A THURSTONE-MOSTELLER MODEL FOR SYMMETRICAL PAIRS

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Paired comparison is a popular method in sensory testing, food technology, market research, assessment of building materials, assessment of architectural designs, educational testing, psychology standardisation etc. Since the number of pairs in such an experiment is large for larger (number of stimuli) the senior author (1967) has proposed fractionation of pairs. This fractionation can be achieved in different ways. A model for analysis of preference data from asymmetrical fractional pairs named standard comparison pairs containing a common stimulus was developed by Sadasivan and Rai (1971). Some methods of fractionating paired comparisons have been developed by Sadasivan and Sundaram (1971). Of these, one of the fractions was found to be very useful because of its balanced nature and was named symmetrical pairs. For t treatments $T_1, T_2, \ldots, T_t$ the symmetrical pairs are $(T_1\ T_2)$, $(T_2\ T_3), \ldots, (T_t\ T_1)$.

The number of pairs in this set is t. We lose information about $t(t-3)/2$ and the amount of reduction is $(t-3)/(t-1)$. For $t=5$ we get two sets of symmetrical pairs. For $t=6$ also there are only two distinct sets of symmetrical pairs. In general, if t is odd we get $(t-1)/2$ sets of symmetrical pairs and for t even we get $(t/2-1)$ sets of symmetrical pairs and one set of $t/2$.

The size of a set of complete pairs is equivalent to $(t-1)/2$ replicates of the same symmetrical pair. The number of comparisons of each stimulus in each set is $(t-1)$. Complete pairs give information about all the possible pairs once whereas in symmetrical pairs each pair occurring in the set gets $(t-1)/2$ replicates. Thus the pairs occurring are estimated with relative precision $(t-1)/2$ in such a design. But we lose information in symmetrical pairs about the non occurring pairs. Again the fraction that is selected by symmetrical pairs is $2/(t-1)$.

Cyclic paired comparison designs can be used for symmetrical pairs as well. Consider the case $n=5$. The 5 symmetrical pairs which can be made when order of presentation is taken into account may be set out in two cyclic sets

(1)	01	12	23	34	40
(2)	04	10	21	32	43

Here the set (s)

$(s=1, 2, \ldots, n-1)$ is of the form

o, s ; $1, s+1$; $t, s+t$; $, n-1, s+n-1$ where $t+s$ is reduced modulo n when necessary. Of these the sets corresponding to $s=1$ and $n-1$ are presented above. Note that in the set the order of presentation has been balanced out. Sets (1) and (2) are equivalent if order of presentation is ignored. For $n=5$ set (1) changes to set (4) for the operator $R\left(5, 4\right)=\begin{pmatrix}0 & 1 & 2 & 3 & 4.\\ 0 & 4 & 3 & 2 & 1\end{pmatrix}$. Thus (1) and (2) differ only in the arbitrary numbers assigned to the objects.

For $n=6$, the symmetrical pairs are set out below :—

(1)	01	12	23	34	45	50
(2)	02	13	24	35	40	51

and also the half set which is also symmetric in character

(3)	03	14	25

The two conditions which a satisfactory sub-set should satisfy are: (*a*) every object should appear equally often, (*b*) the design should be connected so that it is impossible to split the objects into two sets with no comparisons made between objects in one set and objects in the other. The conditions (*a*) and (*b*) are satisfied for $n=5$. But for $n=6$ all the sets satisfy condition (*a*) but only set (1) meets condition (*b*). (2) is not connected and separates into two sub-sets 02, 24, 40 and 13, 35, 51 the reason being that 2 is a factor of t.

