SOME RESULTS ON SIZE IN A MIXED MODEL USING TWO PRELIMINARY TESTS OF SIGNIFICANCE

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1. INTRODUCTION

When sampling is made from a specified population there is no doubt as to the appropriate model to be used in making inferences about the parameters of the model, and the tests of significance are completely determined. In practical fields, an experimenter often faces the problems of incompletely specified models, *i.e.*, when sampling is made from an unspecified population. Such cases of incomplete specification of models can be resolved by performing preliminary test(s) of significance (PTS) initiated by Bancroft[1].

1.1. Some related papers

There are many examples where preliminary test(s) of significance have been used as an aid in determining the model specification to be used in subsequent inferences. The use of preliminary test(s) of significance in model specification and making subsequent inferences have been made in many investigations. Some of the related papers are; Bancroft [1], [2], Bennett [3], Bozivich, Bancroft & Hartley [4], Gupta & Saxena [5], Gupta & Srivastava [6], Kitagawa [7], Paull [9], Saxena [10], [11], [12], Saxena & Gupta [13], Saxena & Srivastava [14], Singh [15], Srivastava [16], Srivastava & Bancroft [17], Srivastava & Bozivich [18], Srivastava & Gupta[19] and Tailor & Saxena [20].

1.2. Statement of the problem and objective of the present study

Suppose we are given four independent mean squares: v_4, v_3, v_2 and v_1 based on n_4, n_3, n_2 and n_1 degrees of freedom respectively. It is desired to test the null hypothesis Ho: $E(v_4)=E(v_3)$ against the alternative H_1: $E(v_4)>E(v_3)$ when given a priori that $E(v_3)\geqslant E(v_1)$

and $E(v_2) \geqslant E(v_1)$. Now one of the ways of testing Ho against H_1 is to compare v_4 and v_3 by the F-statistic and to reject Ho if the observed value of the ratio v_4/v_3 comes out to be significant. It is, however, suspected though not known with certainty that $E(v_3)$ might be equal to $E(v_1)$ and $E(v_2)$ to $E(v_1)$. In these instances, we can obtain better test (s) based on PTS for testing Ho. The present investigation is concerned with an examination of the power function and the study of some theoretical results on size of the SPT procedure for the following mixed model of a Split-Plot experiment in RBD using two PTS.

$$y_{ijk}=\mu+b_i+\tau_j+e_{ij}+\gamma_k+\delta_{jk}+e'_{ijk} \qquad \begin{array}{l} i=1, 2, \ldots, r, \\ j=1, 2, \ldots, p, \\ k=1, 2, \ldots, q, \end{array} \quad \ldots(1.3)$$

where y_{ijk} denote the observation of kth sub-plot of the jth whole plot in the ith block. τ_j, γ_k and δ_{jk} are the fixed effects such that

$$\sum_j \tau_j=\sum_k \gamma_k=0,\ \sum_j \delta_{jk}=0 \text{ for all } k \text{ and } \sum_k \delta_{jk}=0 \text{ for all } j.$$

The random components b_i, e_{ij} and e'_{ijk} are independently distributed as normal with zero means and respective variances σ_b^2, σ_e^2 and $\sigma_{e'}^2$. The analysis of variance resulting from model (1.3) is shown in Table 1, where,

$$\sigma_\tau^2=\frac{1}{p-1}\sum_j \tau_j^2 \text{ and } \sigma_\gamma^2=\frac{1}{q-1}\sum_k \gamma_k^2$$

are finite population variances.

