A PROPERTY OF THE OPTIMUM SOLUTION SUGGESTED BY PAULSON[1] FOR THE *K*-SAMPLE SLIPPAGE PROBLEM FOR THE NORMAL DISTRIBUTION*

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1. PAULSON obtained an optimum solution to the K-sample slippage problem for the normal distributions with common variances. He considered the case when one of the populations might have slipped as regards its mean to the right by a specified amount Δ $(\Delta > 0)$, the means of the remaining populations being equal. This restriction of only one population slipping is relaxed here and it is shown that this procedure is unbiased in the sense that the probability of incorrect choice never exceeds probability of correct choice among the $K + 1$ decisions namely:—

$$D_0 \text{ the decision that } m_1 = m_2 = \ldots . = m_k.$$

and

$$D_i \text{ the decision that } m_i = \text{Max } (m_1, m_2, m_3, \ldots, m_k)$$

where π_i is the i-th population distributed as $N\ (m_i, \sigma^2)$.

2. Without loss of generality we may confine our attention to a single observation from each of the populations, *i.e.*, x_i is a single observation from π_i and is independently normally distributed with mean m_i and common variance σ^2/n and an observation S independently distributed with the probability density function

$$f(S, \sigma) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{1}{2}nS^2/\sigma^2} \left(\frac{nS^2}{2\sigma^2}\right)^{(n-3)/2} \frac{nS}{\sigma^2}$$

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[1] Paulson, E., "An optimum solution to the K-sample slippage problem for the normal distribution," *Annals of Math. Stat.*, 1952.

* Part of the thesis submitted by the author for Diploma in Agricultural Statistics—I.C.A.R., 1957.

Then the procedure d suggested by Paulson is defined as:—

Take decision D_i if

$$\frac{(x_M - \bar{x})}{\sqrt{K(n-1)S^2 + n\sum_{i=1}^{K}(x_i - \bar{x})^2}} \geqslant C/x_M = x_i, \; i = 1, 2, \ldots K$$

and D_0 if

$$\frac{(x_M - \bar{x})}{\sqrt{K(n-1)S^2 + n\sum_{i=1}^{K}(x_i - \bar{x})^2}} < C,$$

where

$$Pr\left[\frac{(x_M - \bar{x})}{\sqrt{K(n-1)S^2 + n\sum_{i=1}^{K}(x_i - \bar{x})^2}} \geqslant C/\omega_0\right] = \alpha,$$

ω_0 is ω: $(m_1 = m_2 = \ldots = m_K, \sigma^2)$, ω being the parameter point $(m_1, m_2, \ldots, m_K, \sigma^2)$

$$x_M = \text{Max} \;\; (x_1, x_2 \ldots, x_K), \; \bar{x} = \sum_{1}^{K} \frac{x_i}{K}.$$

The unbiased property that the probability of incorrect choice never exceeds the probability of correct choice is equivalent to showing that

(i) $Pr[D_0/\omega_0] \geqslant Pr[D_0/\omega]$

(ii) $Pr[D_i/\omega_i] \geqslant Pr[D_j/\omega_i]$, $\;\; i, j = 1, 2, \ldots, K, i \neq j$

(iii) $Pr[D_i/\omega_i] \geqslant Pr[D_i/\omega_0]$, $\;\; i = 1, 2, \ldots, K$

$\omega_i = \omega: [m_i = \text{Max}(m_1, m_2, m_3, \ldots, m_K), \sigma^2]$.

3. In order to prove the first property we have only to show that $Pr(t_M \leqslant c)$ is maximum for ω_0 for all σ,

where

$$t_M = \frac{(x_M - \bar{x})}{\sqrt{K(n-1)S^2 + n\sum_{i=1}^{K}(x_i - \bar{x})^2}}$$

