AN EXAMINATION OF VARIOUS INEQUALITY RELATIONS AMONG PARAMETERS OF THE BALANCED INCOMPLETE BLOCK DESIGN

By K. Kishen

Department of Agriculture, U.P., Lucknow

AND

C. R. Rao

Statistical Laboratory, Calcutta

1. Introduction

The balanced incomplete block design first introduced in agricultural experimentation by Yates (1936) is a solution of the combinatorial problem of arranging v varieties (or treatments) in b blocks of k $(< v)$ varieties (or treatments) each such that every variety (or treatment) occurs in r blocks and every pair of varieties (or treatments) occurs in λ blocks. From the combinatorial point of view, it is a λ-2-k tactical configuration which is a particular case of the complete λ-μ-k configurations, with $\mu = 2$ (Bose, 1939). The five parameters v, b, k, r, λ are not all independent but satisfy the two well-known equations

$$bk = vr \tag{1.1}$$

$$r(k-1) = \lambda(v-1) \tag{1.2}$$

When it is possible to arrange a balanced incomplete block design in r sets of n blocks each such that each set constitutes a complete replication, the design is called a resolvable balanced incomplete block design (Bose, 1942). Obviously, therefore,

$$v = nk \tag{1.3}$$

$$b = nr \tag{1.4}$$

Various inequality relations among parameters of the balanced incomplete block design, regardless of resolvability, and the resolvable balanced incomplete block design have been derived by Fisher (1940), Bose (1942) and Nair (1943). Considering, first, the balanced incomplete block design (irrespective of its resolvability), the inequalities given by Fisher and Nair are respectively

$$b \geqslant v \tag{1.5}$$

$$b \geqslant 1 + \frac{k(r-1)^2}{r-k+\lambda(k-1)} \tag{1.6}$$

In Section 2, a new proof has been given of Fisher's inequality (1.5).

In Section 3, the following two new inequalities have been derived:

$$b \geqslant v + r - k \tag{1.7}$$

$$b \geqslant 1 + \frac{(v-k)(b-r-1)^2}{(b-v-r+k)+(b-2r+\lambda)(v-k-1)} \tag{1.8}$$

It has been shown that whilst the inequality (1.7), like the inequality (1.6), gives an arithmetically closer limit for 'b' than the inequality (1.5), these are stronger than Fisher's inequality only in the arithmetical sense and not for seeking the combinatorial solution to our problem inasmuch as no integral numbers b, v, k, r, λ satisfying the equations (1.1) and (1.2) exist such that

$$v \leqslant b < I \tag{1.9}$$

where I denotes the right-hand side of any of the two inequalities (1.6) and (1.7). Consequently, $b \geqslant v$ implies that $b \geqslant I$. None of the two inequalities is, therefore, combinatorially more stringent than Fisher's inequality.

It has also been demonstrated that whilst the inequality (1.8) gives an arithmetically closer limit for 'b' than either (1.5) or (1.6) for designs for which $v \geqslant 2k$, it is combinatorially not stronger than Fisher's inequality.

For the resolvable balanced incomplete block design, Bose (1942) and Nair (1943) have obtained the following inequalities:

$$b \geqslant v + r - 1 \tag{1.10}$$

$$b \geqslant \frac{rk(r-1)}{r-k+\lambda(k-1)} \tag{1.11}$$

It has been shown that here also no integral numbers $b\,(=nr)$ $v\,(=nk)$, k, r, λ satisfying (1.1) and (1.2) exist such that

$$v \leqslant b < I' \tag{1.12}$$

and

$$v + r - 1 \leqslant b < I'', \tag{1.13}$$

where I' denotes the right-hand side of (1.10) and I'' that of (1.11). It, therefore, follows that Bose's inequality (1.10) is combinatorially not more stringent than Fisher's inequality, and that Nair's inequality (1.11) is combinatorially not more stringent than Bose's inequality, and consequently also than Fisher's inequality.

