A SIMPLIFIED METHOD OF ENUMERATING LATIN SQUARES BY MACMAHON'S DIFFERENTIAL OPERATORS

Part I. The 6 × 6 Latin Squares

BY P. N. SAXENA

Indian Council of Agricultural Research, New Delhi

CONTENTS

§1. Introduction—A Brief Historical Survey

The problem of the enumeration of Latin squares was first undertaken by Euler (1782) in searching for a 6×6 Græco-Latin square. Using an exhaustive process, he found 1, 1, 4 and 56 '*standard*' Latin squares—*i.e.*, those with the letters of the first row and first column in alphabetical order—of sides 2, 3, 4 and 5 respectively.

Cayley (1890) suggested the recognition of groups of squares by means of the permutation which effects passage from one row to another, and he correctly enumerated the Latin squares of sides 2, 3, 4 and 5.

Tarry (1900), using an exhaustive process based on the relations between pairs of rows, showed that there are exactly 17 distinct groups of 6×6 Latin squares, and that they number 9408 in all.

Fisher and Yates (1934) enumerated the 6×6 Latin squares by the method of intramutation and transformation sets, and they also found 9408 standard squares of 17 types, none of the 17 types having Græco-Latin solutions.

Norton (1939), using the method of species and intercalates, presented, in his own words, " an extensive—possibly an exhaustive—study of 7×7 Latin and higher squares". He found 16,927,968 different standard 7×7 Latin squares.

MacMahon (1915) gave a complete algebraic solution of the problem of enumeration of Latin squares of any order. His method is equivalent to determining by *exhaustive* trial all the possible ways of writing a Latin square. After demonstrating the determination of the number of Latin squares of side 4, MacMahon says of his method, 'The calculations for higher orders become impracticable....'

It is the object of this paper to exhibit a simpler alternative general solution, based on MacMahon's operators, of the problem of *directly* enumerating the standard Latin squares of any order; and to establish two new theorems of general applicability introducing considerable further simplifications in the use of the formula. Also the enumeration of the 6×6 Latin squares has, in this part of the paper, been carried out in detail.

§2. MacMahon's Method of Differential Operators

Before presenting the modification of and other simplifications in MacMahon's method, it is now essential to introduce certain notations and definitions, together with an outline leading to his enumerating formula.

(i) *Notation and Definitions*

Def. 1. *Partition.*—Let the ρ-part partition of weight w be denoted as usual by the symbol

$$(p_1^{\pi_1} p_2^{\pi_2} \ldots p_s^{\pi_s}),$$

where

$$p_1\pi_1 + p_2\pi_2 + \ldots + p_s\pi_s = w, \text{ and } \pi_1 + \pi_2 + \ldots + \pi_s = \rho.$$

Def. 2. *Symmetric Function.*—Let further this partition represent the following monomial symmetric function of the a's of 'weight' w:

$$(p_1^{\pi_1} p_2^{\pi_2} \ldots p_s^{\pi_s}) \equiv \Sigma\, (a_1^{p_1} a_2^{p_1} \ldots a_{\pi_1}^{p_1} . a_{\pi_1+1}^{p_2} \ldots a_{\pi_1+\pi_2}^{p_2} \ldots a_{\pi_1+\pi_2+\ldots+\pi_s}^{p_s}),$$

where the p's are fixed and the summation extends over all the ways of assigning the ρ subscripts, including permutations, from the n available.

Def. 3. *Elementary Symmetric Function.*—The quantities

$$a_1, a_2, \ldots a_n$$

defined by

$$a_s = \Sigma a_1 a_2 \ldots a_s \equiv (1^s),\ s = 1, 2, \ldots n;$$

are called 'Unitary' or 'Elementary' symmetric functions.

Def. 4. *Perfect Partition.*—It is a partition such that, using its parts, one and only one partition of every lesser number can be formed.

e.g.—(41^3) is a perfect partition of 7.

The general form of a perfect partition is

$$1^A.(1+A)^B.\{(1+A)(1+B)\}^C.\{(1+A)(1+B)(1+C)\}^D\ldots.$$

Thus, putting

$$A = B = C = \ldots = 1,$$

$$1, 2, 2^2, \ldots 2^{n-1}$$

is a perfect partition of $2^n - 1$, with no repeated part.

Def. 5. *Composition.*—This is a partition in which account is taken of the order of the different parts in the partition.

Thus (21) and (12) are different compositions of 3.

(ii) *The D-Operators*

It is well known that any monomial symmetric function of weight w may be expressed as a linear function of the products of weight w of the elementary symmetric functions $a_1, a_2, a_3, \ldots$, and conversely.