$Pr(t_M \leqslant c)$ is obtained as follows:—

Joint distribution of the ranked observations $x_{[K]} > x_{[K-1]} \ldots > x_{[1]}$ and S is written as

$$\frac{n^{K/2}}{(\sqrt{2\pi}\,\sigma)^K} \sum_{\substack{j_1, j_2, j_3, \ldots, j_K \\ j_1 \neq j_2 \neq j_3 \neq \ldots \neq j_K;\ j_1, j_2, \ldots, j_K = 1, 2, \ldots, K}} \text{Exp.}$$

$$-\frac{n}{2\sigma^2} \sum_{i=1}^{K} (x_{[i]} - m_{j_i})^2 \prod_{i=1}^{K} dx_{[i]} f(S, \sigma)\, dS.$$

On transforming the above by

$$y_{i-1} = \frac{\sqrt{n}}{S\sigma\sqrt{i(i-1)}} \left[(i-1)\, x_{[i]} - \sum_{j=1}^{i-1} x_{[j]}\right], \; i = 2, 3, \ldots, K.$$

$$y_K = \frac{\sqrt{n}}{S\sigma\sqrt{K}} \left[x_{[1]} + x_{[2]} + \ldots x_{[K]}\right], \quad S = S$$

and denoting

$$m_i - \bar{m} = \delta_i,\ i = 1, 2, \ldots, K-1,\ m_K - \bar{m} = -\sum_{i=1}^{K-1} \delta_i = \delta_K.$$

We get after integrating out y_K (y_K is integrated as the definition of the region $t_M \leqslant c$ does not involve it) the joint distribution of $y_1, y_2, \ldots, y_{K-1}$ and S

$$\frac{1}{(\sqrt{2\pi})} f_{K-1}(S, \sigma)\, S^K\, dS \sum_{\substack{j_1, j_2, \ldots, j_K \\ j_1 \neq j_2 \neq \ldots \neq j_K;\ j_1, j_2, \ldots, j_K = 1, 2, \ldots, K}} \text{Exp.}$$

$$-\frac{1}{2}\left\{\sum_{i=2}^{K-1}\left(y_{i-1} S - \frac{(i-1)\,\delta_{j_i} - \delta_{j_1} - \delta_{j_2} - \ldots - \delta_{j_{i-1}}}{\sqrt{i(i-1)}}\right)^2\right.$$

$$\left. + \left(y_{K-1} S - \frac{K\delta_{j_K}}{\sqrt{K(K-1)}}\right)^2\right\} \prod_{i=1}^{K-1} dy_i$$

whence

$$Pr\,(t_M \leqslant c/\delta_1, \ldots \delta_{K-1}) = \int_0^{\infty} f(S, \sigma)\, S^K dS$$

$$\times \int_B \frac{1}{(\sqrt{2\pi})^{K-1}} \sum_{\substack{j_1, j_2, \ldots, j_K \\ j_1 \neq j_2 \neq \ldots \neq j_K;\ j_1, j_2, \ldots, j_K = 1, 2, \ldots, K}} \text{Exp.}$$

$$-\frac{1}{2}\left\{\sum_{i=2}^{K-1}\left(y_{i-1}S-\frac{(i-1)\,\delta_{j_i}-\delta_{j_1}-\delta_{j_2}-\ldots\,\delta_{j_{i-1}}}{\sqrt{i\,(i-1)}}\right)^2\right.$$

$$\left.+\left(y_{K-1}\,S-\frac{K\delta_{j_K}}{\sqrt{K\,(K-1)}}\right)^2\right\}\prod_{i=1}^{K-1}dy_i \qquad (1)$$

B is the space defined by

$$y_2\sqrt{3}>y_1>0,\ y_3\sqrt{2}>y_2>0,\ \ldots,\ y_i\sqrt{\frac{i+1}{i-1}}>y_{i-1}$$

$$>0,\ \ldots\,y_{K-1}\sqrt{\frac{K}{K-2}}>y_{K-2}>0,\ c>t_M$$

$$=\frac{y_{K-1}\sqrt{\dfrac{K-1}{K}}}{\sqrt{\dfrac{Kn\,(n-1)}{\sigma^2}+n\sum\limits_{i=1'}^{K-1}y_i^2}}>0$$

where

$$\int_0^\infty f\,(S,\sigma)\,S^K\,dS\int_B\frac{K!}{(\sqrt{2\pi})^{K-1}}\ \text{Exp.}$$

$$-\frac{1}{2}\left\{\left(\sum_{i=2}^{K-1}y^2_{i-1}\,S^2\right)+\left(y^2_{K-1}S^2\right)\right\}\prod_{i=1}^{K-1}dy_i=1-\alpha.$$

