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Simultaneous Interval Estimation of Variance Components in Two Way Nested Random Model with Unbalanced Data

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Summary

The simulatneous confidence interval of variance components for two way nested unbalanced random model has been obtained using two new statistics under the ususal assumptions of normality and independence of random effects.

Key Words : Two way Nested Unbalanced Random Model, Variance components, Simultaneous Confidence Interval.

Introduction

Simultaneous confidence intervals for variance components in two way nested unbalanced random model has not been reported so far. The point estimation of variance components in two way nested random model with unbalanced data has been discussed by Searle [8]. The problem of interval estimation of variance components and their linear combination in one-way unbalaced random model has been discussed by several workers such as Thomas & Hultquist [12], Seely [9], Graybill & Wang [4], Burdick & Graybill [2] and Sing [10]. The simulatneous confidence interval for variance components for the two way nested balanced random model has been given by Sahai [7].

The object of the present investigation is to develop the simultaneous confidence intervals of variance components for two way nested random model with partially unbalanced data. Due to unbalanced data, the usual anova mean sum of squares, except error mean sum of squares, are not distributed as exact Chi-squares. Employing matrix methods, we have developed two new statistics which have been used in developing the simultaneous confidence intervals of variance components of the model under consideration.

2. *Development of New Statistics*

Consider the unbalanced model

$$Y_{ijk} = \mu + a_i + b_{ij} + e_{ijk}\,, \qquad (1)$$

where $i = 1, 2, \ldots, p (p > 1)$, $j = 1, 2, \ldots, q \; (q > 1)$, $k = 1, 2, \ldots, n_i$ $(n_i > 1)$, such that $N = q \sum n_i$. The random variables a_i, b_{ij} and e_{ijk} are independently and normally distributed with means zero and variances σ_a^2, σ_b^2 and σ_e^2 $[0 < (\sigma_a^2, \sigma_b^2, \sigma_e^2) < \infty]$ and $-\infty < \mu < \infty$ is a constant. The usual sum of squares of ANOVA are

$$S_a^2 = \sum_i \frac{Y_{i..}^2}{qn_i} - \frac{Y_{...}^2}{q\sum n_i},$$

$$S_b^2 = \sum_i \sum_j \frac{Y_{ij.}^2}{n_i} - \sum_i \frac{Y_{i..}^2}{qn_i},$$

and

$$S_e^2 = \sum_i \sum_j \sum_k Y_{ijk}^2 - \sum_i \sum_j \frac{Y_{ij.}^2}{n_i},$$

where

$$Y_{ij.} = \sum_k Y_{ijk}, \; Y_{i..} = \sum_j Y_{ij.} \text{ and } Y_{...} = \sum_i Y_{i..}$$

Now $q(\sum n_i - p) M_e^2 = S_e^2 \sim \sigma_e^2 \chi^2 [q(\sum n_i - p)]$ but $p(q-1) M_b^2 = S_b^2$ is not distributed as $(\sigma_e^2 + \bar{n}\, \sigma_b^2)$. $\chi^2 [p(q-1)]$ for $\sigma_b^2 \neq 0$. Similarly $(p-1) M_a^2 = S_a^2$ is not distributed as $\sigma_e^2 + k_0 \sigma_b^2 + qk_0 \sigma_a^2)\; \chi^2 [p-1]$ for $\sigma_a^2 \neq 0$ and for $\sigma_b^2 \neq 0$, where

$$k_o = \frac{(\sum n_i)^2 - \sum n_i^2}{(p-1)\sum n_i} \text{ and } \bar{n} = \frac{1}{p}\sum_{i=1}^{p} n_i$$

The degree of unbalancedness of (1) can be defined by the ratio $R\; (0 < R < 1)$ which is given by $R = k_0/\bar{n}$. For $R = 0$, the design is highly unbalanced and for $R = 1$, the design is balanced.

The distributions of S_a^2 and S_b^2 have been discussed by Singh [11] but it has been expressed in the form of linear combination of independent Chi-squares with single degree of freedom which is not suitable for the interval estimation of variance components. Therefore some new statistics have to be developed whose distributions are well approximated by some standard distributions.