2. Proof of Fisher's Inequality $b \geqslant v$

A new proof of Fisher's inequality $b \geqslant v$ has been given by Bose (1949). Another proof of it will now be given.

Let us suppose that only the block totals in a balanced incomplete block design represented by $B_1, B_2, \ldots, B_b$ are known, and not the individual values (or yields) for each plot separately. Let m be the true mean, and $t_1, t_2, \ldots, t_v \left(\sum_{i=1}^{v} t_i = 0 \right)$ the varietal effects, block effects being ignored.

Then

$$E(B_i) = km + \sum_{r=1}^{k} t_{i_r} \; (i = 1, 2, \ldots, b) \tag{2.1}$$

where $t_{i_1}, \ldots, t_{i_r}$ are the varietal effects corresponding to the k varieties occurring in the i-th block, are the b independent observational equations in v independent unknowns. We now proceed to show that all the treatment contrasts are estimable, so that $b \geqslant v$ necessarily. These estimates of treatment contrasts are known as inter-block estimates.

Let $\sum_{s=1}^{r} B_{i_s}$ denote the sum of totals for blocks containing variety i, and similarly, let $\sum_{s=1}^{r} B_{j_s}$ denote the sum of totals for blocks containing variety j.

Then

$$E\left(\sum_{s=1}^{r} B_{i_s} \right) = kmr + (r - \lambda)\, t_i \tag{2.2}$$

and

$$E\left(\sum_{s=1}^{r} B_{j_s} \right) = kmr + (r - \lambda)\, t_j \tag{2.3}$$

Consequently, from (2.2) and (2.3), we obtain

$$E\left[\frac{\sum_{s=1}^{r} B_{i_s} - \sum_{s=1}^{r} B_{j_s}}{r - \lambda} \right] = t_i - t_j \tag{2.4}$$

Hence all the treatment contrasts are estimable, whence $b \geqslant v$ necessarily.

3. Two New Inequalities for the Balanced Incomplete Block Design

We now proceed to derive the two new inequalities (1.7) and (1.8) among parameters of the balanced incomplete block design irrespective of resolvability.

(3.1) *The inequality* (1.7).—It is well known that if there exists a combinatorial solution with parameters v, b, k, r, λ, there also exists the complementary solution with parameters v', b', k', r', λ' where

$$v' = v,\ b' = b,\ k' = v - k,\ r' = b - r,\ \lambda' = b - 2r + \lambda \tag{3.1}$$

Now, using Fisher's inequality (1.5) for parameters of the complementary solution, we readily obtain

$$b \geqslant v + r - k \tag{3.2}$$

Since $b \geqslant v$ or $r \geqslant k$, it follows that

$$v + r - k \geqslant v,$$

so that this inequality gives an arithmetically closer limit for 'b' than Fisher's inequality.

(3.2) *The inequality* (1.8).—Using Nair's inequality (1.6) for parameters of the complementary design given above, we readily obtain the new inequality

$$b \geqslant 1 + \frac{(v-k)(b-r-1)^2}{(b-v-r+k)+(b-2r+\lambda)(v-k-1)} \tag{3.3}$$

To prove that this inequality is arithmetically stronger than Fisher's inequality, we have only to show that

$$\frac{(v-k)(b-r-1)^2}{(b-v-r+k)+(b-2r+\lambda)(v-k-1)} \geqslant v - 1.$$

Using the relations (1.1) and (1.2), it would appear that this inequality holds if

$$\frac{(v-1)(vr-kr-k)^2}{k[(r-k)(v-1)+r(v-k-1)^2]} \geqslant v - 1,$$

i.e., if $v^2r^2 + k^2r^2 - 2vkr^2 - vkr + vk^2 - v^2kr - k^3r + 2vk^2r \geqslant 0$,

i.e., if $(r-k)[k(kr-v) + vr(v-2k)] \geqslant 0$,

which is so for designs for which $v \geqslant 2k$, since $r \geqslant k$, and $kr - v = (\lambda - 1)v + (r - \lambda) > 0$ since $\lambda \geqslant 1$ and $r > \lambda$.