The $Pr\,(t_M\leqslant c/\omega)$ (denote it by P) is maximum at ω_0 if, for $i=1,2,\ldots,K-1$

$$\frac{\partial P}{\partial\delta_i}=0 \qquad (2)$$

$$\frac{\partial^2 P}{\partial\delta_i^2}\ \text{is negative} \qquad (3)$$

and the principal minors of $(K-1)$th order matrix

$\left\|\dfrac{\partial^2 P}{\partial\delta_i\partial\delta_j}\right\|$ are negative if of odd order and positive if of even order (4)

at $\delta_i=0,\ i=1,2,\ldots,K-1$, for all values of S^2, since these values of δ_i's define ω_0.

Because of symmetry of δ's in (1)

$$\frac{\partial P}{\partial \delta_i} = \frac{\partial P}{\partial \delta_1} \quad i = 1, 2, \ldots, K-1.$$

$$\frac{\partial^2 P}{\partial \delta_i^2} = \frac{\partial^2 P}{\partial \delta_1^2} \quad i = 1, 2, \ldots, K-1$$

$$\frac{\partial^2 P}{\partial \delta_i \partial \delta_j} = \frac{\partial^2 P}{\partial \delta_1 \partial \delta_2} \quad i \neq j, \; i, \; j = 1, 2, \ldots, K-1.$$

Therefore the conditions (2) and (3) reduce to

$$\frac{\partial P}{\partial \delta_1} = 0 \tag{5}$$

$$\frac{\partial^2 P}{\partial \delta_1^2} \text{ is negative} \tag{6}$$

at $\delta_i = 0, i = 1, 2, \ldots, K-1$.

And the t-th principal minor of the matrix

$$\left\| \frac{\partial^2 P}{\partial \delta_i \partial \delta_j} \right\|$$

reduces to

$$\left(\frac{\partial^2 P}{\partial \delta_1 \partial \delta_2}\right)^t \left\{ \frac{\left(\frac{\partial^2 P}{\partial \delta_1^2} - \frac{\partial^2 P}{\partial \delta_1 \partial \delta_2}\right)}{\frac{\partial^2 P}{\partial \delta_1 \partial \delta_2}} \right\}^{t-1} \left\{ \frac{t + \left(\frac{\partial^2 P}{\partial \delta_1^2} - \frac{\partial^2 P}{\partial \delta_1 \partial \delta_2}\right)}{\frac{\partial^2 P}{\partial \delta_1 \partial \delta_2}} \right\} \tag{7}$$

Writing (1) in the form

$$\int_0^\infty f(S, \sigma)\, S^K dS \int_B \frac{1}{(\sqrt{2\pi})^{K-1}}$$

$$\times \sum_{\substack{j_1, \ldots, j_K \\ j_1 \neq j_2 \neq \ldots \neq j_K;\; j_1, j_2, \ldots, j_K = 1, 2, \ldots, K}} \left\{ \text{Exp.} \right.$$

$$\left. -\frac{1}{2} \sum_{i=1}^{K} \left(A_i S - \delta_{j_i} \right)^2 \right\} \prod_{i=1}^{K-1} dy_i \tag{8}$$

$$y_{i-1} = \frac{1}{\sqrt{i(i-1)}} (\overline{i-1}\, A_i - A_1 - A_2 - \ldots - A_{i-1}),$$

$$A_K = -\sum_{i=1}^{K-1} A_i \qquad i = 2, \ldots K,$$

is equivalent to

$$\int_0^\infty f(S,\sigma)\, S^K dS \int_B \frac{1}{(\sqrt{2\pi})^{K-1}}$$

$$\times \sum_{\substack{j_1, j_2, \ldots, j_K \\ j_1 \neq j_2 \neq \ldots \neq j_K,\ j_1, j_2, \ldots, j_K = 1, 2, \ldots, K,}} \left\{ \text{Exp} - \frac{1}{2} \right.$$