Modifying (1) we get

$$\bar{Y}_{ij.} = \mu + a_i + b_{ij} + \bar{e}_{ij.}, \tag{2}$$

where

$$\bar{Y}_{ij.} = \frac{1}{n_i} \sum_{k=1}^{n_i} Y_{ijk} \text{ and } \bar{e}_{ij.} = \frac{1}{n_i} \sum_{k=1}^{n_i} e_{ijk}.$$

In matrix notation (2) can be written as

$$\bar{Y} = j_{pq}\mu + ZA + B + \bar{E}.$$

where $A \sim N(\phi_p, \sigma_a^2 I_p)$, $B \sim N(\phi_{pq}, \sigma_b^2 I_{pq})$, $\bar{E} \sim N(\phi_{pq}, \sigma_e^2 D)$; $D = \text{diag}\{n_1^{-1} \ldots q \text{ times} \ldots n_p^{-1} \ldots q \text{ times}\}$, j_{pq} is a vector of ones of order pq and ϕ_{rs} is a null matrix of order $r \times s$.

The covariance matrix $\sum$ of $\bar{Y}$ is given by

$$\sum = \sigma_a^2 Z Z' + \sigma_b^2 I_{pq} + \sigma_e^2 D. \tag{3}$$

Employing a transformation $R' = [R'_1 \mid R'_2]$ of order $pq \times pq$ over (3) we get

$$R_1 \bar{Y} \sim N\left[j_p, \sigma_a^2 I_p + \sigma_b^2 R_1 R'_1 + q^{-1} \sigma_e^2 D_1\right],$$

$$R_2 \bar{Y} \sim N\left[\phi_{p(q-1)}, \sigma_b^2 I + \sigma_e^2 D_2\right], \tag{4}$$

where

$$D_1 = \text{diag}\left[n_1^{-1}, n_2^{-1}, \ldots, n_p^{-1}\right],$$

$$D_2 = \text{diag}\left[n_1^{-1}, \ldots n_1^{-1} (q-1) \text{ times}, \ldots, n_p^{-1}, \ldots n_p^{-1} (q-1) \text{ times}\right].$$

Further $R_1\bar{Y}$ and $R_2\bar{Y}$ are independently distributed.

Next, apply an orthogonal transformation H to $R_1 \overline{Y}$ where the orthogonal matrix H is of order p×p consists of $[H'_1 \mid H'_2] = H'$. Here $H_1 = p^{-1/2} j'_p$ and H_2 consists of (p − 1) orthogonal rows in the orthogonal compliment of H_1. Thus

$$H_2 R_1 \overline{Y} \sim N\left[\phi_{p-1}, \sigma_a^2 I_{p-1} + q^{-1} \sigma_b^2 I_{p-1} + q^{-1} \sigma_e^2 H_2 D_1 H'_2\right]. \tag{5}$$

Here $H_2 D_1 H'_2$ is a real symmetric non–diagonal matrix.

In order to diagonalize $H_2 D_1 H'_2$ let ρ be an orthogonal matrix which consist of eigen-vectors of $H_2 D_1 H'_2$.

Then

$$PH_2 R_1 \overline{Y} \sim N\left[\phi_{p-1}, \sigma_a^2 I_{p-1} + q^{-1} \sigma_b^2 I_{p-1} + q^{-1} \sigma_e^2 D_3\right],$$

$$D_3 = \text{diag } (\lambda_1, \lambda_2, \ldots, \lambda_{p-1}) = PH_2 D_1 H'_2 P'.$$

If P_i denotes the i-th row of the matrix P

$$V_i = P_i H_2 R_1 \overline{Y} \sim N\,[\phi, \sigma_a^2 + q^{-1} \sigma_b^2 + q^{-1} \sigma_e^2 \lambda_i],$$

$$i = 1, 2, \ldots, p-1.$$

and

$$\frac{V_i}{(\sigma_a^2 + q^{-1} \sigma_b^2 + q^{-1} \sigma_e^2 \lambda_i)} \sim \chi^2$$

Summing over (p − 1) rows we get

$$\sum_{i=1}^{p-1} V_i^2/(\sigma_a^2 + q^{-1} \sigma_b^2 + q^{-1} \sigma_e^2 \lambda_i) \sim \chi^2 (p-1) \tag{6}$$