TABLE 1

Analysis of Variance of a Crossed and Mixed Model for RBD with Split-Plot

Source of Variation	*Degrees of Freedom*	*Mean Square*	*Expected Mean Square*
Blocks	$r-1$		
Whole-Plot Treatments	$p-1=n_4$	v_4	$\sigma_4^2=\sigma_{e'}^2+q\sigma_e^2+rq\sigma_\tau^2$
Error (I)	$(r-1)(p-1)=n_3$	v_3	$\sigma_3^2=\sigma_{e'}^2+q\sigma_e^2$
Sub-Plot Treatments	$q-1=n_2$	v_2	$\sigma_2^2=\sigma_{e'}^2+rp\sigma_\gamma^2$
Interaction	$(p-1)(q-1)$		
Error (II)	$p(q-1)(r-1)=n_1$	v_1	$\sigma_1^2=\sigma_{e'}^2$

Given four independent mean squares v_i $(i=1, 2, 3, 4)$ based on n_i degrees of freedom as in Table 1 such that $E(v_i)=\sigma_i^2$, we are interested in testing the hypothesis $Ho: \sigma_\tau^2=0$ against the alternative hypothesis $H_1: \sigma_\tau^2>0$ when there is uncertainty whether $\sigma_e^2 \geqslant 0$ and $\sigma_\gamma^2 \geqslant 0$. For the test of Ho, the following sometimes pool test (SPT) is proposed.

Sometimes Pool Test (SPT) Procedure. Reject Ho if at least one of the three mutually exclusive events R_i $(i=1, 2, 3)$ given below occurs :

$$R_1 : \left\{ \frac{V_3}{V_1} < F_1, \quad \frac{V_2}{V_{13}} < F_2, \quad \frac{V_4}{V_{123}} \geqslant F_3 \right\};$$

$$R_2 : \left\{ \frac{V_3}{V_1} < F_1, \quad \frac{V_2}{V_{13}} \geqslant F_2, \quad \frac{V_4}{V_{13}} \geqslant F_4 \right\};$$

$$R_3 : \left\{ \frac{V_3}{V_1} \geqslant F_1, \quad \frac{V_4}{V_3} \geqslant F_5 \right\}.$$

where

$$V_{123}=(n_1v_1+n_2v_2+n_3v_3)/n_{123},\ n_{123}=n_1+n_2+n_3,$$
$$V_{ij}=(n_iv_i+n_jv_j)/n_{ij},\ n_{ij}=n_i+n_j,$$
$$F_1=F(n_3, n_1; \alpha_1),\ F_2=F(n_2, n_{13}; \alpha_2),$$
$$F_3=F(n_4, n_{123}; \alpha_3),\ F_4=F(n_4, n_{13}; \alpha_4),$$
$$F_5=F(n_4, n_3; \alpha_5),$$

and $F(n_i, n_j; \alpha_k)$ denotes upper $100\alpha_k\%$ point of F-distribution with n_i and n_j degrees of freedom.

2. Power and Size of SPT Procedure

We, now, know that v_i $(i=1, 3)$ are distributed as $\chi_i^2\ \sigma_i^2/n_i$, where χ_i^2 is the central chi-square statistics with n_i degrees of freedom and v_2, v_4 are distributed as $\chi_2'^2\ \sigma_1^2/n_2$ and $\chi_4'^2\ \sigma_3^2/n_4$ respectively, where $\chi_2'^2$ is the non-central chi-square statistic with n_2 degrees of freedom and non-centrality parmameter

$$\lambda_2=\frac{1}{2}n_2(\sigma_2^2-\sigma_1^2)/\sigma_1^2,$$

and $\chi_4'^2$ is the non-central chi-square statistic with n_4 degrees of freedom and non-centrality parameter

$$\lambda_4=\frac{1}{2}n_4(\sigma_4^2-\sigma_3^2)/\sigma_3^2.$$

On using the central chi-square approximation to non-central chi-square due to Patnaik (1949), we find that $n_2 v_2$ and $n_4 v_4$ are approximately distributed as $\chi_2^2 \sigma_1^2 c_2$ and $\chi_4^2 \sigma_3^2 c_4$ respectively, where χ_2^2 is the central chi-square statistic based on degrees of freedom ν_2 and the scalar constant c_2, respectively, as

$$\nu_2 = n_2 + 4\lambda_2^2 \Big/ (n_2 + 4\lambda_2) \text{ and } c_2 = 1 + 2\lambda_2/(n_2 + 2\lambda_2),$$