$$\left. \sum_{i=1}^{K} \left(A_{j_i} S - \delta_i \right)^2 \right\} \prod_{i=1}^{K-1} dy_i \tag{9}$$

differentiating with regard to δ_1, we get

$$\int_0^\infty f(S,\sigma)\, S^K dS \int_B \frac{1}{(\sqrt{2\pi})^{K-1}}$$

$$\times \sum_{\substack{j_1, j_2, \ldots, j_K \\ j_1 \neq j_2 \neq \ldots \neq j_K,\ j_1, j_2, \ldots j_K = 1, 2, \ldots, K}} \left\{ \left(A_{j_1} S - A_{j_K} S - 2\delta_1 \right. \right.$$

$$\left. \left. - \sum_{i=2}^{K-1} \delta_i \right) \left(\text{Exp} - \frac{1}{2} \sum_{i=1}^{K-1} (A_{j_i} S - \delta_i)^2 \right) \right\} \prod_{i=1}^{K-1} dy_i \tag{10}$$

$$\left(\frac{\partial P}{\partial \delta_1} \right)_{\substack{\delta_i = 0 \\ i = 1, 2, \ldots, K-1}} = \int_0^\infty f(S,\sigma) S^K dS \int_B \frac{1}{(\sqrt{2\pi})^{K-1}}$$

$$\times \left\{ \text{Exp.} - \frac{1}{2} \sum_{i=1}^{K} A_i^2 S^2 \right\}$$

$$\times \left\{ \sum_{\substack{j_1, j_1, \ldots, j_K \\ j_1 \neq j_2 \neq \ldots \neq j_K;\ j_1, j_2, \ldots j_K = 1, 2, \ldots, K}} \left(A_{j_1} - A_{j_K} \right) S \right\}$$

$$\times \prod_{i=1}^{K-1} dy_i = 0 \tag{11}$$

differentiating (10) with regard to δ_1, and putting $\delta_i = 0, i = 1, 2, \ldots, K-1$

$$\left(\frac{\partial^2 P}{\partial \delta_1^2}\right)_{\substack{\delta_i=0 \\ i=1,2,\ldots,K-1}} = \int_0^\infty f(S,\sigma) S^K dS \int_B \frac{1}{(\sqrt{2\pi})^{K-1}}$$

$$\times \left\{\text{Exp} - \frac{1}{2}\sum_{i=1}^{K} A_i^2 S^2\right\}$$

$$\times \left[\sum_{\substack{j_1, j_2, \ldots, j_K \\ j_1 \neq j_2 \neq \ldots \neq j_K;\, j_1, j_2, \ldots, j_K = 1, 2, \ldots, K}} \left\{\left(A_{j_1} - A_{j_2}\right)^2 S^2 - 2\right\}\right]$$

$$\times \prod_{i=1}^{K-1} dy_i$$

$$= \int_0^\infty f(S,\sigma) S^K dS \int_B \frac{2K\,|K-2}{(\sqrt{2\pi})^{K-1}}$$

$$\times \left\{\text{Exp} - \frac{1}{2}\sum_{i=1}^{K} A_i^2 S^2\right\}\left\{\sum_{i=1}^{K}\left(A_i S\right)^2 - \left(K-1\right)\right\}$$

$$\times \prod_{i=1}^{K-1} dy_i$$

$$= \int_0^\infty f(S,\sigma) S^K dS \int_B \frac{2K\,|K-2}{(\sqrt{2\pi})^{K-1}}$$

$$\times \left\{\text{Exp} - \frac{1}{2}\sum_{i=1}^{K-1} y_i^2 S^2\right\}$$

$$\times \left\{\sum_{i=1}^{K-1} y_i^2 S^2 - (K-1)\right\} \prod_{i=1}^{K-1} dy_i \qquad (12)$$