Let us then consider the symmetric function equality:

$$(p_1^{\pi_1} p_2^{\pi_2} \ldots p_s^{\pi_s}) \equiv \phi\,(a_1, a_2, a_3, \ldots) \equiv \phi \text{ say} \qquad (2.1)$$

where

$$a_i \equiv (1^i).$$

Since, by definition,

$$(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) \equiv x^n - a_1 x^{n-1} + a_2 x^{n-2} + \ldots + (-)^n a_n;$$

multiplication on both sides by $(x-\mu)$ yields:

$$(x-\mu)(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) \equiv x^{n+1} - (a_1+\mu)\,x^n +$$
$$(a_2+\mu a_1)\,x^{n-1} - (a_3+\mu a_2)\,x^{n-2} + \ldots + (-)^n\,(a_n + \mu a_{n-1})x$$
$$+(-)^{n+1}\cdot\mu a_n.$$

Hence, introducing μ in (2.1) we obtain the identity

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots) + \mu^{p_1}(p_1^{\pi_1-1}p_2^{\pi_2}\ldots) + \mu^{p_2}(p_1^{\pi_1}p_2^{\pi_2-1}\ldots) + \ldots$$

$$\equiv \phi\,(a_1 + \mu,\ a_2 + \mu a_1,\ a_3 + \mu a_2,\ \ldots)$$

$$= \phi + \frac{\mu}{1!}\,d_1\phi + \frac{\mu^2}{2!}\,.\,d_1{}^2\phi + \ldots, \text{ by Taylor's expansion,}$$

where

$$d_1 \equiv \frac{\partial}{\partial a_1} + a_1\frac{\partial}{\partial a_2} + a_2\frac{\partial}{\partial a_3} + \ldots \qquad (2.2)$$

$$= \phi + \mu\,.\,D_1\phi + \mu^2\,.\,D_2\phi + \ldots,$$

where

$$D_s \equiv \frac{(d_1{}^s)}{s!} \qquad (2.3)$$

and $(d_1{}^s)$ stands for the symbolic product $d_1 \times d_1 \times \ldots s$ factors.

Equating powers of μ on both sides of the identity, we obtain

$$\left.\begin{array}{l} D_{p_1}(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots) = (p_1^{\pi_1-1}p_2^{\pi_2}p_3^{\pi_3}\ldots), \\ D_{p_2}(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots) = (p_1^{\pi_1}p_2^{\pi_2-1}p_3^{\pi_3}\ldots), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \text{ etc.} \\ D_s(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots) = 0 \text{ if } s \text{ is not included among} \\ \qquad\qquad \text{the parts } p_1, p_2, p_3\ldots, \\ \text{and} \qquad D_s(s) = 1. \end{array}\right\} \qquad (2.4)$$

Since the effect of the operator D_λ on ϕ is to delete a part λ from the partition denoting the symmetric function ϕ, this operator has been called the '*obliterating weight*' operator. It will be noticed that the D-operators obey the commutative law, *viz.*,

$$D_pD_q = D_qD_p.$$

(iii) *Operation of* D_λ *upon a Product of Symmetric Functions*

Let ϕ_s denote a symmetric function, symbolised by a single partition, and expressed as a function of the elementary symmetric functions $a_1, a_2, a_3, \ldots$. Let s take the values $1, 2, \ldots m$ and set

$$\phi = \phi_1 . \phi_2 . \ldots \phi_m .$$

As before the introduction of an additional quantity μ in the set $\alpha_1, \alpha_2, \alpha_3, \ldots,$ transforms

$$\phi \text{ into } (1 + \mu D_1 + \mu^2 D_2 + \ldots)\, \phi,$$

and

$$\phi_s \text{ into } (1 + \mu D_1 + \mu^2 D_2 + \ldots)\, \phi_s, \qquad (s = 1, 2, \ldots m).$$

We thus reach the identity

$$\begin{aligned}(1 + \mu D_1 + \mu^2 D_2 + \ldots)\, \phi \equiv\ & (1 + \mu D_1 + \mu^2 D_2 + \ldots)\, \phi_1 \\ & \times (1 + \mu D_1 + \mu^2 D_2 + \ldots)\, \phi_2 \\ & \times \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ & \times (1 + \mu D_1 + \mu^2 D_2 + \ldots)\, \phi_m.\end{aligned}$$

Equating coefficients of μ^λ on both sides, we obtain

$$D_\lambda . \phi = \Sigma_1 \Sigma_2 D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 \ldots D_{\lambda_s}\phi_s . \phi_{s+1} \ldots \phi_m, \qquad s \leqslant m$$

where

$\Sigma_1 =$ Sum over every partition $(\lambda_1, \lambda_2, \ldots \lambda_s)$ of λ into m *or fewer* parts.

and

$\Sigma_2 =$ Sum over every permutation of every selection of s out of the m factors $\phi_1, \phi_2, \ldots \phi_m$.

