ON ROW-BALANCE IN P.B.I.B. DESIGNS

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1. Introduction

A good number of designs with provisions of two-way elimination of heterogeneity is now available in literature. Actually all these designs have been obtained by rearranging designs with one-way elimination so as to get some balance for the row effects. Thus Latin square is obtained from Randomised Block design and Youden's Square from symmetrical Balanced Incomplete Block design. Recently a number of designs like Youden's Square following from other types of incomplete block designs, have been obtained by the various research workers in this field. Smith and Hartley (1948) showed that from any symmetrical incomplete block design, it is possible to get design with two-way elimination of heterogeneity by suitable arrangement of the treatments in different positions within blocks. Earlier Bose and Kishen (1939) found some designs of this type from symmetrical partially balanced incomplete block designs. Afterwards, extending this idea Shrikhande (1951), Hartley, Shrikhande and Taylor (1953) and Taylor (1957) found some more designs following from balanced incomplete block designs with $b = mv$, and $b \neq mv$ where m is an integer. Shrikhande (1951) found some other designs also following from partially balanced incomplete block designs with $b = mv$ and also $b \neq mv$. In the present note, a class of p.b.i.b. designs with $b \neq mv$ has been obtained, for which two-way elimination is possible. As no inter-block analysis of such row-balanced p.b.i.b. designs is available so far, a method of such analysis has also been included in the present note.

2. Description

Taking, as usual, the parameters of p.b.i.b. designs as v, b, r, k, λ_i $(i = 1, 2, \ldots s)$ n_i $(i = 1, 2, \ldots s)$ and p^k_{ij} $(i, j, k = 1, 2, \ldots s)$;

where

$$bk = vr \tag{1}$$

$$\sum_i n_i = v - 1 \tag{2}$$

$$\sum_i n_i \lambda_i = r(k-1) \tag{3}$$

$$\begin{aligned} \sum_j p^k_{ij} &= n_i \text{ when } i \neq k \\ &= n_i - 1 \text{ when } i = k \end{aligned} \tag{4}$$

$$n_k p^k_{ij} = n_i p^i_{jk} = n_j p^j_{ik} \tag{5}$$

these designs can be devided into 3 broad classes, *viz.*, (i) $b = v$, (ii) $b = mv$, (iii) $b \neq mv$ or $r = mk + \tau$ where $m > 1$ and τ are positive integers. Bose and Kishen (1939) developed the row-balancing of symmetrical p.b.i.b. designs and discussed their analysis without recovery of inter-block information. Shrikhande (1951) extended this idea to the case of p.b.i.b. designs with $b = mv$ as well as $b \neq mv$ and found out two designs for $b \neq mv$. Here we shall discuss about a class of designs with $b \neq mv$ for which two-way elimination of heterogeneity is possible. As $r \neq mk$, arrangement of treatments within blocks so that every treatment is replicated a constant number of times in each row is impossible. The alternative is that some partially balanced arrangement of the treatments within rows may be possible. It may be possible to have such arrangements for many p.b.i.b. designs but an analysis after eliminating row-effects becomes straightforward if the following conditions are satisfied:

(i) each treatment is replicated at least m times and atmost $(m+1)$ times in each row; where m is a positive integer;

(ii) the $(m+1)$th replicate of the treatments (or briefly the odd treatments) can be so arranged among the rows, that the rows form a p.b.i.b. design with parameters

$$v' = v; \; b' = k; \; r', k', \; \lambda_1', \; \lambda_2', \ldots . \lambda_s'; \; n_1', n_2' \ldots . n_s';$$

$$n_1' = n_1, \quad n_2' = n_2, \ldots . n_s' = n_s; \quad p'^k_{ij} = p^k_{ij} \, (i, j, \quad k = 1, 2, \ldots . \, s)$$

or, in other words, with respect to any odd treatment (say α) the remaining odd treatments can be divided into the same s groups as the original p.b.i.b. design such that α and any other treatment of the i-th group ($i = 1, 2, \ldots . s$) occur together in λ_i' ($i = 1, 2 \ldots . s$) blocks.

An example for such p.b.i.b. designs with parameters $v = 12$, $b = 16$, $k = 3$, $r = 4$, $\lambda_1 = 1$, $\lambda_2 = 0$, $n_1 = 8$, $n_2 = 3$, $p^1_{ij} = \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}$ $p^2_{ij} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$ is given on next page.