We shall now demonstrate that this inequality also gives an arithmetically closer limit for 'b' than Nair's inequality (1.6) for designs for which $v \geqslant 2k$.

For this it is enough to show that

$$\frac{(v-k)(b-r-1)^2}{(b-v-r+k)+(b-2r+\lambda)(v-k-1)} \geqslant \frac{k(r-1)^2}{r-k+\lambda(k-1)},$$

which is so if

$$\frac{(v-1)(vr-kr-k)^2}{k[(r-k)(v-1)+r(v-k-1)^2]} \geqslant \frac{k(r-1)^2(v-1)}{(v-1)(r-k)+r(k-1)^2},$$

i.e., if

$$v^3r^2 - 4v^2kr^2 - v^2kr + 2vk^2r - v^3kr + 4v^2k^2r + v^2k^2 - 2vk^3$$
$$+ 6vk^2r^2 - 4k^3r^2 + 4k^4r - 6vk^3r \geqslant 0,$$

i.e., if

$$(v-2k)(r-k)[vr(v-2k)+k(2kr-v)] \geqslant 0,$$

which is so since $r \geqslant k$, and $2kr > kr > v$, and $v \geqslant 2k$ by assumption.

4. Examination of the Stringency of the Inequalities for the Balanced Incomplete Block Design

We shall now show that the inequalities (1.6), (1.7) and (1.8) are combinatorially not more stringent than the Fisherian inequality.

(4.1) *The inequality* (1.6).—If possible, let there exist an integral solution for v, b, k, r, λ satisfying the equations (1.1) and (1.2) such that

$$v \leqslant b < 1 + \frac{k(r-1)^2}{r-k+\lambda(k-1)}.$$

Replacing 'b' by $\frac{vr}{k}$, this gives

$$\frac{vr-k}{k} < \frac{k(r-1)^2(v-1)}{(v-1)(r-k)+r(k-1)^2}.$$

Simplifying, we obtain

$$r(r-k)(v-k)^2 < 0,$$

or $r < k$, which is contrary to the assumption that $r \geqslant k$.

Hence $b \geqslant v$ implies that

$$b \geqslant 1 + \frac{k(r-1)^2}{r-k+\lambda(k-1)},$$

and Nair's inequality is thus combinatorially not more stringent than Fisher's inequality.

(4.2) *The inequality* (1.7).—Suppose, if possible, there exist variable integers v, b, k, r, λ satisfying (1.1) and (1.2) such that

$$v \leqslant b < v + r - k.$$

We, therefore, obtain

$$vr/k < v + r - k,$$

or

$$(v-k)(r-k) < 0,$$

or $r < k$, which contradicts the assumption that $r \geqslant k$. Hence this inequality is also combinatorially not more stringent than Fisher's inequality.

(4.3) *The inequality* (1.8).—If possible, let there exist variable integers v, b, k, r, λ satisfying (1.1) and (1.2) such that

$$v \leqslant b < 1 + \frac{(v-k)(b-r-1)^2}{(b-v-r+k)+(b-2r+\lambda)(v-k-1)}.$$

This gives

$$\frac{vr-k}{k} < \frac{(v-1)(vr-kr-k)^2}{k[(r-k)(v-1)+r(v-k-1)^2]}.$$

Simplifying, we obtain

$$(vr-k)(v-k-1)^2 < (v-1)(v^2r+k^2r-2vkr - vk+2k^2-vr+k),$$

or

$$k^2(r-k) < 0,$$

or $r < k$, again contradicting the assumption $r \geqslant k$. Hence $b \geqslant v$ implies

$$b \geqslant 1 + \frac{(v-k)(b-r-1)^2}{(b-v-r+k)+(b-2r+\lambda)(v-k-1)},$$

and the latter inequality is combinatorially not more stringent than the former.