Writing

$$\Sigma_2 D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 \ldots D_{\lambda_s}\phi_s . \phi_{s+1} \ldots \phi_m \equiv D_{(\lambda_1\lambda_2\ldots\lambda_s)}\, \phi_1 . \phi_2 . \ldots \phi_m \quad (2.5)$$

we shall call

$$D_{(\lambda_1\lambda_2\ldots\lambda_s)}$$

as a '*partition obliterating operator*'.

In terms of these operators, our result may be written:

$$D_\lambda \phi = \Sigma_1 D_{(\lambda_1\lambda_2\ldots\lambda_s)} . \phi. \qquad (2.6)$$

Hence the operation D_λ is performed upon $\phi_1 . \phi_2 . \phi_3, \ldots \phi_m$ by abstracting every partition of λ in all possible ways from the product, one part at most being taken from each factor. Clearly, the partition-operators with more than m parts add nothing to the result.

In terms of compositions, the formula may be written

$$D_{\lambda} (\phi_1 . \phi_2 . \ldots \phi_m) = \Sigma_{\lambda} D_{\lambda_1}\phi_1 . D_{\lambda_2}\phi_2 \ldots D_{\lambda_m}\phi_m, \qquad (2.7)$$

where the factors $\phi_1, \phi_2, \ldots \phi_m$ remain in this order, and

Σ_{λ} = Sum over every composition of λ into *exactly* m parts, admitting zero parts,

and ϕ_s is unaffected by the symbol D_0.

The D-operators are commutative. Hence we may operate, successively and in any order, with any number of operators D_{λ}, D_{μ}, D_{ν}, ... on the product $\phi_1 . \phi_2 . \ldots \phi_m$. Also the order of the factors in the operand is immaterial to the result.

(iv) *Differential Operators and the Enumeration of Diagrams*

We shall now give an example to show how a differential operation may be dissected into several minor operations or processes. A group of operators then leads to various combinations of minor operations. Each combination may be made to correspond with a lattice diagram obeying certain row and column restrictions, and thus the differential formula may be used to enumerate such diagrams.

Consider the expression

$$\left(\frac{\partial}{\partial x_1} . \frac{\partial}{\partial x_2} . \frac{\partial}{\partial x_3}\right)^3 . (x_1 x_2 x_3)^3.$$

The first operation gives 3^3 terms, corresponding to the 27 permuted partitions of $x_1 x_2 x_3$ into exactly three parts, zero being reckoned as a part.

Selecting one of the 27 terms, say

$$\frac{\partial}{\partial x_1} . \frac{\partial}{\partial x_2} (x_1 x_2 x_3) . \frac{\partial}{\partial x_3} (x_1 x_2 x_3) . (x_1 x_2 x_3) = (\,.\,.\,x_3)(x_1 x_2\,.\,)(x_1 x_2 x_3)$$

we have before us one of the 27 minor operations into which the operation of $\frac{\partial}{\partial x_1} . \frac{\partial}{\partial x_2} . \frac{\partial}{\partial x_3}$ upon $(x_1 x_2 x_3)^3$ can be dissected. This may be diagrammatically denoted as a first row:

12	3	.

the diagram denoting that $\frac{\partial}{\partial x_1} \cdot \frac{\partial}{\partial x_2}$ has been performed on the first factor and $\frac{\partial}{\partial x_3}$ upon the second.

The second operation on this term yields 2^3 terms of the type:

$$(\cdot\cdot x_3)\frac{\partial}{\partial x_1}\cdot\frac{\partial}{\partial x_2}(x_1x_2.)\frac{\partial}{\partial x_3}(x_1x_2x_3) = (..x_3)(...)(x_1x_2.) \text{ say.}$$

The two successive operations thus produce the 2-row diagram:

12	3	·
·	12	3

Finally, in the third operation on this term, only one of the 27 minor operations is effective, *viz.*,

$$\frac{\partial}{\partial x_3}(..x_3)(...)\frac{\partial}{\partial x_1}\cdot\frac{\partial}{\partial x_2}(x_1x_2.) = 1.$$

The three successive minor operations thus produce the complete diagram:

12	3	·
·	12	3
3	·	12

It is clear that we can select three successive minor operations in $\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\right)^3(x_1x_2x_3)^3$ ways, since for each such selection unity is the result, and the result for all combinations of three selections must be $\left(\frac{\partial}{\partial x_1}\cdot\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\right)^3(x_1x_2x_3)^3$.