Rows	Blocks															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	5	7	10	4	1	6	3	11	8	12	9	10	7	4	1
2	1	4	8	11	12	9	10	7	6	3	5	2	5	2	11	8
3	3	6	9	12	8	11	2	5	7	10	1	4	9	12	3	6

The arrangement of the odd treatments in the rows are:

Rows	Odd treatments in the rows			
1	1	4	7	10
2	2	5	8	11
3	3	6	9	12

which is a p.b.i.b. design with parameters

$$v' = 12,\ b' = 3,\ r' = 1,\ k' = 4,\ n_1' = 8,\ n_2' = 3,\ \lambda_1' = 0,$$

$$\lambda_2' = 1,\ p'^k_{ij} = p^k_{ij}\ (i, j, k = 1, 2).$$

Theorem 1

The following class of designs given by [Bose and Nair (1939)]

$$v = p\,(p-1),\ b = p^2,\ r = p,\ k = p-1,\ n_1 = p\,(p-2),$$

$$n_2 = p-1,\ \lambda_1 = 1,\ \lambda_2 = 0;\ p_{ij}^1 = \begin{pmatrix} p\,(p-3) & (p-1) \\ p-1 & 0 \end{pmatrix};$$

$$p^2_{ij} = \begin{pmatrix} p\,(p-2) & 0 \\ 0 & p-2 \end{pmatrix}$$

where p is a prime or power of a prime, can be arranged satisfying the above two conditions. The example given above is a special case when $p = 4$.

For proof it is necessary to prove the following lemma.

Lemma 1

The class of designs given above is resolvable.

Proof.—The proof easily follows from the method of construction of the class given by Bose and Nair (1939) which is as follows:—

Let the $p(p-1)$ varieties be arranged as

$$\begin{array}{l} 1, 2, 3, \quad \cdot \quad \cdot \quad \cdot \quad p \\ p+1, p+2, p+3, \quad \cdot \quad 2p \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (p-2)p+1, (p-2)p+2, (p-2)p+3 \ldots . p(p-1). \end{array}$$

Consider the set of $(p-1)$ orthogonal Latin square of side p. In the first orthogonal square, replace the 'p' letters by the varieties 1, 2, . . p and in the second square by $p+1$, $p+2, \ldots . 2p$ and so on for the remaining $(p-3)$ Latin squares of the set. Now superimposing on the first square, the remaining $(p-2)$ squares so formed and taking the cells each containing $p-1$ varieties as blocks, we get the required design. Since each treatment occurs once and only once in each row and column of each of the Latin squares, p blocks corresponding to the cells in any row (or column) of the superimposed Latin square give a complete replication of the treatments. Similarly, the blocks corresponding to the other rows (or columns) account for the other replications. Thus, the blocks get divided into as many sets as there are rows (or columns).—Hence the lemma.

Proof of Theorem 1

Since the design is resolvable, it is possible to separate out a set "S" of p blocks such that each treatment occurs once in this set of blocks. Again as $b = v + p$, the remaining $b - p$ blocks constitute an incomplete block design with $b = v$ and it is always possible to arrange the treatments within the $b - p$ blocks so that each treatment occurs once in each row. As each treatment occurs once in the set "S" also; in the arrangement of the complete design each treatment occurs at least once and at most twice in each row.

Now let us denote the treatments $1, 2, \ldots . p$ by the group "A_1"; $p+1$, $p+2 \ldots . 2p$ by "A_2"; . . . . ; $(p-2)p+1$, $(p-2)p+2, \ldots .$ $p(p-1)$ *by* $A(p-1)$; From the method of construction of the design it is evident that the different blocks of set " S " are obtained by taking one different treatment from each of $(p-1)$ groups. Thus in arranging the treatments within the blocks of " S " in rows it is always possible to bring all the treatments of the same group in the same row. Thus the odd treatments in the different rows also form a p.b.i.b. design with parameters

$v' = v$, $b' = k$, $r' = 1$, $k' = p$, $\lambda_1' = 0$, $\lambda_2' = 1$, $n_1' = n_1$, $n_2' = n_2$ and $p'^{k}_{ij} = p^{k}_{ij}$.

Hence the theorem is proved.