Efficiencies of symmetrical pairs can be compared in different ways. Mckeon (1960) has defined efficiency, E_f, of any paired comparison design as the ratio of the average between object variance of the incomplete pairs. Denote the cyclic paired comparison design by $[g_1, g_2, \ldots, g_m]$ with $g_s=1$ or 0 according as set (s) is or is not included in the design. Then the efficiency of using one set of symmetrical pairs is

$$E_f=\frac{(n-1)^2}{2nb_0}$$

where

$$b_0=\sum_{l=1}^{(1/2)n-1}\left(\frac{1}{\lambda_l}\right)+\frac{1}{2\lambda(\frac{1}{2}n\lambda)} \qquad n \text{ even}$$

$$=\sum_{l=1}^{(1/2)(n-1)}\frac{1}{\lambda_l} \qquad n \text{ odd}$$

and

$$\lambda_l = 1 - \sum_{k=1}^{\frac{1}{2}(n-1)} g_k \operatorname{Cos} 2\pi\, (kl/n) - (-1)^l \tfrac{1}{2}\, g\, (\tfrac{1}{2}\, n) \qquad n \text{ even}$$

$$= 1 - \sum_{k=1}^{\frac{1}{2}(n-1)} g_k \operatorname{Cos} 2\pi\, (Kl/n) \qquad n \text{ odd}$$

Some of these efficiencies are given below for comparison :—

n	r	*set*	E_f
5	2	1	·800
6	2	1	·714
7	2	1	·643
8	2	1	·583
9	2	1	·533
10	2	1	·491
11	2	1	·455
12	2	1	·423
13	2	1	·396
14	2	1	·371
15	2	1	·350

Thus there is a steady decrease in efficiency of symmetrical pairs as n increases. When sets of symmetrical pairs are considered it may happen that a smaller design is more efficient than a poorly chosen larger design. It may some times happen that an unconnected design is more efficient than a connected design.

Efficiencies of the design can also be compared by using the effective pair-wise precision. Under complete pairs with t treatments and n repetitions the rating for each treatment is obtained from $\left(\frac{t-1}{2} \times n\right)$ effective blocks. Under symmetrical pairs with n replications, the rating for any treatment is obtained from effective blocks, If σ_1^2 is the variance per block under complete pairs and σ_2^2 the variance per block under symmetrical pairs the efficiency of symmetrical pairs relative to complete pairs is given by $\frac{2}{t-1}\frac{\sigma_1^2}{\sigma_2^2}$. Estimates of σ_1^2 and σ_2^2 can be worked out from Thurstone-Mosteller models for the cases. It may also be noted that using designs of the same size and assuming experimental errors per pair to be of the same order the relative efficiency for comparison of any pair of the symmetrical pairs becomes $t-1$ showing that the efficiency increases with t.

If each symmetrical pair is presented to a number, n, of judges, the resulting preference data can be analysed by building a model on the lines of Thurstone and Mosteller. Thurstone postulated a subjective continuum over which sensations are jointly normally distributed with equal standard deviations and zero correlations between pairs of stimuli. Mosteller (1951*a*) shows that the assumption of zero correlations may be relaxed to an assumption of equal correlation, with no change of method. Without further loss of generality, we may let the scale of sensation continuum be so chosen that the difference of any stimulus response is normal with mean d and unit variance. The Thurstone-Mosteller model prohibits the declaration of ties. It was postulated by Glen and David (1960) that where the difference between two responses lies below a certain threshold, the judge will be unable to detect it; that is if the difference lies in an interval between ($-r$ and r) the judge will declare a tie. The same model is here modified for symmetrical pair.

Consider an experiment in symmetrical pairs involving t treatments and n replications and let X_i and X_j be single responses of a judge to the i-th and j-th stimuli. Then proceeding under the same assumptions as in Glen and David (1960) we get

$$r_{ij}' = [F^{-1}(a_{ij}) + F^{-1}(a_{ji})]/2 \qquad .. (1)$$

$$S_i' - S_j' = [F^{-1}(a_{ij}) - F^{-1}(a_{ji})]/2$$

where r_{ij}' and $S_i'-S_j'$ are respectively the experimental values of r and S_i-S_j resulting from a comparison of T_i and T_j, $2r$ being the interval centred at the origin of the distribution of X_i-X_j within which the judge will declare a tie and

$a_{ij} = p_{i \cdot ij} + p_{o \cdot ij}$ = proportion of preferences plus ties for i,

$a_{ji} = p_{j \cdot ij} + p_{o \cdot ij}$ = proportion of preferences plus ties for j and $j = (i+1) \bmod t$.