On transforming to polar co-ordinates, the above integral reduces to

$$\int_0^\infty f(S,\sigma) S^K dS \int_{\theta(\theta_1, \ldots \theta_{K-2}) \in w(\theta)} \cos^{K-1}\theta_1 \cos^{K-2}\theta_2 \ldots \cos\theta_{K-3}$$

$$\times d\theta_1 \ldots d\theta_{K-2} \int_{r=0}^{\phi(c,\theta)} \frac{2K\,|K-2}{(\sqrt{2\pi})^{K-1}}$$

$$\times \{\text{Exp} - \tfrac{1}{2}(rS)^2\}\{r^2S^2 - (K-1)\}\, r^{K-2}\, dr. \qquad (13)$$

The transformed domain of integration lies in the positive quadrant and can be shown to be given by $\theta \leqslant r \leqslant \phi(c, \theta)$ and $\theta \epsilon \omega(\theta)$, where

$$\phi(c, \theta) = \frac{c}{\sigma} \sqrt{\frac{Kn(n-1)}{\frac{K-1}{K} \sin^2 \theta_1 - nc^2}}$$

and $\omega(\theta)$ is given by

$$0 < \theta_{K-i} < \cot^{-1} \sqrt{\frac{i+1}{i-1}} \operatorname{cosec} \theta_{K-i+1}, \; i = 3, 4, \ldots K-1;$$

$$0 < \theta_{K-2} < \cot^{-1} \sqrt{3}.$$

Therefore the sign of the integral (13) depends only on the sign of the integral with respect to r. Since S^2 is positive, S^2 can be taken as 1 without any loss of generality for the consideration of the sign of integral, as will be apparent from a transformation $rS = Z$.

Taking c' for $\phi(c, \theta)$, which is positive, the integral

$$\int_0^{c'} \{r^2 - (K-1)\} r^{K-2} e^{-\frac{1}{2}r^2} dr$$

$$= \int_0^{c'} (r^K - Kr^{K-1}) c^{-\frac{1}{2}r^2} dr + K \int_0^{c'} (r^{K-1} - (K-1) r^{K-2}) e^{-\frac{1}{2}r^2} dr$$

$$+ (K-1)^2 \int_0^{c'} \{r^{K-2} - (K-2) r^{K-3}\} e^{-\frac{1}{2}r^2} dr + (K-1)^2$$

$$\times (K-2) \int_0^{c'} \{r^{K-3} - (K-3) r^{K-4}\} e^{-\frac{1}{2}r^2} dr + (K-1)^2$$

$$\times (K-2)(K-3) \int_0^{c'} \{r^{K-4} - (K-4) r^{K-3}\} e^{-\frac{1}{2}r^2} dr$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ (K-1)^2 (K-2)(K-3) \ldots 3 \int_0^{c'} (r^2 - 2r) e^{-\frac{1}{2}r^2} dr$$

$$+ (K-1)^2 (K-2)(K-3) \ldots 2 \int_0^{c'} (r-1) e^{-\frac{1}{2}r^2} dr$$

$$= \left| -r^K e^{-\frac{1}{2}r^2} \right|_0^{c'} + K \left| -r^{K-1} e^{-\frac{1}{2}r^2} \right|_0^{c'}$$

$$+ (K-1)^2 \left| -r^{K-2} e^{-\frac{1}{2}r^2} \right|_0^{c'}$$

$$+ (K-1)^2 (K-2) \left| -r^{K-3} e^{-\frac{1}{2}r^2} \right|_0^{c'}$$

$$+ (K-1)^2 (K-2)(K-3) \mid -r^{k-4} e^{-\frac{1}{2}r^2} \mid_0^{c'}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ (K-1)^2 (K-2)(K-3) \ldots\ldots 3 \mid -r^2 e^{-\frac{1}{2}r^2} \mid_0^{c'}$$
$$+ (K-1)^2 (K-2)(K-3) \ldots\ldots\ldots 2 \mid -re^{-\frac{1}{2}r^2} \mid_0^{c'}$$

is negative for all c', each of the terms being negative, therefore (13) is negative and so $\partial^2 P/\partial\delta_1^2$ is negative.