Similarly it can be shown that the class of p.b.i.b. designs given by Bose and Nair (1939) with parameters

$$v = pq;\ \ b = p^2;\ \ r = p;\ \ k = q;\ \ n_1 = p(q-1);$$

$$n_2 = p - 1;\ \lambda_1 = 1,\ \lambda_2 = 0;\ p^1_{jk} = \begin{pmatrix} p(q-2) & p-1 \\ p-1 & 0 \end{pmatrix},$$

$$p^2_{jk} = \begin{pmatrix} p(q-1) & 0 \\ 0 & p-2 \end{pmatrix}$$

for values of p, q given below can be arranged satisfying the above two requirements. Shrikhande's design (1951) with $v = 15$, $b = 25$, $r = 5$, $k = 3$, $n_1 = 10$, $n_2 = 4$, $\lambda_1 = 1$, $\lambda_2 = 0$; $p^1_{jk} = \begin{pmatrix} 5 & 4 \\ 4 & 0 \end{pmatrix}$, $p^2_{jk} = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$ is a particular case of the class with $p = 5$ and $q = 3$. In the case of designs with $b - v = \alpha p$, where α is an integer and greater than one, the arrangement of the odd treatments in the rows will be such, that the treatments complementary* to the odd treatments should belong to the groups defined above.

	Values of p					
	$p = 4$	$p = 5$	$p = 7$	$p = 8$	$p = 9$	$p = 10$
Values of q	3	4	6	7	8	9
	2	3	4	4	5	5
		2	3	3	4	3
			2	2	3	2
					2	

* Treatments which do not occur $m + 1$ times in the rows.

3. Method of Analysis

We shall discuss here the analysis of two associates class of p.b.i.b. designs with $r = mk + \tau$.

Let

$$y_{ijt} = \mu + b_j + r_i + v_t + \epsilon_{ij\,(t)}$$

where

b_j = jth block effect, $j = 1, 2, \ldots b$

v_t = tth treatment effect, $t = 1, 2, \ldots v$

r_i = ith row effect, $i = 1, 2, \ldots k$

μ = grand mean.

$y_{ij\,(t)}$ = observed yield of tth treatment in ith block and jth row

and $\epsilon_{ij\,(t)}$, s are randomly distributed with zero mean and variance σ^2. The normal equations by the method of least square are

$$V_t = \sum_{ij} y_{ij\,(t)} = r\,(\mu + v_t) + S_t\,(r_i) + S_t\,(b_j) \tag{7}$$

$$B_j = \sum_{i\,(t)} y_{ij\,(t)} = k\,(\mu + b_j) + S_j\,(v_t) \tag{8}$$

$$R_i = \sum_{j\,(t)} y_{ij(t)} = b\,(\mu + r_i) + S_i\,(v_t) \tag{9}$$

$$G = \Sigma\, y_{ij\,(t)} = bk\mu \tag{10}$$

where

$S_t\,(r_i)$ = Sum of row effects over those rows containing the odd replicate of v_t;

$S_t\,(b_j)$ = Sum of block effects over those blocks containing v_t;

$S_i\,(v_t)$ = Sum of all the odd replicates in the ith row;

$S_j\,(v_t)$ = Sum of all the treatments in the jth block.

Let us define

$$Q_t = \left(V_t - \frac{1}{k}\,S_t\,(B_j) - \frac{1}{b}\,S_t\,(R_i) + \frac{\tau G}{bk}\right).$$

Hence substituting from the normal equations, Q_t becomes

$$Q_t = \left\{\frac{r(k-1)+\lambda_1}{k} + \frac{\lambda_1' - r'}{b}\right\} v_t$$

$$+ \left\{\frac{\lambda_1 - \lambda_2}{k} + \frac{\lambda_1' - \lambda_2'}{b}\right\} \sum v_{t_2}; \tag{11}$$

where Σv_{t_2} = sum of all the treatments which are 2nd associates of v_t. Summing (11) over all the treatments which are 2nd associates of v_t we have

$$\sum Q_{t_2} = \left[\left\{\frac{\lambda_1 - \lambda_2}{k} + \frac{\lambda_1' - \lambda_2'}{b}\right\} p^1_{21}\right] v_t$$

$$+ \left[\left\{\frac{r(k-1)+\lambda_1}{k} + \frac{\lambda_1' - r'}{b}\right\}\right.$$

$$\left. + \left\{\frac{\lambda_1 - \lambda_2}{k} + \frac{\lambda_1' - \lambda_2'}{b}\right\}\left\{p^2_{22} - p^1_{22}\right\}\right] \sum v_{t_2} \tag{12}$$