Given data of this form for each of the symmetrical pairs we would like to determine the least square estimates of r and S_i. Since this solution is not possible due to lack of independence we use angular transformations to estimate the values. For large samples we can assume

$$F(a)=1/2 \int_{-a}^{\pi/2} \text{Cos } y dy = (1+\sin . a)/2 \qquad \ldots(2)$$

where $-\pi/2 \leqslant a \leqslant \pi/2$. Then by following a similar procedure we find instead of (1)

$$r'_{ij}=[\sin^{-1}(2a_{ij}-1)+\sin^{-1}(2a_{ji}-1)]/2 \qquad \ldots(3)$$

$$S'_i-S'_j=[\sin^{-1}(2a_{ij}-1)-\sin^{-1}(2a_{2ij}-1) \qquad \ldots(4)$$

For large samples ρ_{ij}, the correlation between $\sin^{-1}(2a_{ij}-1)$ and $\sin^{-1}(2a_{ji}-1)$ is approximately,

$$\rho_{ij}=\sqrt{\frac{\pi_{i\cdot ij}\;\pi_{j\cdot ij}}{(1-\pi_{i\cdot ij})\;(1-\pi_{j\cdot ij})}} \qquad \ldots(5)$$

and hence

$$\text{Var }(r'_{ij})=(1+\rho_{ij})/2n \qquad \ldots(6)$$

$$\text{Var }(S_i'-S_j')=(1-\rho_{ij})/2n \qquad \ldots(7)$$

For $\pi_{0\,ij}\neq 0$ these variances will not, in general be homogeneous over the symmetrical pairs. However, in the absence of extreme comparisons, departures from homogeneity will be relatively small. It is expected that estimates obtained from an unweighted least square solution as in (2) will serve as good first approximations to the results of a weighted analysis.

Unweighted Analysis of Balanced Experiments

Let us assume that r is a common value which applies in all comparisons. Assume the variances (6) and (7) to be homogenous. To get initial estimates of the parameters r and $S_i(i=1, 2, \ldots\ldots, t)$ define the observations

$$G_{ij},\; H_{ij}\;[j=(i+1) \bmod t)]$$

as

$$r'_{ij}=G_{ij} \qquad \ldots(8)$$

and

$$S'_i-S'_j=H_{ij}. \qquad \ldots(9)$$

From the G_{ij} we obtain the least square estimates of r as

$$r^*=\Big(\sum_{s} G_{ij}\Big)/t \qquad \ldots(10)$$

From the H_{ij} we determine the least squares estimate S_s^* of S_i such that

$$Q_2 = \sum_{s} (S_i - S_j - H_{ij})^2$$

is a minimum for

$$S_i = S_i^*(i=2,\ldots,t)$$

and $$S^*_t = 0.$$

Q_2 expressed in matrix form is

$$Q_2 = (Y-XB)'\,(Y-XB)$$

where $$Y' = (H_{12}, H_{23}, \ldots, H_{t1}),$$

$$B' = (S_2, \ldots, S_t),$$

and X, at $(t-1)$ matrix containing 1's, -1's and 0's as described below.

Corresponding to each element in the vector, γ there is a row of X, while the column of X may be regarded as associated respectively with the elements of the vector B'. The row corresponding to H_{ij} has $+1$ in the column corresponding to S_i and -1 in the column corresponding to S_j. All elements not otherwise mentioned are zero. The required vector of least square estimates is

$$B^* = (X'\,X)^{-1}\,X'Y \text{ where } X'X \text{ is} \qquad \ldots(11)$$

given by

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \ldots(12)$$

This is circulant and symmetric and hence inversion is possible.

