ON SOME UNBIASED PRODUCT TYPE STRATEGIES

By

P. C. GUPTA AND DHIRESH ADHVARU

South Gujarat University, Surat

(Received : September, 1979)

SUMMARY

The product strategy H_p=(Srs, $\bar{y}_p$) where $\bar{y}_p=\bar{y}_s\ \bar{x}_s/\bar{x}$, $\bar{x}$, proposed by Murthy [5] is biased. In this paper three unbiased product type strategies obtained by combining H_p=(Srs, $\bar{y}_p$) and H_{p_n}=(Srs, $\bar{y}_{p_n}$) where $\bar{y}_{p_n}=\bar{p}_s/\bar{x}$; $\bar{p}_s=\sum_s x_i y_i/n$, are proposed. Expression for their variances are obtained in section 3 and some empirical comparisions have been carried out in the last section.

1. INTRODUCTION

In survey sampling when a relationship between the study variable and the auxiliary variable is known to exist, there are two main streams for utilising the known auxiliary information in efficient as well as practical manner. One is to use the available auxiliary information at the design stage while the other is to use it at the estimation stage to construct more efficient strategies than H=(Srs, $\bar{y}_s$) (sampling design alongwith the estimator being called a "Strategy") where srs means simple random sampling without replacement and $\bar{y}_s$ is the sample mean of the study variable y. We note that H is optimal if the prior knowledge is symmetric with respect to the lebels and in such cases incorporation of auxiliary information attached to the labels does not improve upon it. Apart from such situations auxiliary information can be used at any of the stages, either design or estimation or both. At the design stage it would usually be the choice of sampling design such that the selection probabilities depend suitably on the auxiliary characteristic as in various pps designs. Many procedures use auxiliary information at the estimation stage among them are the classical ratio and regression strategies H_R=(Srs. $\bar{y}_R$) and H_{1r}=(Srs, $\bar{y}_{1r}$) where $\bar{y}_R=\bar{y}_s\ \bar{x}/\bar{x}_s$ and $\bar{y}_{1r}=\bar{y}_s+b(\bar{x}-\bar{x}_s)$, $\bar{x}$ and b being population mean of the auxiliary characteristic x and sample regression coefficient of y on x respectively and $\bar{x}_s$ is the sample mean of x. Murthy [5] proposed the product strategy H_P=(Srs. $\bar{y}_p$) where $\bar{y}_p=\bar{y}_s\, \bar{x}_s/\bar{X}$, as complementary to the ratio strategy H_R from efficiency point of view. Under different conditions both of them use auxiliary information in an efficient manner but both the strategies are biased. One line to make the strategy unbiased is to modify the sampling procedure such that the same estimator becomes unbiased and the other is to modify the form of the estimator by correcting it for the bias. Hartley and Ross [3] corrected the mean of the ratio strategy H_{R_n}=(Srs, $\bar{y}_{R_n}$) where $\bar{y}_{R_n}=\bar{r}_s\, \bar{X}$; $\bar{r}_s=n^{-1}\sum_s y_i/x_i$, for its bias and obtained the unbiased ratio-type strategy H_{HR}=(Srs, $\bar{y}_{HR}$) where $\bar{y}_{HR}=\bar{X}\,\bar{r}_s+\frac{n(N-1)}{N(n-1)}(\bar{y}_s-\bar{x}_s\,\bar{r}_s)$. In the present paper three unbiased product type strategies H_p^{*}, H_p^{**} and H_p^{***} are proposed combining the product of mean and mean of the product strategies H_p=(Srs, $\bar{y}_p$) and H_{P_n}=(Srs, $\bar{y}_{P_n}$) and explicit expressions for their variances are obtained.

2. Unbiased Product-Type Strategies

Consider the mean of the product strategy H_{P_n}=(Srs, $\bar{y}_{P_n}$) where $\bar{y}_{P_n}=\bar{p}_s/\bar{X}$; $\bar{p}_s=\sum_s y_i\, x_i/n$. This is a biased strategy.

To estimate its bias we prove the following lemma.

Proposition 2.1. The unbiased estimator of the bias in H_{P_n} is

$$b(H_{P_n})=\frac{N-1}{N}\cdot\frac{n}{n-1}(\bar{y}_{P_n}-\bar{y}_p) \qquad \ldots(2.1)$$

Proof : We have

$$B(H_{P_n})=E(H_{P_n})-\bar{Y}$$

$$=\sum_1^N (y_i\, x_i/N\bar{X})-\bar{Y}$$

$$=\frac{N-1}{N}\,\frac{S_{yx}}{\bar{X}} \qquad \ldots(2.2)$$

here

$$S_{yx}=\sum_1^N (y_i-\bar{Y})(x_i-\bar{X})/(N-1)$$

Hence from the theory well known, the unbiased estimator of (2.1) is

$$b(H_{P_n})=\frac{N-1}{N}\,\frac{s_{yx}}{\bar{x}}$$

$$=\frac{N-1}{N}\,\frac{1}{(n-1)\,\bar{x}}\,(\sum_{s} y_i x_i - n\bar{y}_s\bar{x}_s)$$

$$=\frac{n(N-1)}{N(n-1)}\,(\bar{y}_{P_n}-\bar{y}_p)$$

From the above proposition the following theorem is straight forward.