Thus the number $\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\right)^3 (x_1x_2x_3)^3 = (3!)^3$ enumerates diagrams with the following property:

(i) The diagrams have three rows and three columns.

(ii) The numbers 1, 2, 3 occur in each row and in each column, but the number of entries in each cell of the diagram is unrestricted.

In general, it may be noticed that the successive operators correspond to the rows and the factors of the operand correspond to the columns of our lattices. It is then apparent that a different arrangement of the factors in the operand or the operator, will yield different types of diagrams, but obviously the enumerating number is not changed.

(v) *The Latin Squares Enumerating Formula of MacMahon*

Let I_n denote the totality of Latin squares of order n, whether standard or not standard, and R_n the totality of standard Latin squares of the n-th order. Then it is well known that

$$I_n = n!\,(n-1)!\,R_n \tag{2.8}$$

We shall now indicate the derivation of MacMahon's general formula for I_n in the form

$$I_n = D^n{}_{2^n-1}(1.\,2.\,2^2.\,\ldots\,2^{n-1})^n. \tag{2.9}$$

This can be easily seen from the particular case of $n = 4$ as follows:

Consider the expression

$$D^4{}_{a+b+c+d}\,(abcd)^4,$$

where the D's are 'obliterating weight' operators and $(abcd)$ denotes the symmetric function

$$\Sigma \alpha_1{}^a\,\alpha_2{}^b\alpha_3{}^c\alpha_4{}^d.$$

Further, in order that $D_{a+b+c+d}$ pick out the numbers a, b, c, d, one from each factor of the operand, we must give such values to a, b, c, d that $(abcd)$ is the *only* partition of $a + b + c + d$ into 4 *or fewer* parts, which involves the parts a, b, c, d *repeated or not*. The simplest solution is obtained when the numbers a, b, c, d are so chosen that $(abcd)$ is a 'perfect' partition of the number $a + b + c + d$.

Since now $(abcd)$ is the only partition of $a + b + c + d$ into 4 or fewer parts, the summation sign Σ_1 in formula (2.6) drops out, and we have the operator equivalence

$$D_{a+b+c+d} \equiv D_{(abcd)} \tag{2.10}$$

In this case, therefore, the formula (2.7) reduces to

$$D_{a+b+c+d}\,(abcd)^4 = \Sigma D_a\,(abcd).D_b\,(abcd).D_c\,(abcd).D_d\,(abcd), \tag{2.11}$$

where the sum is taken over every composition, which is obtainable from the single partition $(abcd)$.

In other words, we have to delete every permutation of a, b, c, d dropping one and only one letter from each factor in the symmetric-function product:

$$(abcd) \,.\, (abcd) \,.\, (abcd) \,.\, (abcd),$$

and keeping the factors in this order. We have, therefore, broken up the operation $D_{a+b+c+d}$ into 4! minor operations.

A typical term would be

$$(a.cd)\,(.bcd)\,(abc.)\,(ab.d).$$

This may be diagrammatically represented as a 1st row:

b	a	d	c

On each such term, we apply the second operation $D_{a+b+c+d}$. The letters dropped out give us the second row of our diagrams of the type:

b	a	d	c
c	b	a	d

and the symmetric-function product term is now

$$(a..d)\,(..cd)\,(.bc.)\,(ab..).$$

On every such term, we apply the 3rd operation $D_{a+b+c+d}$. The letters deleted generate the 3rd row of our diagrams of the type

b	a	d	c
c	b	a	d
a	d	c	b

and the symmetric-function term reduces to

$$(...d)\,(..c.)\,(.b..)\,(a...).$$

The 4th operation $D_{a+b+c+d}$ now generates the 4th row and gives us completed diagrams of the type:

b	*a*	*d*	*c*
c	*b*	*a*	*d*
a	*d*	*c*	*b*
d	*c*	*b*	*a*

reducing the symmetric-function term to unity.

The number

$$I_4 = D^4_{a+b+c+d}\,(abcd)^4$$

therefore enumerates diagrams of the type:

(i) The diagrams have four rows and four columns.

(ii) Every letter *a, b, c, d* occurs only once in each row and once in each column.

i.e., the Latin squares of order four.

