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SOME USEFUL PLANS FOR LARGE NUMBER OF TREATMENTS WITH SMALL NUMBER OF REPLICATES

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SUMMARY

Research workers, while handling large number of treatments, often face difficulties in getting suitable designs with small number of replications (say, 2 or 3). Keeping this end in view, a simple technique of obtaining such designs with practicable block sizes is presented here. It is interesting to note that the analysis of these designs remains the same as that of the conventional incomplete block designs. A list of plans with parameters ($t \geqslant 18, r = 2$ or 3, $k \leqslant 16$) is also appended for ready reference.

Keywords: *C*-design; dual design; Kronecker product of matrices; Variance-covariance matrix.

Introduction

Research workers in certain fields face difficulties in getting suitable plans with reasonably small number (say, 2 or 3) of replications of treatments. Even when the number of treatments is not too large, the available list of incomplete block designs may not include a design with the number of treatments that the experimenter is actually interested in or may supply him with plans of designs which require too many replications. As pointed out by Calinski [2], Verdooren [6] could not find a suitable plan for comparing 18 varieties of wheat using 3 replicates only. Several such situtations may be cited where the experimenters are unable to find appropriate designs for their experiments and are compelled to use sub-standard designs.

In the present communication, we shall first prove a theorem concerning construction of a class of incomplete block designs called C-designs (Saha [4]) from the existing C-designs and the result of this theorem will be applied to evolve some useful plans for comparing large number of treatments with small number of replicates. Actually, it will be seen that some existing incomplete block designs (viz., balanced incomplete block, partially balanced incomplete block, etc.) with small number of replicates or duals of such designs having small (2 or 3) block sizes may conveniently be utilized to obtain plans whose treatment numbers and block sizes are suitable multiples of those of the basic designs while the number of blocks and replicates remain unchanged. Moreover, since the derived designs are found to satisfy the property of C-designs, their analyses remain simple and straightforward as given by Calinski [2]. As the basic designs used in this paper are equireplicate and proper incomplete block designs, we shall confine ourselves with equireplicate and proper C-designs only.

2. Preliminary Concepts

2.1. *C-design*

A block design $N(t, b, r, k)$ having t treatments, b blocks, r replications and k blocksizes is defined as an equireplicate and proper C-design, if it satisfies

$$M_0^2 = \mu M_0$$

where, $M_0 = (1/rk)\, N N' - (r/n)\, J$ and μ is a constant $(0 \leqslant \mu \leqslant 1)$, defined by Jones [3] as a measure of relative loss of information due to partially confounding the treatment contrasts with blocks of the design; N is the $(t \times b)$ incidence matrix of the design and N' its transpose; J is the $(t \times t)$ matrix of unit elements and n is the total number of observations,

2.2. *Dual design*

If N is the incidence matrix of a design D, the design D' which has N' as its incidence matrix, is said to be the dual design of D. In other words, a block design D' obtained from another block design D by changing the treatments and blocks of D to respectively the blocks and treatments of D', is called the dual design of D.

2.3. *Kronecker product of matrices*

If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is another matrix of order $p \times q$, then the Kronecker product, denoted by $A \otimes B$, is the $mp \times nq$ matrix given by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

3. Method

In this section, we first prove a theorem concerning generation of new C-designs from existing C-designs. Then we discuss briefly the analytical outlines of an equireplicate and proper C-design.

3.1. *Construction of C-design*

The method of construction is presented below :

THEOREM 3.1. *Let $N_{t\times b}$ (t, b, r, k) be an equireplicate and proper C-design called basic design. Then there always exists another equireplicate and proper C-design $N^*_{ct\times b} = (1_{c\times 1} \otimes N_{t\times b})$ with parameters $t^* = ct$, $b^* = b$, $r^* = r$, $k^* = ck$, where c is a positive integer greater than zero.*

Proof. The parameters of the resulting C-design N^* are obvious. We are required to show only that N^* is also a C-design. Since N is a C-design, then evidently $M_0^2 = \mu M_0$, where μ and M_0 are as defined in 2.1.