$$\text{Also, } X'Y = \begin{bmatrix} H_{23} + H_{21} \\ H_{34} + H_{32} \\ Ht_1 + Ht, t-1 \end{bmatrix} \qquad \ldots(13)$$

and hence

$$B^* = (S^*_2\; S^*_3\; S_t)'. \qquad \ldots(14)$$

The above analysis can be extended to the case of h judges with different thresholds. A parameter r_k is associated with the k-th judge, $(k=1, 2, \ldots, h)$. The estimate r_k is found to be the mean of the G_{ij} values pertaining to the k-th judge. Any estimate

S_i^* ($i=1, 2,\ldots, t$) for the overall experiment is the *AM* of the estimates S^*_i for the separate judges. The analysis, therefore, consists of applying the procedure of this section to the data for each judge, or group of judges for which a r_k is postulated. Separate r^*_k and S^*_i are obtained, the latter being pooled as above.

Weighted Analysis of Balanced Experiments

From (6), (7), (8) and (9) we may write

$$\text{Var}\,(G_{ij})=(1+\rho_{ij})/2n \qquad \ldots(15)$$

$$\text{Var}\,(H_{ij})=(1-\rho_{ij})/2n \qquad \ldots(16)$$

If there exists heterogeneity in these variances each G_{ij} and H_{ij} must be weighted in proportion to the inverse of its variance. This requires estimates of ρ_{ij}. Proceeding along the same lines as in reference (2) this estimate is found to be

$$r_{ij}=\sqrt{\frac{(1-a^*_{ij})\,(1-a^*_{ji})}{a^*_{ij}\,A^*_{ji}}} \qquad \ldots(17)$$

Also

$$\text{Var}\,(G_{ij})=(1+r_{ij})/2n$$

and $$\text{Var}\,(H_{ij})=(1-r_{ij})/2n. \qquad \ldots(19)$$

The weights for G_{ij} and H_{ij} are respectively,

$$V_{ij}=1/(1+r_{ij}) \qquad \ldots(18)$$

and $$W_{ij}=1/(1-r_{ij}).$$

Let us denote the estimates to be obtained from the weighted analysis by r^{**} and $S_i^{**}(i=2,\ldots, t)$ with $S_1^{**}=0$. The quantity that is minimised in determining r^{**} is

$$\sum_s V_{ij}\,(r-G_{ij})^2$$

and leads to the solution,

$$r^{**}=\sum_s V_{ij}\;G_{ij}/\sum_s V_{ij} \qquad \ldots(20)$$

For S_i^{**}, Q_2 is replaced by :

$$Q_{2w}=\sum_s W_{ij}\;(S_i-S_j-H_{ij})^2.$$

Representing Q_{2w} in quadratic form as:

$Q_{2w}=(Y-XB)'\;W\;(Y-XB)$ where W is the diagonal matrix of W_{ij}'s defined in (19), the vector of S_i^{**} is given by

$$B^{**}=(X'WX)^{-1}\,X'WY \text{ where}$$

$$(X'WX)=\begin{bmatrix} +W_{12}+W_{23}, & -W_{23}, & 0 & 0 & 0 \\ -W_{23}, & W_{23}+W_{34}, & -W_{34}, & 0 & \dots & 0 \\ \cdot\cdot & \dots & \dots & \dots & \dots \\ \cdot\cdot & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & -W_{t-1,t} & W_{t-1,t}+W_{t,1} \end{bmatrix} \quad ...(21)$$

and

$$(X'WY)=\begin{bmatrix} \sum\limits_{h\epsilon i_2}^{1,3} W_2h & H_{2h} \\ \sum\limits_{h\epsilon I_3} W_3h & H_{3h} \\ \sum\limits_{h\epsilon i_4} W_4h & H_{4h} \\ \dots & \dots \\ \sum\limits_{h\epsilon i_5} W_th & H_{th} \end{bmatrix} \quad ...(22)$$

where $\sum\limits_{h\,\epsilon\, I_h}$ is the summation of the associates $k(=k-1$ and $k+1)$.

But one cannot write down a simple general form for $[X'WX]^{-1}$.