The reasoning is obviously quite general, and I_n the totality of $n \times n$ Latin squares, is enumerated by

$$I_n = D^n_{p_1+p_2+\ldots+p_n}(p_1, p_2, \ldots p_n)^n$$

where $(p_1, p_2, \ldots p_n)$ is a perfect partition of the number $p_1+p_2+\ldots+p_n$. The only perfect partition involving distinct parts is given by the powers of 2

$$1, 2, 2^2, 2^3, \ldots$$

Hence, finally, the general formula may be written in the form:

$$\left.\begin{aligned} I_n &= D^n_{2^n-1}\,(1.2.2^2\ldots2^{n-1})^n \\ \text{with}\qquad R_n &= \frac{I_n}{n!\,(n-1)!} \end{aligned}\right\} \qquad (2.12)$$

(vi) *Illustration*

Ex. 1. *Enumeration of 4 × 4 Latin Squares.*—The enumerating formula is, taking $n = 4$,

$$I_4 = D_{15}^4\,(1.2.4.8)^4,$$

Using the equivalence $D_{15} \equiv D_{(1\cdot 2\cdot 4\cdot 8)}$, the 1st operation gives:

$$4! \times (248)\,(148)\,(128)\,(124).$$

The 2nd operation gives

$$4! \times [(48)\,(48)\,(12)\,(12)+(24)\,(48)\,(18)\,(12)+(28)\,(48)\,(12)\,(14)+$$
$$+(28)\,(14)\,(28)\,(14)+(24)\,(18)\,(28)\,(14)+(48)\,(14)\,(28)\,(12)$$
$$+(28)\,(14)\,(18)\,(24)+(24)\,(18)\,(18)\,(24)+(48)\,(18)\,(12)\,(24)].$$

The 3rd operation gives, in order:

$$4! \times [4 + 2 + 2 + 4 + 2 + 2 + 2 + 4 + 2] \times (1)\,(2)\,(4)\,(8).$$

The 4th operation gives, finally

$$4! \times [4 + 2 + 2 + 4 + 2 + 2 + 2 + 4 + 2] \times 1.$$

or $$I_4 = 4! \times 24,$$

and $$R_4 = \frac{4! \times 24}{4! \times 3!} = 4.$$

Thus there are 4 standard squares of side four.

§ 3. The Modifications of MacMahon's Formula

It is clear that to find R_n, the number of standard squares, we must first evaluate I_n, the number of standard and non-standard squares by formula (2.12). Since it is known that

$$I_5 = 56 \times 5! \times 4! = 161{,}280,$$

the calculations even for R_5 become almost unmanageable. We shall now derive a formula which gives R_n directly, without the necessity of first evaluating I_n.

Consider the formula for the 4×4 Latin squares:

$$D^4_{a+b+c+d}\ (abcd)^4.$$

Take the 1st operation $D_{a+b+c+d}$. Since we want standard Latin squares, the 1st row must be

$$a,\ b,\ c,\ d$$

in alphabetical order, *i.e.*, we delete 'a' from the 1st factor, 'b' from the 2nd, 'c' from the 3rd and 'd' from the 4th.

Hence the formula for building up the last three rows becomes

$$D_{a+b+c+d}\, D_{a+b+c+d}\, D_{a+b+c+d}\, (bcd)\,(acd)\,(abd)\,(abc).$$

Since the 1st column must also be in alphabetical order,

In the 2nd operation we must delete 'b' from the 1st factor and the formula becomes

$$D_{a+b+c+d}\, D_{a+b+c+d}\, (cd)\, D_{a+c+d}\, (acd)\,(abd)\,(abc).$$

In the 3rd operation we must delete 'c' from the 1st factor and the formula becomes

$$D_{a+b+c+d}\, (d)\, D_{a+b+d}\, .\, D_{a+c+d}\, (acd)\,(abd)\,(abc).$$

Finally, in the 4th operation, we must delete 'd' from the 1st factor and the formula becomes

$$R_4 = D_{a+b+c} . D_{a+b+d} . D_{a+c+d}\, (acd)\,(abd)\,(abc). \tag{3.1}$$

The method of proof is quite general, and noting that the operators are commutative, we may state our result in the form of a theorem:

THEOREM 1.—The number of standard Latin squares of side n is enumerated by the formula

$$R_n = D_{p_1+p_3+\cdots+p_n} . D_{p_1+p_2+p_4+\cdots+p_n} \ldots D_{p_1+p_2+\cdots+p_{n-1}}$$
$$(p_1, p_3, \ldots p_n)\,(p_1, p_2, p_4, \ldots p_n) \ldots (p_1, p_2, \ldots p_{n-1}) \tag{3.2}$$

where $(p_1, p_2, p_3, \ldots p_n)$ is the perfect partition $(1.2.4.8.16\ldots 2^{n-1})$.