where ΣQ_{t_2} = sum of Q_t's of those v_t's which are 2nd associates of v_t. From equations (11) and (12) the solution for v_t is

$$v_t = \frac{1}{\triangle}\left\{(B_{22} - Q_t - B_{12}\, \Sigma Q_{t_2}\right\} \tag{13}$$

where

$$A_{12} = \frac{r(k-1)+\lambda_1}{k} + \frac{\lambda_1' - r'}{b}$$

$$B_{12} = \frac{\lambda_1 - \lambda_2}{k} + \frac{\lambda_1' - \lambda_2'}{b}$$

and

$$\left.\begin{aligned} A_{22} &= B_{12}\, p^1_{21} \\ B_{22} &= A_{12} + B_{12}(p^2_{22} - p^1_{22}) \\ \triangle &= A_{12} B_{22} - B_{12} A_{22}. \end{aligned}\right\} \tag{14}$$

Now the sum of squares due to treatments can be obtained from $\Sigma v_t Q_t$ and the different variance components are presented in Table I.

TABLE I

Analysis of Variance

Sources		d.f.	S.S.
Blocks	..	$b-1$	$\frac{1}{k}\sum_i B_i^2 - \frac{G^2}{bk}$
Rows	..	$k-1$	$\frac{1}{b}\Sigma R_i^2 - \frac{G^2}{bk}$
Treatments	..	$v-1$	$\Sigma v_t Q_t$
Residual	..	$bk-v-b+2$	On substraction
TOTAL	..	$bk-1$	$\Sigma y^2_{ij(t)} - \frac{G^2}{bk}$.

Variances of different treatment contrasts are given by

$$\mathrm{Var}\,(v_{t_1} - v_{t_2}) = \frac{2\sigma^2}{\triangle}(B_{22}) \tag{15}$$

when v_{t_1} and v_{t_2} are 1st associates;

$$\mathrm{Var}\,(v_{t_1} - v_{t_2}) = \frac{2\sigma^2}{\triangle}(B_{22} + B_{12}). \tag{16}$$

when v_{t_1} and v_{t_2} are 2nd associates, and σ^2 is estimated by the residual mean square whose expectation is $(bk - v - k - b + 2)\,\sigma^2$.

3.1. RECOVERY OF INTER-BLOCK INFORMATION

Assuming the block effects to be normally distributed with zero mean and variance σ_1^2, it is possible to increase the accuracy of the experiment by extracting the information on treatment comparisons given by the block totals (B_j). Maximising:

$$\frac{1}{\sigma^2}\sum (y_{ij(t)} - \mu - r_i - b_j - v_t)^2 + \frac{1}{k\,(\sigma^2 + k\sigma_1^2)}$$

$$\times \left\{ \sum_j \left[(B_j - k\mu - S_j(v_t) \right]^2 \right\}. \tag{20}$$

with respect to v_t and using the equations given by (7), (8), (9) and (10) we have

$$Q_t = -f\left[\frac{S_t(B_j)}{k} - \frac{rG}{bk}\right] + \left[\frac{r(k-1) + \lambda_1 + f(r-\lambda_1)}{k} + \frac{1}{b}(\lambda_1' - r')\right] v_t + \left[\frac{(\lambda_1 - \lambda_2)(1-f)}{k} + \frac{\lambda_1' - \lambda_2'}{b}\right] \sum v_{t_2}$$

or

$$P_t = Q_t + Q_t' = \left[\frac{r(k-1) + \lambda_1 + f(r-\lambda_1)}{k} + \frac{\lambda_1' - r'}{b}\right] v_t + \left[\frac{(\lambda_1 - \lambda_2)(1-f)}{k} + \frac{\lambda_1' - \lambda_2'}{b}\right] \sum v_{t_2} \qquad (21)$$

where

$$Q_t' = \frac{1}{k} S_t(B_j) - \frac{rG}{bk},$$

$$f = \frac{\sigma^2}{\sigma^2 + k\sigma_1^2}$$

and

$$P_t = Q_t + Q'_t.$$

Summing (21) over all treatments which are 2nd associates of v_t we have

$$\Sigma P_{t_2} = \left[\frac{(\lambda_1 - \lambda_2)(1-f)}{k} + \frac{(\lambda_1' - \lambda_2')}{b}\right] p^1_{21} v_t + \left[\left\{\frac{r(k-1) + \lambda_1 + f(r-\lambda_1)}{k} + \frac{\lambda_1' - r'}{b}\right\} + \left\{\frac{(\lambda_1 - \lambda_2)(1-f)}{k} + \frac{\lambda_1' - \lambda_2'}{b}\right\} \times \{p^2_{22} - p^1_{22}\}\right] \sum v_{t_2}. \qquad (22)$$