Since $(X'WX)$ is necessarily symmetric the inverse can be found by the Doolittle method.

Now we proceed to estimate the variance and covariance of the S_i^{**} $(i=1, 2, \dots\dots, t)$ and variance r^{**} using large sample theory. From (15) and (18) we may write $\hat{\text{Var}}\,(G_{ij})=1/(2n\ V_{ij})$ which in conjunction with (20) yields $\hat{\text{Var}}\,(r^{**})=\sum\limits_{s}(V_{ij}^2/2n\ V_{ij})/$ $(\sum\limits_{s} V_{ij}^2)=\dfrac{1}{2n\sum\limits_{s} V_{ij}}$ since the comparisons on the pairs are made independently. In the same fashion we may write :

$$\hat{\text{Var}}\,(H_{ij})=\frac{1}{2n\ W_{ij}} \text{ so that}$$

$\widehat{\text{Var}}\,(W_{ij}\,H_{ij}) = W_{ij}/2n$. Combining this result with (22) we have the estimated variance matrix associated with the vector :

$$B^{**} \text{ as } \underset{B^{**}}{\Sigma} = (X'WX)^{-1} \underset{X'WY}{\Sigma} (X'WX)^{-1} = (X'WX)^{-1}/2n.$$

If, r^*, r^{**} and the respective S^*_i and S_i^{**} are not in close agreement, one may conclude that there is heterogeneity in Var (G_{ij}) and in Var (H_{ij}) and the weighted analysis should be carried out. In such a case one may use the r^{**} and the S_i^{**} to determine an improved set of weights and repeat the procedure described above. This iterative procedure may be carried through till two successive stages give approximately the same result. Variances and covariances of the estimates should be obtained by the iterative procedure only at the last stage.

We can also consider the weighted analysis of balanced experiments when a different r is postulated for each of R judges. The estimate r_k^{**} associated with the k-th judge ($k=1, 2, \ldots\ldots, R$) is found to be a weighted AM of the G_{ij} values pertainingto the k-th judge. In determining the S_i^{**} the matrices $X'WX$ and $X'WY$ are shown to be respectively the sums of the matrices (21) and (22) for the separate judges.

Testing Validity of the Model

Using the test of goodness of fit, we compare the observed numbers in each category with the expected numbers derived from the solution. If the discrepancies are small we consider the solution to be internally consistent. For the binomial situation in which ties are not allowed, a test has been presented by Mosteller (1951). When ties are admitted, a trinomial distribution is associated with each pair. In our case, the data constitute t independent trinomial distributions since the comparisons on the pairs are made independently. Now we proceed to calculate the expected numbers in each of the categories. We denote those values of r and S_i by r'' and S_i'' ($i=2,\ldots\ldots, t$) with $S_i''=0$. Using these values, we determine the expected values of a_{ij} and a_{ji} as a''_{ji} and a''_{ji} satisfying the relations,

$$a_{ij}'' = [1+\sin(r''+S_i''-S_j'')]/2 \text{ and}$$
$$a_{ji}'' = [1+\sin(r''-S_i''+S_j'')]/2.$$

Let $n_i''._{ij}$ be the expected number of preferences for T_i, $n_j''._{ij}$ be the expected number of preferences for T_j and $n''_{0\cdot ij}$ be the expected number of ties when T_i and T_j are compared, where

$n_{i\cdot i}''+n_{j\cdot ij}''+n_{0\cdot ij}''=n$. But $n_{i\cdot j}''+n_{0\cdot ij}''=na_{ij}''$ and $n_{j\cdot ij}''+n_{0\ ij}''=na_{ji}''$.