We shall, for brevity, drop the p's and write only the suffixes. Since the p's have to be deleted, it follows that on working with the same rules of operation of the D's, it is immaterial whether we delete the numbers

$$1, 2, 2^2, \ldots 2^{n-1}$$

or the numbers

$$1, 2, 3, \ldots n.$$

The formula (3.2) may then be written

$$R_n = D_{(134\ldots n)}\, D_{(1245\ldots n)} \ldots D_{(123\ldots n-1)}$$
$$(134\ldots n)\,(1245\ldots n)\ldots(123\ldots n-1). \tag{3.3}$$

We shall now illustrate the working of this formula for the enumeration of 4 × 4 Latin squares. The work may be compared with *Ex.* 1 based on MacMahon's formula.

Ex. 1*a*. *Enumeration of* 4 × 4 *Latin squares* (*Aliter*).—From (3.3), we have

$$R_4 = D_{(134)} . D_{(124)} . D_{(123)} (134) (124) (123).$$

Operation $D_{(134)}$ gives the terms

$$(34) (12) (12) + (13) (24) (12) + (14) (12) (23).$$

Operation $D_{(124)}$ gives the terms

$$(3) (2) (1) + (3) (1) (2) + (3) (2) (1) + (1) (2) (3).$$

Operation $D_{(123)}$ gives, finally

$$R_4 = 1 + 1 + 1 + 1 = 4.$$

Further Simplifications

It is very simple to apply the general formula (3.3) to the enumeration of the 5 × 5 Latin squares.

However, we shall now prove two further general theorems, which apply to squares of any order, and introduce considerable additional simplifications in the enumeration.

It is convenient to introduce the idea in terms of the formula for enumerating the 5×5 Latin squares:

$$R_5 = D_{a+c+d+e}\, D_{a+b+d+e}\, D_{a+b+c+e}\, D_{a+b+c+d}\, (acde)\, (abde)\, (abce)\, (abcd).$$

Consider the operation of $D_{a+c+d+e} \equiv D_{(acde)}$. Clearly, we can break up this operation into the following four parts:

Operation	Terms obtained
(i) When '*a*' is deleted from the 1st factor	$(cde)\ D_{(cde)}\ (abde)(abce)(abcd)$
(ii) " '*c*' " "	$(ade)\ D_{(ade)}\ (abde)(abce)(abcd)$
(iii) " '*d*' " "	$(ace)\ D_{(ace)}\ (abde)(abce)(abcd)$
(iv) " '*e*' " "	$(acd)\ D_{(acd)}\ (abde)(abce)(abcd)$

The result of the operation $D_{(acde)}$ is then the sum of the terms in (i), (ii), (iii) and (iv).

Let us now denote an element θ in the i-th row and j-th column of a Latin square by the symbol

$$(i, j, \theta).$$

Consider now the totality of standard Latin squares of order 5, containing the elements

$$(2, 2, a), (2, 2, c), (2, 2, d), (2, 2, e);$$

i.e., the number of ways of writing Latin squares under the following constraints:

a	*b*	*c*	*d*	*e*,	*a*	*b*	*c*	*d*	*e*,	*a*	*b*	*c*	*d*	*e*,	*a*	*b*	*c*	*d*	*e*
b	*a*	.	.	.	*b*	*c*	.	.	.	*b*	*d*	.	.	.	*b*	*e*	.	.	.
c	.	.	.	.	*c*	.	.	.	.	*c*	.	.	.	.	*c*	.	.	.	.
d	.	.	.	.	*d*	.	.	.	.	*d*	.	.	.	.	*d*	.	.	.	.
e	.	.	.	.	*e*	.	.	.	.	*e*	.	.	.	.	*e*	.	.	.	.

Denote these numbers by the symbols

$$A_5 \quad , \quad C_5 \quad , \quad D_5 \quad , \quad E_5$$

respectively.

Clearly,

$$R_5 = A_5 + C_5 + D_5 + E_5.$$

We shall now prove that

$$C_5 = D_5 = E_5.$$

Proof.—It follows from (i), (ii), (iii), (iv) above that

$$A_5 = D_{a+b+d+e} D_{a+b+c+e} D_{a+b+c+d} \cdot (cde)\ D_{(cde)}\ (abde)\ (abce)\ (abcd).$$

$$C_5 = D_{a+b+d+e} D_{a+b+c+e} D_{a+b+c+d} \cdot (ade)\ D_{(ade)}\ (abde)\ (abce)\ (abcd).$$

$$D_5 = D_{a+b+d+e} D_{a+b+c+e} D_{a+b+c+d} \cdot (ace)\ D_{(ace)}\ (abde)\ (abce)\ (abcd).$$

$$E_5 = D_{a+b+d+e} D_{a+b+c+e} D_{a+b+c+d} \cdot (acd)\ D_{(acd)}\ (abde)\ (abce)\ (abcd).$$

Notice that the result in any given case of performing the indicated operations is a numerical constant independent of the choice of *a*, *b*, *c*, *d*, *e*. Hence, the result would be the same, if in any one formula, we replace *a*, *b*, *c*, *d*, *e* by any permutation of these letters.