Differentiating with respect to δ_2 and putting $\delta_i = 0$, $i = 1, 2, \ldots,$ $K-1$, we get

$$\left(\frac{\partial^2 P}{\partial\delta_1 \partial\delta_2}\right)_{\substack{\delta_i=0 \\ i=1,2,\ldots,K-1}} = \int_0^\infty f(S, \sigma) S^K dS \int_B \frac{K \mid K-2}{(\sqrt{2\pi})^{K-1}}$$
$$\times \left\{ \text{Exp} - \frac{1}{2} \sum_{i=1}^{K} A_i^2 S^2 \right\}$$
$$\times \left\{ \sum_{i=1}^{K} (A_i S)^2 - (K-1) \right\} \prod_{i=1}^{K-1} dy_i.$$

Comparing with (12) we find that its value is half of $\partial^2 P/\partial\delta_1^2$ and is negative.

Conditions (5) and (6) are satisfied because of (11) and (13). From the above it follows that the t-th principal minor of the matrix $\|\partial^2 P/\partial\delta_i \partial\delta_j\|$ as expressed at (7) is positive when t is even and negative when t is odd. This completes the proof of properties (2) to (4) and hence (i) is proved.

4. To prove the second property we will first obtain $Pr(D_i/\omega_i)$ and $Pr(D_j/\omega_i)$. $Pr(D_i/\omega_i)$ is obtained by writing down the joint distribution of S and ordered observations $x_{[K]} > x_{[K-1]} > \ldots > x_{[1]}$ under the restriction that $x_{[K]}$ is drawn from the i-th population, and can be readily written down from (9)

$$Pr(D_i/\omega_i) = \int_0^\infty f(S, \sigma) S^K dS \int_{B'} \frac{1}{(\sqrt{2\pi})^{K-1}} e^{-\frac{1}{2}(A_1 S - \delta_i)^2}$$
$$\times \sum_{\substack{j_1, j_2, \ldots, j_K \text{ except } j_i \\ j_1 \neq j_2 \neq \ldots \neq j_K;\ j_1, j_2, \ldots, j_K = 2, 3, \ldots, K}} e^{-\frac{1}{2} \sum_{l=1}^{K} (A_{jl} S - \delta_l)^2}$$
$$\times \prod_{i=1}^{K-1} dy_i$$

$$= \int_0^\infty f(S, \sigma)\, S^K dS \int_{B'} \frac{1}{(\sqrt{2\pi})^{K-1}}$$

$$\times \sum_{\substack{j_1, \ldots, j_K \text{ except } j_i \\ j_1 \neq j_2 \neq \ldots \neq j_K;\ j_1, \ldots, j_K = 2 \ldots K}} e^{-\frac{1}{2}(A_1 S - \delta_i)^2 - \frac{1}{2}(A_{jj} S - \delta_j)^2}$$

$$\times e^{-\frac{1}{2} \sum\limits_{\substack{l=1 \\ l \neq i \neq j}}^{K} (A_{jl} S - \delta_l)^2} \prod_{i=1}^{K-1} dy_i \tag{14}$$

B' is defined by

$$y_2 \sqrt{3} > y_1 > 0, \quad y_3 \sqrt{2} > y_2 > 0, \ldots, y_i \sqrt{\frac{i+1}{i-1}}$$

$$> y_{i-1} > 0,\ y_{K-1} \sqrt{\frac{K}{K-2}} > y_{K-2} > 0,$$

$$\sqrt{\frac{K-1}{nK}} > t_M > c.$$

Similarly,

$$Pr(D_j / \omega_i) = \int_0^\infty f(S, \sigma)\, S^K dS \int_{B'} \frac{1}{(\sqrt{2\pi})^{K-1}}$$

$$\times \sum_{\substack{j_1, \ldots, j_K \text{ except } j_j,\ j_1 \neq j_2 \neq \ldots \neq j_K \\ j_1, j_2, \ldots, j_K = 2, \ldots K}} e^{-\frac{1}{2}(A_1 S - \delta_j)^2 - \frac{1}{2}(A_{j_i} S - \delta_i)^2}$$