By definition, the M_0-matrix of N^* is given by

$$M_0^* = (1/r^*k^*)\, N^*_{ct\times b}\, N^{*'}_{b\times ct} - (r^*/cn)\, J_{ct\times ct}$$

$$= (1/rck)\,(1_{c\times 1} \otimes N_{t\times b})\,(1'_{1\times c} \otimes N'_{b\times t}) - (r/cn)\, J_{ct\times ct}$$

$$= (1/rck)\,(J_{c\times c} \otimes (NN')_{t\times t}) - (r/cn)\,(J_{c\times c} \otimes J_{t\times t})$$

$$= (1/c)\, J_{c\times c} \otimes [(1/rk)\,(NN')_{t\times t} - (r/n)\, J_{t\times t}]$$

i.e., $M_0^* = (1/c)\, J_{c\times c} \otimes M_0$

Now,

$$M_0^{*2} = ((1/c)\, J_{c\times c} \otimes M_0)\,((1/c)\, J_{c\times c} \otimes M_0)$$
$$= \mu\, ((1/c)\, J_{c\times c} \otimes M_0)$$

Hence

$$M_0^{*2} = \mu M_0^*$$

which shows that the resulting design N^* is a C-design. Also, the constant 'μ' for the resulting design remains the same as that of the original C-design.

It may be mentioned here that the work of Calinski [2] and Saha [4] reveals that the following binary block designs and their duals belong to the class of equireplicate and proper C-designs :

(i) Balanced incomplete block (BIB) designs,
(ii) Affine resolvable incomplete block designs,
(iii) Semi-regular group divisible (GD) designs,
(iv) Singular GD designs,
(v) The class of T_2-designs with parameters $t = \binom{n}{2}, b = n, r = 2,$ $k = (n - 1), \lambda_1 = 1, \lambda_2 = 0$, and
(vi) The class of L_i designs with $r + (s - i) \lambda_1 - (s - i + 1) \lambda_2 = 0$ or, $(r - i \lambda_1) + (i - 1) \lambda_2 = 0$.

Therefore, all the designs listed above and their duals could be used as basic designs in Theorem 3.1 to derive a large number of new C-designs.

3.2. *Analytical Outlines*

The analysis of C-designs remains the same as that of the conventional incomplete block designs. However, a brief account of the same in the present context is provided here. To compute the adjusted treatment sum of squares (s.s.), one has to first find out the variance-covariance matrix of the least square estimate of treatment effects. Following Tocher [5] and Calinski [2], the said variance-covariance matrix (Ω), for an equireplicate and proper C-design with parameter set (t, b, r, k) can be found out, under the usual fixed effects additive model, as

$$\Omega = [I + (1 - \mu)^{-1} M_0]/r \qquad (3.2.1)$$

where μ and M_0 are as defined earlier. Then the least square estimate of treatment effects (ΩQ) and the adjusted treatment s.s. ($Q'\Omega Q$) can be calculated as usual by noting that Q (the adjusted treatment total) $= T - N B/k$, where T and B are the vectors of unadjusted treatment and block totals respectively and N is the incidence matrix of the design. Other s.s. viz., s.s. due to Total, Block and Error could be found out in usual manner. Hence the following analysis of variance table could be set up for testing the null hypothesis of equality of treatment effects :

Sources of variation	*d.f.*	*s.s.*	*m.s.*
Between blocks (unadj)	$b-1$	$\sum_i B_i^2/t - CF$	
,, Treatments (adj.)	$t-1$	$Q'\Omega Q$	
Error	$t(r-1)-(b-1)$	By subtraction	
Total	$rt-1$	$\sum_{ij} y_{ij}^2 - CF$	

In fact, the analysis of *C*-design is simple provided Ω is obtained easily. So far as the expression (3.2.1) is concerned, once the constant 'μ' for a *C*-design having known, the calculation of Ω does not pose much of difficulty. In general, the μ-value of a *C*-design could be worked out from the relation : $M_0^2 = \mu M_0$. But the process seems to be a lengthy one. Alternatively, the same could be obtained through the use of the corollary 4 of Theorem 2 in Saha [4]. The μ-values of the designs listed at the end of Section 3.1 which are obtained through the said corollary are given below for ready reference.

Designs with parameters	*μ-value*
BIB design (t, b, r, k, λ)	$(r-\lambda)/rk$
Affine resolvable design (t, b, r, k, q_1, q_2)	$(k-q_1)/rk$
Semi-regular GD $(t, b, r, k, m, n, \lambda_1, \lambda_2)$	$(r-\lambda_1)/rk$
Singular GD $(t, b, r, k, m, n, \lambda_1, \lambda_2)$	$(rk - v\lambda_2)/rk$
T_2-Design $\left(t = \binom{n}{2}, b=n, r=2, k=n-1, \lambda_1 = 2, \lambda_2 = 0\right)$	$(n-2)/(2n-2)$
L_i-Design $(t = s^2, b, r, k, , , i)$	(i) $((r - i\lambda_1) + (i-1)\lambda_2)/rk$, when $r + (s-i)\lambda_1 - (s-i+1)\lambda^2 = 0$ (ii) $(r + (s-i)\lambda_1 - \lambda_2(s-i+1))/rk$, when $(r - i\lambda_1) + (i-1)\lambda_2 = 0$.