We note that the interchange of the two letters 'c' and 'd' converts the formula for C_5 into the formula for D_5—the order of the factors in the operator or the operand being immaterial. It immediately follows that

$$C_5 = D_5.$$

Similarly, the interchange of 'd' and 'e' shows that

$$D_5 = E_5.$$

We have therefore established that

$$C_5 = D_5 = E_5$$

and consequently

$$R_5 = A_5 + 3 . C_5. \tag{3.4}$$

The method of proof is quite general, and can be immediately extended to squares of any order. We may, therefore, state our general result in the form of:

THEOREM 2.—Let the universe of standard Latin squares of order n, containing the letters $a, c, d, e, f, \ldots$ in the cell (2, 2) be denoted by $A_n, C_n, D_n, E_n, F_n, \ldots$

Then

$$C_n = D_n = E_n = F_n = \ldots \tag{3.5}$$

and

$$R_n = A_n + (n - 2) . C_n \tag{3.6}$$

It will be noticed that the enumeration of

$$R_n = A_n + C_n + D_n + E_n + F_n + \ldots$$

has been reduced to the enumeration of A_n and C_n only.

We now pass on to simplifications in the enumeration of A_n or C_n. These simplifications result from a recognition of what may be termed '*equinumerous subsets*' of terms at various stages of the differential operations.

Def. 6.—If among the set of terms obtained after a D-operation we can find a subset of terms, each member of which gives, on performing the remaining D-operations, the same numerical result, then the subset consists of '*equinumerous*' terms.

For recognising such subsets, we shall prove the following theorem:

THEOREM 3.—If two terms of a set, obtained after a D-operation, are connected by a permutation which leaves the remaining D-operators invariant, then the two terms are equinumerous.

Proof.—Denoting the remaining set of operations by the symbol δ, and the two terms by α, β, we have the formulæ:

$$\delta(\alpha) = N_1$$

$$\delta(\beta) = N_2 \text{ say.}$$

The permutation converts α into β and keeps δ invariant. It therefore converts $\delta(\alpha)$ into $\delta(\beta)$.

Since the numerical result of any operation is independent of permutations within its letters, it follows that

$$\delta(\alpha) = \delta(\beta)$$

i.e., $$N_1 = N_2. \quad (3.7)$$

We shall now illustrate the use of these theorems on the enumeration of Latin squares of order 5.

Ex. 2. *Enumeration of* 5 × 5 *Latin Squares.*—Formula (3.3) gives

$R_5 = D_{(1345)} D_{(1245)} D_{(1235)} D_{(1234)}$ (1345) (1245) (1235) (1234)

The operation $D_{(1345)}$ gives the following four sets of terms:

(i) (345).(124) (125) (123)... t_1 (Deleting 1 from the 1st factor)
(345).(125) (123) (124)... t_2

(ii) (145).(245) (123) (123)... t_3
(145).(124) (235) (123)... t_4 („ 3 „)
(145).(125) (123) (234)... t_5

(iii) (135).(245) (123) (124)... t_6
(135).(124) (235) (124)... t_7 („ 4 „)
(135).(124) (125) (234)... t_8

(iv) (134).(245) (125) (123)... t_9
(134).(125) (235) (124)... t_{10} („ 5 „)
(134).(125) (125) (234)... t_{11}

From Theorem 2, it will be noted that the sets (ii), (iii) and (iv) are equivalent, so that it is unnecessary to write down the sets (iii) and (iv), and that

$$A_5 = D_{(1245)} D_{(1235)} D_{(1234)} [t_1 + t_2]$$

and

$$C_5 = D_{(1245)} D_{(1235)} D_{(1234)} [t_3 + t_4 + t_5].$$

Since the order of the factors in the operand is immaterial, it follows that

$$t_1 = t_2.$$

Next note that t_4 and t_5 are connected by the transposition* (45), which leaves the operator set

$$D_{(1245)} D_{(1235)} D_{(1234)}$$

invariant.

