN' matrix is of degree 4 w.r.t. the decomposition $5=1+2+2$.

SUMMARY

A method for the construction of orthogonal mate of NN' matrices of *PBIB* designs with two associate classes has been obtained. Through illustrations, it has been shown that by this method we may obtain more m.o. F-squares than obtainable from a set of m.o. latin squares of the same order.

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