χ_4^2 is the central chi-square statistics based on degrees of freedom ν_4 and the scalar constant c_4, respectively, as

$$\nu_4 = n_4 + 4\lambda_4^2 \Big/ (n_4 + 4\lambda_4) \text{ and}$$

$$c_4 = 1 + 2\lambda_4/(n_4 + 2\lambda_4).$$

The joint density of the independent mean squares v_1, v_2, v_3 and v_4 (using Patnaik's approximation) will be given by

$$h(v_1, v_2, v_3, v_4) = K^* v_1^{\frac{1}{2}n_1 - 1} v_2^{\frac{1}{2}\nu_2 - 1} v_3^{\frac{1}{2}n_3 - 1} v_4^{\frac{1}{2}\nu_4 - 1}$$

$$\times \exp\left[-\frac{n_1 v_1}{2\sigma_1^2} \left\{ 1 + \frac{n_2 v_2}{n_1 v_1 c_2} + \frac{n_3 v_3}{n_1 v_1 \theta_{31}} + \frac{n_4 v_4}{n_1 v_1 c_4 \theta_{31}} \right\} \right] \quad \ldots(2.1)$$

where K^* is a normalizing constant and $\theta_{31} = \sigma_3^2 \Big/ \sigma_1^2$.

Introducing the new variates

$$u_1 = \frac{n_4 v_4}{n_1 v_1 c_4 \theta_{31}}, \quad u_2 = \frac{n_2 v_2}{n_1 v_1 c_2}, \quad \ldots(2.2)$$

$$u_3 = \frac{n_3 v_3}{n_1 v_1 \theta_{31}}; \quad w = \frac{n_1 v_1}{2\sigma_1^2};$$

and integrating w from 0 to ∞, the joint density of the new variates u_1, u_2 and u_3 is obtained as follows :

$$f(u_1, u_2, u_3) = K \frac{u_1^{\frac{1}{2}\nu_4 - 1} u_2^{\frac{1}{2}\nu_2 - 1} u_3^{\frac{1}{2}n_3 - 1}}{(1 + u_1 + u_2 + u_3)^{\frac{1}{2}(n_1 + \nu_2 + n_3 + \nu_4)}}, \quad \ldots(2.3)$$

where

$$K = \frac{\Gamma\{\frac{1}{2}(n_1 + \nu_2 + n_3 + \nu_3)\}}{\Gamma(\frac{1}{2}n_1)\Gamma(\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}n_3)\Gamma\frac{1}{2}\nu_4)} \quad \ldots(2.4)$$

The power of the SPT procedure is the probability P given by

$$P=P\left(\bigcup_{i=1}^{3} R_i\right)=\sum_{i=1}^{3} P(R_i) =\sum_{i=1}^{3} P_i, \qquad \ldots(2.5)$$

since $R_i \cap R_j=\phi$ for all $i \neq j$; where

$$P_1=\int_{u_3=0}^{a} \int_{u_2=0}^{b(1+u_3\theta_{31})} \int_{u_1=c(1+u_2c_2+u_3\theta_{31})}^{\infty} f(u_1, u_2, u_3)\, du_3\, du_2\, du_1, \quad \ldots(2.6)$$

$$P_2=\int_{u_3=0}^{a} \int_{u_2=b(1+u_3\theta_{31})}^{\infty} \int_{u_1=d\,(1+u_3\theta_{31})}^{\infty} f(u_1, u_2, u_3) du_3 du_2 du_1, \quad \ldots(2.7)$$

$$P_3=\int_{u_3=a}^{\infty} \int_{u_2=0}^{\infty} \int_{u_1=eu_3}^{\infty} f\,(u_1, n_2, u_3)\, du_3\, du_2\, du_1, \quad \ldots(2.8)$$

$$a=\frac{n_3F_1}{n_1\theta_{31}}, \quad b=\frac{n_2\, F_2}{n_{13}\; c_2}, \quad c=\frac{n_4\; F_3}{n_{123}\; c_4\; \theta_{31}},$$

$$d=\frac{n_4\, F_4}{n_{13}\; c_4\; \theta_{31}}, \qquad e=\frac{n_4\, F_5}{n_3\; c_4}.$$

For $\lambda_4=0$ ($\lambda_4=0 \Rightarrow c_4=1$ and $\nu_4=n_4$), (2.5) will give the size of the test.