$$e^{-\frac{1}{2} \sum\limits_{\substack{l=1 \\ l \neq i \neq j}}^{K} (A_{jl} S - \delta_l)^2} \times \prod_1^{K-1} dy_i$$

$$= \int_0^\infty f(S, \sigma)\, S^K dS \int_{B'} \frac{1}{(\sqrt{2\pi})^{K-1}}$$

$$\times \sum_{\substack{j_1, \ldots, j_K, \text{ except } j_i,\ j_1 \neq j_2 \neq \ldots \neq j_K \\ j_1, \ldots, j_K = 2, \ldots, K}} e^{-\frac{1}{2}(A_1 S - \delta_j)^2 - \frac{1}{2}(A_{jj} S - \delta_i)^2}$$

$$\times e^{-\frac{1}{2} \sum\limits_{\substack{l=1 \\ l \neq i \neq j}}^{K} (A_{jl} S - \delta_l)^2} \prod_1^{K-1} dy_i$$

We have to show that

$$Pr\,(D_i/\omega_i) - Pr\,(D_j/\omega_i) \geqslant 0$$

i.e.,

$$\int_0^\infty f(S,\sigma)\, S^K dS \int_{B'} \frac{1}{(\sqrt{2\pi})^{K-1}}$$

$$\times \left\{ \sum_{\substack{j_1,\ldots,j_K \text{ except } j_i,\, j_1,\ldots,j_K=2,\ldots,K \\ j_1\neq j_2\neq\ldots\neq j_K}} e^{-\frac{1}{2}\sum\limits_{l=1,\, l\neq i\neq j}^{K} (A_{j_l}S-\delta_l)^2} \right.$$

$$\times\; e^{-(A_1{}^2S^2+A_{j_j}{}^2S^2)-(\delta_i{}^2+\delta_j{}^2)}\, e^{A_1S\delta_i+A_{j_j}S\delta_j}$$

$$\left. \times \left(1-e^{(\delta_j-\delta_i)(A_1-A_{j_j}S)}\right)\right\} \prod_{i=1}^{K-1} dy_i \geqslant 0$$

which is so as

$$\left[1-e^{(\delta_j-\delta_i)(A_1-A_{j_j})S}\right]$$

is positive, $A_1 - A_{j_j}$ being always positive and $(\delta_j - \delta_i)$ being negative [because $m_i = \text{Max}\,(m_1, m_2, m_3, m_4, \ldots, m_K)$ for ω_i]. This proves the second property.

5. The third property, *viz.*,

$$Pr\,(D_i/\omega_i) \geqslant Pr\,(D_i/\omega_0)$$

is established as follows:—

We see from (9) and (14) that

$$\sum_L Pr\,(D_L/\omega) = 1 - Pr\,(D_0/\omega) \qquad (15)$$

As

$$Pr\,(D_i/\omega_i) \geqslant Pr\,(D_j/\omega_i) \quad \text{(property ii)}$$

from (15), we get

$$Pr\,(D_i/\omega_i) \geqslant \frac{1}{K}\sum_L Pr\,(D_L/\omega_i) = \frac{1}{K}\,[1 - Pr\,(D_0/\omega_i)] \qquad (16)$$

and as

$$Pr(D_0/\omega_i) \leqslant Pr(D_0/\omega_0) \quad \text{(property i)}$$

We get from (16)

$$Pr(D_i/\omega_i) \geqslant \frac{1}{K}[1 - Pr(D_0/\omega_0)]$$

$$= \frac{1}{K}\sum_{L} Pr(D_L/\omega_0)$$

$$= Pr(D_i/\omega_0) \quad \text{(from 15)}$$

because $Pr(D_i/\omega_0)$ is same for all i, and this can be easily seen by putting δ's equal to zero in (14).

SUMMARY

For the K-sample slippage problem Paulson suggested an optimum solution under certain restrictions on the parameter space. It is shown here that this procedure is desirable even when these restrictions are relaxed as this procedure is an unbiased one.

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