Now $\text{tr}\left[H_2 D_1 H'_2\right] = \text{tr}\left[D_1 H'_2 H_2\right]$

$$= \frac{p-1}{p} \sum_i^p \frac{1}{n_i} = \sum_i^p \lambda_i.$$

and therefore $\overline{\lambda} = \dfrac{\sum \lambda_i}{p-1} = \dfrac{1}{p} \sum \dfrac{1}{n_i} = \dfrac{1}{\overline{\overline{n}}}$, where $\bar{n}$ is the harmonic mean of $n_1, n_2, \ldots, n_p$ observations. Since $\bar{\lambda} = \frac{1}{\bar{n}}$, and therefore λ_i's are usually small, replacing λ_i's by $\bar{\lambda}$ in (6) we get

$$W_1 = q\bar{n} \sum_{i=1}^{p-1} V_i^2 / \left(\sigma_e^2 + \bar{n}\,\sigma_b^2 + q\,\bar{n}\,\sigma_a^2\right) \sim \text{(approx)}\ \chi^2\ (p-1). \tag{7}$$

Similarly from (4) we get

$$R_2 \bar{Y} \sim \text{(approx)}\ N\left[\phi_{p(q-1)},\ \sigma_b^2 I + \bar{n}^{-1} \sigma_e^2 I\right]$$

and therefore

$$W_2 = \bar{n}\,\bar{Y}' R'_2 R_2 \bar{Y} / (\sigma_e^2 + \bar{n}\,\sigma_b^2) \sim \text{(approx)}\ \chi^2\ [p\,(q-1)], \tag{8}$$

Now $\bar{Y}' R'_1 H'_2 P' P H_2 R_1 \bar{Y} = (p-1) S_1^2$, where

$$S_1^2 = \frac{1}{p-1}\left[\sum_i \bar{Y}_{i..}^2 - \frac{1}{p}(\bar{Y}_{i..})^2\right],\ \bar{Y}_{i..} = \frac{1}{qn_i}\sum_j \sum_k Y_{ijk}$$

and it does not depend upon orthogonal matrix P.

Similarly $\bar{Y}' R'_2 R_2 \bar{Y} = p\,(q-1) S_2^2$, where

$$S_2^2 = \frac{1}{p\,(q-1)}\left[\sum_i \sum_j \bar{Y}_{ij.}^2 - q^{-1} \sum_i \left(\sum_j \bar{Y}_{ij.}\right)^2\right];$$

$$\bar{Y}_{ij.} = \frac{1}{n_i} \sum_{k=1}^{n_i} Y_{ijk}.$$

Thus $q\bar{n}\,S_1^2$ and $\bar{n}\,S_2^2$ can be used in lieu of M_a^2 and M_b^2 respectively for the construction of confidence intervals for σ_a^2 and σ_b^2. Further

$$G_1 = \frac{q\,S_1^2/(\sigma_e^2 + \bar{n}\,\sigma_b^2 + q\bar{n}\,\sigma_a^2)}{S_2^2/(\sigma_e^2 + \bar{n}\,\sigma_b^2)} \sim \text{(approx)}\ F\ [p-1,\ p\,(q-1)],$$

$$G_2 = \frac{\bar{n}\,S_2^2/(\sigma_e^2 + \bar{n}\,\sigma_b^2)}{M_e^2/\sigma_e^2} \sim \text{(approx)}\ F\left[\ p\,(q-1),\ \left(\sum n_i - p\right)\right]$$

and

$$G_3 = \frac{\bar{n}\, q\, S_1^2/(\sigma_e^2 + \bar{n}\,\sigma_b^2 + q\,\bar{n}\,\sigma_a^2)}{M_e^2/\sigma_e^2} \sim \text{(approx)}$$

$$F\left[p-1,\ q\left(\sum n_i - p\right)\right].$$

For the balanced data $R = 1 \Rightarrow \bar{n} = n$, $q\,\bar{n}\,S_1^2 = M_a^2$, $\bar{n}\,S_2^2 = M_b^2$ have exact χ^2– distribution with appropriate degrees of freedom. Similarly G_1, G_2 and G_3 follow exact F–distribution with appropriate degrees of freedom.