3. Theoretical Results on Size

For $\lambda_2 = \lambda_4=0$ (*i.e.*, $c_2=c_4=1$) and $\theta_{31}=1$, the expressions, (2.6), (2.7) and (2.8) reduce to

$$S_1 =\int_{u_3=0}^{h_1} \int_{u_2=0}^{h_2\,(1+u_3)} \int_{u_1=h_3\,(1+u_2+u_3)}^{\infty} g(u_1, u_2, u_3)\; du_3\; du_2\; du_1, \qquad \ldots(3.1)$$

$$S_2 =\int_{u_3=0}^{h_1} \int_{u_2=h_2(1+u_3)}^{\infty} \int_{u_1=h_4\,(1+u_3)}^{\infty} g(u_1, u_2, u_3)\; du_3\; du_2\; du_1, \quad \ldots(3.2)$$

$$S_3 =\int_{u_3=h_1}^{\infty} \int_{u_2=0}^{\infty} \int_{u_1=h_5u_3}^{\infty} g(u_1, u_2, u_3)\; du_3\; du_2\; du_1, \qquad \ldots(3.3)$$

where

$$g(u_1, u_2, u_3) = G\,\frac{u_1^{\frac{n_4}{2}-1}\,u_2^{\frac{n_2}{2}-1}\,u_3^{\frac{n_3}{2}-1}}{(1+u_1+u_2+u_3)^{\frac{1}{2}(n_1+n_2+n_3+n_4)}}$$

$$G = \frac{\Gamma\{\frac{1}{2}(n_1+n_2+n_3+n_4)\}}{\Gamma(\frac{1}{2}n_1)\,\Gamma(\frac{1}{2}n_2)\,\Gamma(\frac{1}{2}n_3)\,\Gamma(\frac{1}{2}n_4)},$$

$$h_1 = \frac{n_3F_1}{n_1},\quad h_2 = \frac{n_2F_2}{n_{13}},\quad h_3 = \frac{n_4F_3}{n_{123}}, \qquad \ldots(3.4)$$

$$h_4 = \frac{n_4F_4}{n_{13}},\ h_5 = \frac{n_4F_5}{n_3};$$

$$n_{ij} = n_i+n_j,\ n_{ijk}=n_i+n_j+n_k \qquad \ldots(3.4)$$

$$n_{1234} = n_1+n_2+n_3+n_4.$$

Theorem 1. For $\lambda_2=\lambda_4=0$ (*i.e.*, $c_2=c_4=1$) and $\theta_{31}=1$, the lower bound for size of the SPT procedure is

$$(1-\alpha_1)\,(1-\alpha_2)\,\alpha_3,$$

Proof : Under the conditions of the theorem, the size of the SPT procedure is given by

$$S = S_1+S_2+S_3. \qquad \ldots(3.5)$$

Since S_i $(i=1,2,3)$ are non-negative quantities, hence

$$S \geqslant S_1 \qquad \ldots(3.6)$$

On applying the following transformation

$$u_3 = x_3,\ u_2=(1+x_3)\,x_2,\ u_1=(1+x_3)\,(1+x_2)\,x_1$$

to (3.1) and after simplification, we get

$$S_1=G\int_0^{h_1}\frac{x_3^{\frac{n_3}{2}-1}\,dx_3}{(1+x_3)^{\frac{1}{2}(n_1+n_3)}}\int_0^{h_2}\frac{x_2^{\frac{n_2}{2}-1}\,dx_2}{(1+x_2)^{\frac{1}{2}n_{123}}}\int_{h_3}^{\infty}\frac{x_1^{\frac{n_4}{2}-1}\,dx_1}{(1+x_1)^{\frac{1}{2}n_{1234}}},$$

where n_{123} and n_{1234} are defined in (3.4). By using the well-known relation