Theorem 2.1. The strategy

$$H_p^{*}=(\text{Srs},\ \bar{y}_{P_n})$$

where

$$\bar{y}_P=\frac{n(N-1)}{N(n-1)}\bar{y}_P-\frac{N-n}{N(n-1)}\bar{y}_{P_n} \qquad \ldots(2.3)$$

is unbiased.

Again for the bias of H_{P_n} we prove the following

Proposition 2.2. The bias in H_{P_n} is also given by

$$B(H_{P_n})=\frac{n(N-1)}{N-n}\quad E(\bar{y}_p-\bar{y}_S) \qquad \ldots(2.4)$$

Proof: We have, from (2.1),

$$B(H_{P_n})=\frac{N-1}{N}\,\frac{S_{yx}}{\bar{x}}$$

$$=\frac{n(N-1)}{N-n}\,\frac{Cov(\bar{y}_s,\ \bar{x}_s)}{\bar{x}}$$

$$=-\frac{n(N-1)}{N-n}\,B(H_P)$$

$$=\frac{n(N-1)}{N-n}\,E(\bar{y}_p-\bar{y}_s),$$

Hence the following theorem is straight forward.

Theorem 2.2. The strategy

$$H_p^{**}=(\text{Srs},\ \bar{y}_p^{**})$$

where

$$\bar{y}_p^{**}=\bar{y}_{P_n}-\frac{n(N-1)}{N-n}\,(\bar{y}_p-\bar{y}_s) \qquad \ldots(2.5)$$

is unbiased,

And noting that

$$B(\bar{y}_p)=\frac{N-n}{n(N-1)}E(\bar{y}_{P_n}-\bar{y}_s)$$

We have

Theorem 2.3. The strategy

$$H_P^{****}=(\text{Srs}, \bar{y}_P^{***})$$

where
$$\bar{y}_p^{***}=\bar{y}_p-\frac{N-n}{n(N-1)}(\bar{y}_{p_n}-\bar{y}_s)$$

is unbiased.

Remark 2.1. In keeping with Hartley-Ross unbiased strategy we can put $\bar{y}_p^{*}$, $\bar{y}_p^{**}$ and $\bar{y}_p^{***}$ in the following form

$$\bar{y}_p^{*}=\frac{n(N-1)}{N(n-1)}\cdot\frac{\bar{y}_s\bar{x}_s}{\bar{x}}-\frac{N-n}{N(n-1)}\cdot\frac{\bar{p}_s}{\bar{x}}$$

$$\bar{y}_P^{**}=\frac{n(N-1)}{N-n}\frac{\bar{y}_s}{\bar{x}}(\bar{x}-\bar{x}_s)+\frac{\bar{p}_s}{\bar{x}}$$

and

$$\bar{y}_P^{***}=\frac{\bar{y}_s\bar{x}_s}{\bar{x}}-\frac{N-n}{n(N-1)}\frac{\bar{y}_s\bar{x}_s+\bar{p}_s}{\bar{x}}$$

Remarks 2.2. We note here in passing that Adhvaryu [1] obtained the unbiased product type strategy H_P^{*} by a different approach but did not discuss it further.

3. The Variances of the Proposed Strategies

In this section we shall obtain the variances of the proposed strategies and their consistent estimators. For that we give the following results. Proofs involve some routine algebra and hence omitted to save space.

Proposition 3.1. The variance of H_{P_n} is

$$V(H_{P_n})=\frac{1-f}{n}\frac{S_P^2}{\bar{x}^2} \qquad \text{...(3.1)}$$

where

$$S_P^2=\sum_1^N (P_i-\bar{p})^2/N-1\,;\ \bar{p}=\sum_1^N p_i/N;\ p_i=y_ix_i$$

Proposition 3.2. The covariance between H_P and H_{P_n} is

$$\text{Cov } (H_P . H_{P_n}) = \frac{1-f}{n\bar{x}} (S_{yp} + RS_{xp}) \quad \text{...(3.2)}$$

where

$$S_{zp} = \sum_{1}^{N} (z_i - \bar{z})(p_i - \bar{p}) / (N-1) \text{ for } z = x, y$$

and $$R = \bar{y}/\bar{x}.$$

Proposition 3.3. The covariance between H and H_P is

$$\text{Cov}(H, H_P) = \frac{1-f}{n} (S_y^2 + RS_{yx}) \quad \text{...(3.3)}$$

where expression for S_y^2 is same as that of S_{yx} for $y=x$.

Proposition 3.4. The covariance between H and H_{P_n} is

$$\text{Cov}(H, H_{P_n}) = \frac{1-f}{n} \frac{S_{yp}}{\bar{x}} \quad \text{...(3.4)}$$

From the above proposition, after some routine algebra and reorganization of terms following theorem is obvious.