The distributions of S_1^2, S_2^2, G_1, G_2 and G_3 can be obtained by Robbins and Pitman [6] method.

It has been observed that for a number of unbalanced designs, χ^2–distribution is an excellent approximation for σ_a^2/σ_e^2, $\sigma_b^2/\sigma_e^2 > 0.25$ and from 1% to 99% levels of significance. Similarly it has been observed that for a number of unbalanced designs F-distribution is an excellent approximation for σ_a^2/σ_e^2, $\sigma_b^2/\sigma_e^2 > 0.25$ and from 1% to 99% levels of significance.

3. *Simultaneous Interval Estimation*

Let

$$\chi_1^2 = \chi_l^2(\alpha_1; r_1),\ \chi_2^2 = \chi_u^2(\alpha_1; r_1),\ r_1 = p-1,$$

$$\chi_3^2 = \chi_l^2(\alpha_2; r_2),\ \chi_4^2 = \chi_u^2(\alpha_2; r_2),\ r_2 = p\,(q-1,)$$

and $F_1 = F_l(1-\alpha_1; r_1, r_3),\ F_2 = F_u(1-\alpha_1; r_1, r_3),$

$$r_3 = q\left(\sum n_i - p\right),$$

$$F_3 = F_l(1-\alpha_2; r_2, r_3),\ F_4 = F_u(1-\alpha_2; r_2, r_3).$$

Using $q\bar{n}\,S_1^2$, $\bar{n}\,S_2^2$ and M_e^2 the usual 100 $(1-\alpha_1)$ percent confidence intervals for functions $(\sigma_e^2 + \bar{n}\,\sigma_b^2 + q\,\bar{n}\,\sigma_a^2)$ and $\left(\frac{\sigma_b^2}{\sigma_e^2} + q\,\frac{\sigma_a^2}{\sigma_e^2}\right)$ are given by

$$\frac{r_1 q \bar{n} S_1^2}{\chi_2^2} \leq (\sigma_e^2 + \bar{n}\,\sigma_b^2 + q\,\bar{n}\,\sigma_a^2) \leq \frac{r_1 q \bar{n} S_1^2}{\chi_1^2} \tag{9}$$

and
$$\frac{1}{\bar{n}}\left[\frac{q\,\bar{n}\,S_1^2}{M_e^2 F_2} - 1\right] \leq \frac{\sigma_b^2}{\sigma_e^2} + q\frac{\sigma_a^2}{\sigma_e^2} \leq \frac{1}{\bar{n}}\left[\frac{q\,\bar{n}\,S_1^2}{M_e^2 F_1} - 1\right] \tag{10}$$

Similarly the $100(1-\alpha_2)$ percent confidence intervals for $(\sigma_e^2 + \bar{n}\,\sigma_b^2)$ and σ_b^2/σ_e^2 are given by

$$\left[\frac{r_2\,\bar{n}\,S_s^2}{\chi_4^2} \leq \left(\sigma_e^2 + \bar{n}\,\sigma_b^2\right) \leq \frac{r_2\bar{n}S_2^2}{\chi_3^2}\right] \tag{11}$$

and
$$\left[\frac{1}{\bar{n}}\left(\frac{\bar{n}S_2^2}{M_e^2 F_4} - 1\right) \leq \frac{\sigma_b^2}{\sigma_e^2} \leq \frac{1}{\bar{n}}\left(\frac{\bar{n}\,S_2^2}{M_e^2 F_3} - 1\right)\right]. \tag{12}$$

The set of simultaneous confidence interval from (9) and (11) with confidence coefficient $(1-\beta_1)$ of functions $(\sigma_e^2 + \bar{n}\,\sigma_b^2 + q\bar{n}\,\sigma_a^2)$ and $(\sigma_e^2 + \bar{n}\,\sigma_b^2)$ is given by

$$\left\{\frac{r_1\,q\,\bar{n}\,S_1^2}{\chi_2^2} \leq (\sigma_e^2 + \bar{n}\,\sigma_b^2 + q\,\bar{n}\,\sigma_a^2) \leq \frac{r_1\,q\,\bar{n}\,S_1^2}{\chi_1^2}\right.$$

$$\left.\frac{r_2\bar{n}S_2^2}{\chi_4^2} \leq \left(\sigma_e^2 + \bar{n}\,\sigma_b^2\right) \leq \frac{r_2\bar{n}S_2^2}{\chi_3^2}\right\},$$

where $1-\beta_1 = (1-\alpha_1).(1-\alpha_2).$ (13)