As the μ-value of the new *C*-design derived through each of the above designs remains the same as that of the basic design, the μ-values given above can be used while analysing the designs suggested in this paper.

4. Results and Discussion

Example 4.1. The singular GD design with parameters $t = 6, b = 3,$

$r = 2$, $k = 4$, $m = 3$, $n = 2$, $\lambda_1 = 2$, $\lambda_2 = 1$ whose block contents and incidence matrix are given below is a *C*-design (6, 3, 2, 4).

Blocks	*Contents*
I.	(1, 2, 3, 4)
II.	(1, 2, 5, 6)
III.	(3, 4, 5, 6)

Incidence Matrix

$$N_{6\times3} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now taking $c = 3$, we have, by Theorem 3.1,

$$N_{18\times3} = (1_{3\times1} \otimes N_{6\times3})$$

which is the incidence matrix of a new *C*-design with parameters $t = 18$, $b = 3$, $r = 2$, $k = 12$ and is a rare plan of a design that can be used to compare 18 treatments taking only 2 replicates of each treatment. Below are given the block contents of the derived *C*-design after remembering the treatments.

Blocks	*Contents*
I.	(1, 2, 3, 4, 7, 8, 9, 10, 13, 14, 15, 16)
II.	(1, 2, 5, 6, 7, 8, 11, 12, 13, 14, 17, 18)
III.	(3, 4, 5, 6, 9, 10, 11, 12, 15, 16, 17, 18)

By using the semi-regular GD design with parameters $t = 6$, $b = 9$, $r = 3$, $k = 2$, $m = 2$, $n = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$ as basic design and taking $c = 3$, Theorem 3.1 leads to the *C*-design (18, 9, 3, 6) by which 18 treatments could by compared using 3 replicates only. Earlier, Calinski [2] gave a solution of this plan in a different way. The present one may, therefore, be treated as an alternative solution of the problem confronted by Verdooren [6] as described in Section 1. In the same way, this particular semi-regular GD design will give rise to plans of designs in practicable block sizes for 12, 24, 30, 36, 42, 48 treatments having 3 replicates only, when $c = 2$, 4, 5, 6, 7, and 8 respectively.

Thus we can conclude that by using the incomplete block designs (as listed at the end of Section 3.1) having small number of replicates (or, dual of these designs with small block sizes), a good number of incomplete block designs for large number of treatments with small number of replicates could be obtained by suitably choosing the values of c. A list of such incomplete block designs alongwith the sources (i.e., basic designs) and c-values are given in the Appendix.

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Appendix

Plans of Designs for $t \geqslant 18$ with $r = 2$ or 3 and $k < 16$

Serial No.	Derived designs *t*	*b*	*r*	*k*	Basic designs*	Value of *c*
1.	18	3	2	12	(a) Design no. S 1 (b) Design no. S 12	$c = 3$ $c = 2$
2.	18	9	3	6	(a) Design no. SR (b) Design no. SR 12	$c = 3$ $c = 2$
3.	18	4	2	9	Design no. SR 1	$c = 3$
4.	20	6	3	10	(a) BIBD (4, 6, 3, 2, 1) (b) DBIBD (6, 10, 5, 3, 2)	$c = 5$ $c = 2$
5.	20	5	2	8	(a) DBIBD (5, 10, 4, 2, 1) (b) Design no. T 1	$c = 2$ $c = 2$
6.	20	10	3	6	Design no. T 6	$c = 2$
7.	21	7	3	9	BIBD (7, 7, 3, 3, 1)	$c = 3$
8.	24	3	2	16	Design no. S 1	$c = 4$
9.	24	6	3	12	(a) Desing no. S 6 (b) BIBD (4, 6, 3, 2, 1) (c) Design no. S 23 (d) DSR 2	$c = 3$ $c = 6$ $c = 2$ $c = 3$
10.	24	9	3	8	(a) Design no. SR 3 (b) Design no. SR 20 (c) DBIBD (9, 12, 4, 3. 1)	$c = 4$ $c = 2$ $c = 2$
11.	24	4	2	12	Design no. SR 1	$c = 4$
12.	27	9	3	9	Design no. SR 12	$c = 3$
13.	28	7	3	12	(a) Design no. S 40 (b) BIBD (7, 7, 3, 3, 1)	$c = 2$ $c = 4$
14.	28	6	3	14	BIBD (4, 6, 3, 2, 1)	$c = 7$
15.	30	9	3	10	Design no. SR 3	$c = 5$
16.	30	6	3	15	DBIBD (6. 10, 5, 3, 2)	$c = 3$
17.	30	10	3	9	Design no. T 6	$c = 3$
18.	30	15	3	6	Design no. T 28	$c = 2$
19.	30	4	2	15	Design no. SR 1	$c = 5$
20.	30	5	2	12	(a) DBIBD (5, 10, 4, 2, 1) (b) Design no. T 1	$c = 3$ $c = 3$