Theorem 3.1. The variances of the proposed strategies H_P^*, H_P^{**} and H_P^{***} are

$$V(H_P^*) = \frac{n(1-f)(N-1)^2}{N^2 (n-1)^2} (S_y^2 + R^2 S_x^2 + 2RS_{yx}) + \frac{(1-f)^3}{n(n-1)} \frac{S_p^2}{\bar{x}^2} - \frac{2(1-f)^2 (N-1)}{N(n-1)^2} \frac{S_{yp} + RS_{xp}}{\bar{x}} \quad \text{...(3.5)}$$

$$V(H_P^{**}) = \frac{1-f}{n} \frac{S_p^2}{\bar{x}^2} + \frac{n(N-1)^2}{N(N-n)} R^2 S_x^2 - \frac{2(N-1)}{N} \frac{RS_{xp}}{\bar{x}} \quad \text{...(3.6)}$$

and

$$V(H_P^{***}) = \frac{1-f}{n} \Big[(a+1)^2 S_y^2 + 2R(a+1) S_{yx} + R^2 S_x^2 + \frac{a^2 S_p^2}{\bar{x}^2} - \frac{2a}{\bar{x}} ((a+1) S_{yp} - S_{xp}) \Big] \quad \text{...(3.7)}$$

where $$a = \frac{N-n}{n(N-1)}.$$

Remark 3.1 : The consistent estimators of the variances (3.5), (3.6) and (3.7) can be obtained by replacing R, S_y^2, S_x^2, S_p^2, S_{yp} and and S_{xp} by the sample statistics R, s_y, s_x, s_P^2 and s_{xp} where $R=\bar{y}_s/\bar{x}_s$, $s_{uv}=(n-1)^{-1}\sum_S (u_i-\bar{u}_s)(v_i-\bar{v}_s)$ and $s_u^2=s_{uv}$ i f $u=v$ respectively in the expressions.

4. Some Empirical Comparisons

The empirical efficiency of the usual unbiased strategy H, product strategy H_P and the three proposed strategies H_P^*, H_P^{**} and H_P^{***} are compared in this section by considering three populations. Population I consists of the data on quit (x) and unemployment (y) rate in the U.S. manufacturing between 1960-72 (Damodar Gujarati [2] p. 59) and population II and III consist of data on per capita consumption (y) and deflated prices (x) of two varieties of meat *viz.* beef (II) and lamb (III).

For the above mentioned three populations the summary table of values of necessary population parameters is given below :

Techniques *Jour. Amer. Assn.* **53, 98-101**

Parameter / Population	R	$\bar{p}$	ρ	S_y^2	S_x^2	$\bar{x}$
I	2.651	9.892	−0.8081	2.191	0.2754	1.912
II	0.94	5573.722	−0.780	87.775	137.621	77.362
III	0.0615	329.387	−0.752	0.224	56.804	73.450

$\bar{y}$	S_{xy}	S_{xp}	S_{yp}	S_p^2
5.069	−0.627	0.148	1.074	12.555
72.625	−86.110	3771.770	−704.980	308857.360
4.518	−2.683	44.772	5.432	629.971

and the calculated variances of usual unbiased strategy H, produc strategy H_P and the three proposed strategies are given in the following table :

Variance / Population	N	n	$V(H)$	$MSE(H_p)$	$V(H_P^*)$	$V(H_P^{**})$	$V(H_P^{***})$
I	13	5	0,269	0.098	0.102	13.409	0.102
II	16	4	16.458	8.905	38.046	493.981	14.850
III	16	4	0.042	0.021	0.029	1.079	0.072

So it will be observed here that for all the populations the product strategy H_P gives the least variance but it is biased. Among the unbiased strategies two proposed strategies H_P^* and H_P^{***} are having almost the same variance for population I. While for the second population H_P^{***} comes out to be better than the rest and H_P^{***} fares well as compared to the rest for the third population. The strategy H_P^{**} does not perform well among the proposed strategies for all the three populations considered. Thus H_P^* and H_P^{***} come out to be uniformly superior to H_P^{**} as a result of this empirical study which is, of course, of limited scope.

References

[1] Adhvaryu, Dhiresh (1974) : 'A note on some unbiased estimators' (Abstract *Inst. of Math. Stat. Bull.* 3, 171.

[2] Gujarati, D. (1978) : *Basic Econometrics,* McGraw Hills Co. Fub. New York.

[3] Hartley, H.O. and Ross, A. 1954 : Unbiased ratio estimators. *Nature,* 174, 270-271.

[4] Maddala, G.S. (1977) : *Econometrics* McGraw Hills Pub. Co. New York

[5] Murthy, M.N. (1964) : Product method of estimation. *Sankhya,* 26, A, 69-74.

[6] Raj, Des (1958) : On the relative accuracy of some sampling techniques. *Jour. Amer. Stat. Assn.,* 53, 98-101.