Here the equality holds because (9) and (11) are independent. Similarly from (10) and (12) a set of simultaneous confidence interval with confidence coefficient $(1-\beta_2)$ of $(\sigma_b^2/\sigma_e^2 + q\,\sigma_a^2/\sigma_e^2)$ and σ_e^2/σ_e^2 is

$$\left[\frac{1}{\bar{n}}\left(\frac{\bar{n}qS_1^2}{M_e^2 F_2} - 1\right) \leq \left(\frac{\sigma_b^2}{\sigma_e^2} + q\frac{\sigma_a^2}{\sigma_e^2}\right) \leq \frac{1}{\bar{n}}\left(\frac{\bar{n}\,q\,S_1^2}{M_e^2 F_1} - 1\right)\right],$$

$$\frac{1}{\bar{n}}\left(\frac{\bar{n}\,S_2^2}{M_e^2 F_4} - 1\right) \leq \frac{\sigma_b^2}{\sigma_e^2} \leq \frac{1}{\bar{n}}\left(\frac{\bar{n}S_s^2}{M_e^2 F_3} - 1\right)\Bigg],$$

where $1-\beta_2 \geq (1-\alpha_1).(1-\alpha_2).$ (14)

The inequality is used because (10) and (12) are not independent and it has been obtained using improved form of Bonferroni's inequality (Millier [5] p. 101).

Thus for any fixed error variance component we can construct a 100 $(1 - \beta_1)$ percent confidence region for (σ_a^2, σ_b^2) as

$$B_1\left(\sigma_e^2\right) = \left[\sigma_a^2, \sigma_b^2) : \frac{1}{\bar{\bar{n}}}\left(\frac{qr_1\bar{n}\, S_1^2}{\chi_2^2} - \sigma_e^2\right) \leq \left(\sigma_b^2 + q\,\sigma_a^2\right)\right.$$

$$\leq \frac{1}{\bar{\bar{n}}}\left(\frac{r_1\, q\bar{n}S_1^2}{\chi_1^2} - \sigma_e^2\right), \ \frac{1}{\bar{\bar{n}}}\left(\frac{r_2\,\bar{n}S_2^2}{\chi_4^2} - \sigma_e^2\right)$$

$$\left.\leq \sigma_b^2 \leq \frac{1}{\bar{\bar{n}}}\left(\frac{r_2\,\bar{n}S_2^2}{\chi_3^2} - \sigma_e^2\right)\right].$$

Similarly a 100 $(1 - \beta_2)$ percent confidence region for (σ_a^2, σ_b^2) as

$$B_2\left(\sigma_e^2\right) = \left[(\sigma_a^2, \sigma_b^2) : \frac{\sigma_e^2}{\bar{\bar{n}}}\left(\frac{q\bar{n}S_1^2}{M_e^2\, F_2} - 1\right) \leq (\sigma_b^2 + q\,\sigma_a^2)\right.$$

$$\leq \frac{\sigma_e^2}{\bar{\bar{n}}}\left(\frac{q\bar{n}S_1^2}{M_e^2\, F_1} - 1\right), \ \frac{\sigma_e^2}{\bar{\bar{n}}}\left(\frac{\bar{n}S_2^2}{M_e^2\, F_4} - 1\right) \leq \sigma_b^2$$

$$\left.\leq \frac{\sigma_e^2}{\bar{\bar{n}}}\left(\frac{\bar{n}S_2^2}{M_e^2\, F_3} - 1\right)\right]$$