5. Examination of the Stringency of the Inequalities for the Resolvable Balanced Incomplete Block Design

We shall now prove that Bose's inequality (1.10) for the resolvable balanced incomplete block design is combinatorially not more stringent than Fisher's inequality, and that Nair's inequality (1.11) is combinatorially not more stringent than Bose's inequality, and consequently also than Fisher's inequality.

(5.1) *Bose's inequality* (1.10).—If possible, let there exist variable integers $v\,(=nk)$, $b\,(=nr)$, k, r, λ satisfying (1.1) and (1.2) such that

$$v \leqslant b < v+r-1 \tag{5.1}$$

From (1.2), we obtain

$$\lambda n + \frac{\lambda(n-1)}{k-1} = r \tag{5.2}$$

whence, since λn is integral, $n > 1$ and $k > 1$, it follows that $\frac{\lambda(n-1)}{k-1}$ is a positive integer. (5.3)

Now, from (5.1), we have

$$b < v + r - 1,$$

or

$$nr < \frac{r(k-1)}{\lambda} + r,$$

or

$$\frac{\lambda(n-1)}{k-1} < 1,$$

which is contrary to (5.3). Hence no variable integers exist satisfying the inequality

$$b < v + r - 1,$$

and consequently also the inequality

$$v \leqslant b < v + r - 1.$$

Hence $b \geqslant v$ implies that $b \geqslant v + r - 1$, and Bose's inequality, therefore, is combinatorially not more stringent than Fisher's inequality.

(5.2) *Nair's inequality* (1.11).—Suppose, if possible, there exists an integral solution for $v (= nk)$, $b (= nr)$, k, r, λ satisfying (1.1) and (1.2) such that

$$v + r - 1 \leqslant b < \frac{rk(r-1)}{r - k + \lambda(k-1)} \quad (5.4)$$

Since $b \geqslant v + r - 1$, we have, as before,

$$\frac{\lambda(n-1)}{k-1} \geqslant 1 \text{ and integral.} \quad (5.5)$$

Now, from (5.4), we obtain

$$b < \frac{rk(r-1)}{r - k + \lambda(k-1)},$$

or

$$n[r - k + \lambda(k-1)] < k(r-1),$$

or

$$(n-1)(r - \lambda - k) < 0,$$

or, since $n > 1$,

$$r - \lambda - k < 0,$$

or, using (5.2),

$$k\left[\frac{\lambda(n-1)}{k-1} - 1\right] < 0,$$

or

$$\frac{\lambda(n-1)}{k-1} < 1,$$

which is contrary to the result (5.5).

Hence $b \geqslant v + r - 1$ implies that $b \geqslant \dfrac{rk(r-1)}{r - k + \lambda(k-1)}$, and, therefore, Nair's inequality is combinatorially not stronger than Bose's inequality, and consequently also than Fisher's inequality.

6. Conclusion and Summary

The above investigation shows that for the parameters of the balanced incomplete block design, whether non-resolvable or resolvable, the known inequality relations derived by Bose, Nair and in Section 3 of the present paper are, from the combinatorial point of view, not more stringent than Fisher's inequality $b \geqslant v$, which consequently comes out as the fundamental inequality among the parameters of the balanced incomplete block design.

References

1. Bose, R. C. .. "On the construction of balanced incomplete block designs," *Annals of Eugenics*, 1939, **9**, 353–99.

2. ——— .. "A note on the resolvability of balanced incomplete block designs," *Sankhya*, 1942, **6**, 105–10.

3. ——— .. "A note on Fisher's inequality for balanced incomplete block designs," *Ann. Math. Stat.*, 1949, **20**, 619–20.

4. Fisher, R. A. .. "An examination of the different possible solutions for a problem in incomplete blocks," *Annals of Eugenics*, 1940, **10**, 52–75.

5. Nair, K. R. .. "Certain inequality relationships among the combinatorial parameters of incomplete block designs," *Sankhya*, 1943, **6**, 255–59.

6. Yates, F. .. "Incomplete randomized blocks," *Annals of Eugenics*, 1936, **7**, 121–40.