From the above relations:

$$x^2 = \sum_s \Big[(n_{i\cdot ij} - n_{i\cdot ij}'')^2/n_{i\cdot ij}'' + (n_{j\cdot ij} - n_{j\cdot ij}'')^2/n_{j\cdot ij}'' + \frac{(_{0\cdot ij} - n_{0\cdot ij}'')^2}{n_{0\cdot ij}''} \Big].$$

For large values of the expected numbers the quantity above is distributed approximately as X^2 with degrees of freedom determined as below. There are t symmetrical pairs which yield two independent observations each. From the data we have estimated t parameters *viz.* r and $(t-1)$ values of S_i $(i=2, \;\; , t)$. Thus the degrees of freedom for X^2 are $2t \cdot t = t$.

For the case in which R judges are involved, a separate r is assumed for each judge and expected numbers obtained. A separate X^2 test is made for each judge, the degrees of freedom being t. If an overall test is desired, the judges being considered independent, the sum of squares may be pooled over the R judges. In this case the degrees of freedom are $2Rt-(R+t-1)$.

An Illustrative Example

This model is illustrated with the help of an experiment conducted in the quality testing laboratory of the Indian Agricultural Research Institute, New Delhi. Five improved varieties of wheat namely: 1. Kalyan Sona, 2. Sonalika, 3. Choti Lerma, 4. Sharbati Sonera and 5. N.P. 718 were used for the test. Five judges were selected by the duo-trio test. Then the symmetrical pairs were presented at random to each judge in a random order. A preferred variety is given the score 1 and the non-preferred the score 0. In case of ties each variety was given the score 1/2. The experiment was replicated thrice. The results pooled over the judges are given below:

Pair	$n_{i\cdot ij}$	$n_{0\cdot ij}$	$n_{j\cdot ij}$	Total
1, 2	5	3	7	15
2, 3	6	4	5	15
3, 4	6	3	6	15
4, 5	4	3	8	15
5, 1	6	2	7	15
Total	27	15	33	75

Using these data we have computed the vector $X'Y$ and hence the vector $S_i^*(i=2, 3, 4, 5)$ with $S_1^*=0$.

$$\begin{bmatrix} S_2^* \\ S_3^* \\ S_4^* \\ S^*_5 \end{bmatrix} = 1/5 \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0{\cdot}299 \\ -0{\cdot}116 \\ -0{\cdot}281 \\ -0{\cdot}213 \end{bmatrix} = \begin{bmatrix} 0{\cdot}100 \\ -0{\cdot}099 \\ -0\ 183 \\ 0{\cdot}015 \end{bmatrix}$$

Thus $S_i^*=0$, $S_2^*=0{\cdot}100$, $S_3^*=-0{\cdot}099$, $S_4^*=-0{\cdot}183$ and $S_5^*=0.015$. Hence the ratings are:

1. Sonalkia, 2. NP718, 3. Kalyansona, 4. Chotilerma and 5. Sharbati Sonera.

Using large sample theory the standard error for r and S_i's is $\sqrt{1/2n}=0{\cdot}182$. Thus the varieties are not differing significantly among themselves in palatability. The ties are also due to chance.

REFERENCES

1. David, H.A. (1963) : The structure of cyclic paired comparison designs. Journal of the Australian Mathematical Society, Vol. III ; Part I. pp. 117-127.

2. Glen, W. A. and David H.A. (1960) : Ties in paired comparison experiments using a modified Thurstone-Mosteller Model. Biometrics 16, 86-109.

3. Mc Keon, J. J. (1960) : Some cyclical Incomplete Paired comparison designs. Tech. Rep. No. 24. Psychometric Lab. University of North Carolina.

4. Mosteller (1951) : Remarks on the method of paired comparisons III Psychometrika 16, 203-18.

5. Rao, P.V. and Kuppor, L.L. (1967) : Ties in paired comparison experiments—a generalisation of the Bradley—Terry Model. JASA 62-194-204.

6. Sadasivan, G. (1970) : Designs for paired and triad comparisons J.I.S.A.S., 22, 32-48.

7. Sadasivan, G. and Rai, S.C. (1973). : A Bradley Terry Model for standard comparison pairs., Sankhya Series B.

8. Sadasivan, G. and S.S. Sundaram (1971). : Some methods of fractionating paired comparisons. (Under publication Australian Journal of Statistics).