(*contd. on page* 31)

(*contd. from page* 30)

Serial No.	*Derived design* *t*	*b*	*r*	*k*	*Basic design**	*Value of c*
21.	30	6	2	10	(a) DBIBD (6, 15, 5, 2, 1)	$c = 2$
					(b) Design no. T 20	$c = 2$
22.	32	6	3	16	(a) Design no. S 6	$c = 4$
					(b) Design no. S 54	$c = 2$
					(c) BIBD (4, 6, 3, 2, 1)	$c = 8$
23.	32	16	3	6	L_i design (16, 16, 3, 3, 0, 1, 4, 3)	$c = 2$
24.	35	7	3	15	BIBD (7, 7, 3, 3, 1)	$c = 5$
25.	36	9	3	12	(a) Design no. SR 3	$c = 6$
					(b) ,, ,, SR 12	$c = 4$
26.	40	5	2	16	(a) DBIBD (5, 10, 4, 2, 1)	$c = 4$
					(b) Design no. T 1	$c = 4$
27.	40	10	3	12	Design no. T 6	$c = 4$
28.	42	9	3	14	Design no. SR 3	$c = 7$
29.	42	7	2	12	Design no. T 31	$c = 2$
30.	45	9	3	15	Design no. SR 12	$c = 5$
31.	45	6	2	15	Design no. T 20	$c = 3$
32.	45	15	3	9	Design no. T 28	$c = 3$
33.	48	9	3	16	(a) Design no. SR 3	$c = 8$
					(b) ,, ,, SR 20	$c = 4$
34.	48	16	3	9	L_i design (16, 16, 3, 3, 0, 1, 4, 3)	$c = 3$
35.	50	15	3	10	DSR 36	$c = 2$
36.	50	10	3	15	Design no. T 6	$c = 5$
37.	56	8	2	14	Design no. T 32	$c = 2$
38.	60	15	3	12	Design no. T 28	$c = 4$
39.	64	16	3	12	L_i design (16, 16, 3, 3, 0, 1, 4, 3)	$c = 4$
40.	64	24	3	8	DSR 61	$c = 1$
41.	72	18	3	12	DSR 45	$c = 2$
42.	72	9	2	16	Design no. T 33	$c = 2$
43.	75	15	3	15	Design no. T 28	$c = 5$
44.	80	16	3	15	L_i design (16, 16, 3, 3, 0, 1, 4, 3)	$c = 5$

(*contd. on page* 32)

(contd. from page 31)

Serial No.	Derived designs t	b	r	k	Basic designs*	Value of c
45.	100	30	3	10	DSR 73	$c = 1$
46.	128	24	3	16	DSR 61	$c = 2$
					Some Additional Designs with k = 18 or 20	
1.	63	7	2	18	Design no. T 31	$c = 3$
2.	90	10	2	18	„ „ T 35	$c = 2$
3.	108	18	3	18	DSR 45	$c = 3$
4.	110	11	2	20	Design no. T 36	$c = 2$
5.	200	30	3	20	DSR 73	$c = 2$

**Note* : The design nos. of basic designs referred herein are as per Bose *et al.* [1] from where the parameters and plans of basic designs could be obtainable.

S = Singular GD design : SR = Semi-regular GD design

T = Triangular PBIB design : DSR = Dual Design of SR

BIBD = Balanced Incomplete Block design

DBIBD = Dual design of BIBD.