$$\text{Prob}\,\{F(p, q)\leqslant F_0\} = I\left(x; \frac{p}{2}, \frac{q}{2}\right);\ x = \frac{pF_0}{(q+p\,F_0)},$$

between F-integral and normalized incomplete beta function, we obtain

$$S_1 = (1-\alpha_1)\,(1-\alpha_2)\,\alpha_3.$$

Hence from (3.6)

$$S \geqslant (1-\alpha_1)\,(1-\alpha_2)\,\alpha_3. \qquad \ldots(3.7)$$

Theorem 2 : For $c_2 = c_4 = \theta_{3_1} = 1$, the upper bound for size of the SPT procedure is

$$(1-\alpha_1)(1-\alpha_2)\alpha_3+(1-\alpha_1)\alpha_4+\alpha_5.$$

Proof : By increasing the range of integration for u_2, s_2 given by (3.2) can be written as

$$S_2 \leqslant \int_{u_3=0}^{h_1} \int_{u_2=0}^{\infty} \int_{u_1=h_4(1+u_3)}^{\infty} g(u_1, u_2, u_3)\, du_3\, du_2\, du_1. \quad \ldots(3.8)$$

On applying the following transformation

$$u_1 = (1+w_3)\,w_1,\ u_2=(1+w_1)(1+w_3)\,w_2,\ u_3=w_3$$

and after simplification, we obtain

$$S_2 \leqslant G \int_0^{h_1} \frac{w_3^{\frac{n_3}{2}-1}\, dw_3}{(1+w_3)^{\frac{1}{2}n_{13}}} \int_0^{\infty} \frac{w_2^{\frac{n_2}{2}-1}\, dw_2}{(1+w_2)^{\frac{1}{2}n_{1234}}} \int_{h_4}^{\infty} \frac{w_1^{\frac{n_4}{2}-1}\, dw_1}{(1+w_1)^{\frac{1}{2}n_{134}}} \quad \ldots(3.9)$$

i.e., $S_2 \leqslant (1-\alpha_1)\,\alpha_4$.

Similarly on increasing the range of integration for u_3, s_3 given by (3.3) can be written as

$$S_3 \leqslant G \int_0^{\infty} \int_0^{\infty} \int_{h_5 u_3}^{\infty} g(u_1, u_2, u_3)\, du_3\, du_2\, du_1\, .$$

on applying the transformation

$$u_3 = X,\ u_2=Y,\ u_1=XZ$$

and integrating X and Y, we obtain,

$$S_3 \leqslant \frac{1}{B\left(\frac{n_3}{2}, \frac{n_4}{2}\right)} \int_{h_5}^{\infty} \frac{Z^{\frac{n_4}{2}-1}\, dz}{(1+Z)^{\frac{1}{2}(n_3+n_4)}} \quad \ldots(3.10)$$

i.e., $S_3 \leqslant \alpha_5$.

on combining (3.7), (3.9) and (3.10) ; we see that (3 5) yields

$$S \leqslant (1-\alpha_1)(1-\alpha_2)\alpha_3+(1-\alpha_1)\alpha_4+\alpha_5.$$

Corollary : For $c_2 = c_4 = \theta_{3_1} = 1$ and all the preliminary levels of significance are equal to α_p (*i e.*, $\alpha_1=\alpha_2=\alpha_p$) and all the final levels of significance are equal to α_f (*i.e.*, $\alpha_3=\alpha_4=\alpha_5=\alpha_f$), then the size of the SPT procedure lies between

$$(1-\alpha_p)^2\,\alpha_f \text{ and } [(1-\alpha_p)^2+(2-\alpha_p)]\,\alpha_f.$$

Proof : Obvious from Theorems 1 and 2.

Summary

In this paper a sometimes pool test (SPT) is obtained for the test of a hypothesis of no treatment effect in an analysis of variance (ANOVA) problem for a mixed model of a Split-plot experiment in Randomised Block Design (RBD). The lower and upper bounds of size are determined in terms of preliminary and final levels of significance.

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