The boundaries of these intervals are the linear function of the nuisance parameter. Denote these two regions by $B_1\,(\sigma_e^2)$ and $B_2\,(\sigma_e^2)$ and let $(\sigma_a^2, \sigma_b^2) \in B_1\,(\sigma_e^2)$ and $B_2\,(\sigma_e^2)$ and let $(\sigma_a^2, \sigma_b^2) \in B_i\,(\sigma_e^2)$ stands for the i-th interval includes (σ_a^2, σ_b^2). For any given value of σ_e^2

$$P\left[(\sigma_a^2, \sigma_b^2) \in B_1\,(\sigma_e^2) \cap B_2\,(\sigma_e^2) \,\middle|\, \sigma_e^2\right]$$

$$= P\left[(\sigma_a^2, \sigma_b^2) \in B_1\,(\sigma_e^2) \,\middle|\, \sigma_e^2\right]$$

$$+ P\left[(\sigma_a^2, \sigma_b^2) \in B_2\,(\sigma_e^2) \,\middle|\, \sigma_e^2\right]$$

$$- P\left[(\sigma_a^2, \sigma_b^2) \in B_1\,(\sigma_e^2) \cup B_2\,(\sigma_e^2) \,\middle|\, \sigma_e^2\right]$$

$$\geq 1 - \beta_1 - \beta_2.$$

For any given σ_e^2, the simultaneous confidence intervals for (σ_a^2, σ_b^2) can be obtained by taking the projection of the confidence regions on the coordinate axes. Thus the simultaneous confidence interval for (σ_a^2, σ_b^2) determined by $B_1(\sigma_e^2)$ with a confidence coefficient $(1-\beta_1)$ are

$$\frac{1}{q\bar{n}}\left(\frac{r_1 q\bar{n} S_1^2}{\chi_2^2} - \sigma_e^2\right) \le \sigma_a^2 \le \frac{1}{q\bar{n}}\left(\frac{r_1 q\bar{n} S_1^2}{\chi_1^2} - \sigma_e^2\right),$$

$$\frac{1}{\bar{n}} f_1(S_1^2, S_2^2) \le \sigma_b^2 \le \frac{1}{\bar{n}} f_2(S_1^2, S_2^2). \qquad (15)$$

where

$$f_1(S_1^2, S_2^2) = \min\left[\left(\frac{r_2\bar{n}S_2^2}{\chi_4^2} - \sigma_e^2\right), \left(\frac{r_1 q\bar{n}S_1^2}{\chi_2^2} - \sigma_e^2\right)\right]$$

$$f_2(S_1^2, S_2^2) = \min\left[\left(\frac{r_2\bar{n}S_2^2}{\chi_3^2} - \sigma_e^2\right), \left(\frac{r_1 q\bar{n}S_1^2}{\chi_1^2} - \sigma_e^2\right)\right].$$

Similarly the simultaneous confidence interval for (σ_a^2, σ_b^2) determined by $B_2(\sigma_e^2)$ with a confidence coefficient $(1-\beta_2)$ is

$$\left[\frac{\sigma_e^2}{q\bar{n}}\left(\frac{q\bar{n}S_1^2}{M_e^2 F_2} - 1\right) \le \sigma_a^2 \le \frac{\sigma_e^2}{q\bar{n}}\left(\frac{q\bar{n}S_1^2}{M_e^2 F_1} - 1\right),\right.$$

$$\left.\frac{\sigma_e^2}{\bar{n}} g_1(S_1^2, S_2^2, M_e^2) \le \sigma_b^2 \le \frac{\sigma_e^2}{\bar{n}} g_2(S_1^2, S_2^2, M_e^2)\right]. \qquad (16)$$

where

$$g_1(S_1^2, S_2^2, M_e^2) = \min\left[\left(\frac{\bar{n}S_2^2}{M_e^2 F_4} - 1\right), \left(\frac{q\bar{n}S_1^2}{M_e^2 F_2} - 1\right)\right.$$

$$g_2(S_1^2, S_2^2, M_e^2) = \min\left[\left(\frac{\bar{n}S_2^2}{M_e^2 F_3} - 1\right), \left(\frac{q\bar{n}S_1^2}{M_e^2 F_1} - 1\right)\right.$$