From equations (21) and (22) we have

$$v_t = \frac{B'_{22} P_t - B'_{12}\, \Sigma P_{t_2}}{\Delta'} \qquad (23)$$

where

$$A'_{12} = \frac{r(k-1) + \lambda_1 + f(r - \lambda_1)}{k} + \frac{(\lambda_1' - r')}{b}$$

$$B'_{12} = \frac{(\lambda_1 - \lambda_2)(1-f)}{k} + \frac{(\lambda_1' - \lambda_2')}{b}$$

$$A'_{22} = B'_{12} p^1_{21}$$

$$B'_{22} = A'_{12} + B'_{12}(p^2_{22} - p^1_{22})$$

$$\triangle' = A'_{12} B'_{22} - A'_{22} B'_{12}$$

and ΣP_{t_2} = sum of P_t's of those v_t's which are 2nd associates of v_t.

Now

$$\text{Var}(v_{t_1} - v_{t_2}) = \frac{2\sigma^2}{\triangle'}(B'_{22})$$

when v_{t_1} and v_{t_2} are 1st associates; and

$$\text{Var}(v_{t_1} - v_{t_2}) = \frac{2\sigma^2}{\triangle'}\{B'_{22} + B'_{12}\}$$

when v_{t_1} and v_{t_2} are 2nd associates.

As indicated earlier, the estimate of σ^2 can be obtained from the residual mean square of Table I whose expectation is σ^2. Proceeding in the same way as Taylor (1957) has done for *B.I.B.* design, the estimate of σ_1^2 can be obtained from the expected value of Block S.S. of Table II (below):

TABLE II

Sources	d.f.	S.S.
Rows and Treatments ..	$v + k - 2$	$\frac{1}{k}\sum R_i^2 + \sum v_t{}^* P_t$
Blocks ..	$b - 1$	On substraction
Residual ..	$bk-b-k-v+2$	Same as Table I
TOTAL ..	$bk - 1$	$\sum y^2_{ijt} - \frac{G^2}{bk}$

* v_t is the solution of the equations

$$(rb - r + \lambda_1')\, v_t + (\lambda_1' - \lambda_2')\, \Sigma v_{t_2} = P_t$$

and

$$(\lambda_1' - \lambda_2')\, p^1_{21}\, v_t + [(rb - r + \lambda_1') + (\lambda_1' - \lambda_2')(p^2_{22} - p^1_{22})] \times \Sigma v_{t_2} = \Sigma P_t$$

Expected value of block S.S. for the class of designs defined above (Taking block effects to be random) becomes

$$(b-1)\,k\sigma_1^2 - \left\{\frac{br\,(v-k)}{br-\tau+\lambda_2'} + \frac{b\,(\lambda_2'-\lambda_1')\,(v-p)\,(r-\lambda_1)}{(br-\tau+\lambda_2')\,(br-\tau+\lambda_2'+p\,(\lambda_1'-\lambda_2'))}\right\}\sigma_1^2 + (b-1)\,\sigma^2.$$

Equating the calculated value to its expected value the required estimate of σ_1^2 is obtained.

4. SUMMARY

A class of partially balanced incomplete block designs with $b \neq mv$ for which two-way elimination of heterogeneity is possible has been obtained. Methods of analysis with and without recovery of inter-block information has been also given for such designs.

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6. REFERENCES

1. Bose, R. C. and Nair, K. R. .. *Sankhya*, 1939, **4**, 337–72.
2. ——— and Kishen, K. .. *Science and Culture*, 1939, **5**, 136–37.
3. Hartley, H. O., Shrikande, S. S. and Tayler, W. B. *Annals of Math. Stat.*, 1953, **24**, 123–28.
4. Smith, C. A. B. and Hartley, H. O. *J.R.S.S.*, Series B, 1948, **10**, 262–63.
5. Shrikhande, S. S. .. *Annals of Math. Stat.*, 1951, **22**, 235–47.
6. Taylor, W. B. .. *Biometrics*, 1957, **13**, 1–12.