For any given sample results $B_1(\sigma_e^2)$ and $B_2(\sigma_e^2)$ will form a region of intersection which when projected onto the σ_a^2 axis will be bounded by the intersection of the upper limits and the intersection of the lower limits of the intervals of σ^2/σ_a^2 in the simultaneous intervals for (σ_a^2, σ_b^2) determined by B_1 (σ_e^2) and B_2 (σ_e^2). Similarly, the region of intersection of B_1 (σ_e^2) and B_2 (σ_e^2) when projected onto the σ_b^2 axis will be bounded by the intersection of the upper limits and the intersection of the lower limits of the intervals of σ_b^2 in the simultaneous intervals for (σ_a^2, σ_b^2). These points of intersection will form a set of simultaneous confidence intervals which will include (σ_a^a, σ_b^2), whenever the region of intersection of B_1 (σ_e^2) and B_2 (σ_e^2) includes (σ_a^2, σ_b^2), regardless of the true value of σ_e^2. Thus we get

$$P\left[\frac{r_1 q\bar{n}S_1^2 - 4_1 M_e^2 F_2}{q\bar{n}\chi_2^2} \le \sigma_a^2 \le \frac{rq\bar{n}S_1^2 - r_1 M_e^2 F_1}{q\bar{n}\chi_1^2},\right.$$

$$\left. h_1(S_1^2, S_2^2, M_e^2) \le \sigma_b^2 \le h_2(S_1^2, S_2^2, M_e^2)\right] \ge 1 - \beta_1 - \beta_2, \tag{17}$$

where

$$h_1(S_1^2, S_2^2, M_e^2) = \min\left[\frac{r_2\bar{n}S_2^2 - r_2 M_e^2 F_4}{\bar{n}\chi_4^2}, \frac{r_1 q\bar{n}S_1^2 - r_1 M_e^2 F_2}{\bar{n}\chi_2^2}\right],$$

$$h_2(S_1^2, S_2^2, M_e^2) = \min\left[\frac{r_2\bar{n}S_2^2 - r_2 M_e^2 F_3}{\bar{n}\chi_3^3}, \frac{r_1 q\bar{n}S_1^2 - r_1 M_e^2 F_1}{\bar{n}\chi_1^2}\right].$$

This determines the required simultaneous confidence intervals for σ_a^2 & σ_b^2.

4. *Numerical Illustration*

Consider the example from Graybill (1961, pp. 358) with the modification that $p = 4$, $q = 2$, $n_1 = 5$, $n_2 = 4$, $n_3 = 3$ $n_4 = 4$,

$$\hat{\sigma}_a^2 = 61.4739, \quad \hat{\sigma}_b^2 = 5.0315 \text{ and } \hat{\sigma}_e^2 = 20.0529,$$

$$r_1 = 3, \; r_2 = 4, \; r_3 = 24, \; \bar{n} = 3.871, \; S_1^2 = 66.5798,$$

$$S_2^2 = 10.2118 \text{ and } M_e^2 = 20.0529, \; \alpha_1 = \alpha_2 = 0.02$$

with equal tail probability we get

$$\chi_1^2 = \chi_l^2\ (0.99;3) = 0.11483,\ \chi_2^2 = \chi_u^2\ (0.01;3) = 11.343,$$

$$\chi_3^2 = \chi_l^2\ (0.99,4) = 0.29711,\ \chi_4^2 = \chi_u^2\ (0.01,4) = 13.277,$$

$$F_1 = F_l\ (0.99;3,24) = 0.037,\ F_2 = F_u\ (0.01;3,24) = 4.7181,$$

$$F_3 = F_l\ (0.99;4,24) = 0.071793,\ F_4 = F_u\ (0.01;4,24) = 4.2184.$$

The simulaneous confidence intervals about the function of σ_a^2, σ_b^2 and σ_e^2 is given as

$$P\,[\ 136.30519 \le (\sigma_e^2 + 3.871\ \sigma_b^2 + 7.742\ \sigma_a^2) \le 13466.711,$$

$$11.90928 \le (\sigma_e^2 + 3.871\ \sigma_b^2) \le 532.19182\ ] = 0.98.$$

and $$P\,[\ 1.1491031 \le \frac{\sigma_b^2}{\sigma_e^2} + 2\frac{\sigma_a^2}{\sigma_e^2} \le 179.21236,$$

$$0 \le \frac{\sigma_b^2}{\sigma_e^2} \le 45.988628\ ] \ge 0.98.$$

The simultaneous confidence interval for σ_a^2, σ_b^2 formed from the boundary intersection are

$$P\,[\ 14.374413 \le \sigma_a^2 \le 1736.8916, 0 \le 132.47472\ ] \ge 0.96.$$

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