It follows from Theorem 3, that t_4 and t_5 are equinumerous. Hence

$$A_5 = D_{(1245)} D_{(1235)} D_{(1234)} [2t_1].$$

$$C_5 = D_{(1245)} D_{(1235)} D_{(1234)} [t_3 + 2t_4].$$

We now consider the operations on t_1, t_3, t_4 in turn:

Term t_1.—(123) (124) (125) (345).

$D_{(1245)}$ gives

$$(23)\,(14)\,(12)\,(35) + (23)\,(12)\,(15)\,(34) + (13)\,(24)\,(12)\,(35) + (13)\,(12)\,(25)\,(34).$$

Last two operations give

$$1 + 1 + 1 + 1 = 4.$$

Term t_3.—(123) (123) (145) (245).

$D_{(1245)}$ gives

$$2\,[(23)\,(13)\,(15)\,(24) + (23)\,(13)\,(14)\,(25)].$$

Last two operations give

$$2.[1 + 1] = 4.$$

Term t_4.—(123) (124) (145) (235).

* Interchange of two letters.

$D_{(1245)}$ gives:

$$(23)(14)(15)(23) + (23)(12)(14)(35) + (13)(24)(15)(23) + (13)(12)(45)(23).$$

Last two operations give

$$2 + 1 + 1 + 2 = 6.$$

It will be noticed that the last two operations can be performed simply by inspection. The last operation need not be performed, as there is only one way of writing the last row, when the previous rows are written down. The penultimate operation is easily performed, when we notice that the part '5' can be deleted from only one factor.

We thus obtain

$$A_5 = 2 \times 4 = 8$$

$$C_5 = 4 + 2 \times 6 = 16.$$

From formula (3.4),

$$R_5 = A_5 + 3.C_5$$
$$= 8 + 3.16$$
$$= 56.$$

§4. The Enumeration of 6 × 6 Latin Squares

The formula (3.3) gives

$$R_6 = D_{(13456)} D_{(12456)} D_{(12356)} D_{(12346)} D_{(12345)}$$
$$(13456)(12456)(12356)(12346)(12345).$$

Consider the operation of $D_{(13456)}$.

This operation can be split up into five parts according as 1, 3, 4, 5 or 6 is deleted from the first factor. The first two parts give us the following two sets of terms corresponding to A_6 and C_6:

We have

$$A_6 = D_{(12456)} D_{(12356)} D_{(12346)} D_{(12345)}$$

$$[(1234)^2.(1256)^2.(3456)+ \qquad t_1$$
$$+ 2.(1234)(1236)(1245)(1256).(3456) \qquad 2t_2$$
$$+ (1235)^2.(1246)^2.(3456) \qquad t_3$$
$$+ 2.(1234)(1235)(1246)(1256).(3456) \qquad 2t_4$$
$$+ (1236)^2.(1245)^2.(3456) \qquad t_5$$
$$+ 2.(1235)(1236)(1245)(1246).(3456).] \qquad 2t_6$$

$$C_6 = D_{(12456)} D_{(12356)} D_{(12346)} D_{(12345)}$$

[2.(1234) (1235) (1236).(1456) (2456)+	$2t_7$
+ (1234)^2.(1256).(1456) (2356)	t_8
+ (1234) (1235) (1246).(1456) (2356)	t_9
+ (1234) (1236) (1245).(1456) (2356)	t_{10}
+ (1234) (1235) (1256).(1456) (2346)	t_{11}
+ (1235)^2.(1246).(1456) (2346)	t_{12}
+ (1235) (1236) (1245).(1456) (2346)	t_{13}
+ (1234) (1236) (1256).(1456) (2345)	t_{14}
+ (1235) (1236) (1246).(1456) (2345)	t_{15}
+ (1236)^2.(1245).(1456) (2345).]	t_{16}

We can then use formula (3.6), which for $n = 6$ gives

$$R_6 = A_6 + 4 . C_6. \qquad (4.1)$$

To detect equinumerous terms among the t_i, we have now to use connecting permutations which keep the operator set

$$\xi = D_{(12456)} D_{(12356)} D_{(12346)} D_{(12345)}$$

invariant.

We obtain the following equinumerous subsets:

TABLE I

Equinumerous subsets with ξ invariant

Subset ..	t_1	t_3	t_5			
Connecting Permutation ..	(45)	(56)				
Subset ..	t_2	t_4	t_6			
Connecting Permutation ..	(56)	(46)				
Subset ..	t_8	t_{12}	t_{16}			
Connecting Permutation ..	(45)	(56)				
Subset ..	t_9	$\overrightarrow{t_{10}}$	t_{11}	$\overrightarrow{t_{13}}$	t_{14}	t_{15}
Connecting Permutation ..	(56)	(456)	(46)	(456)	(45)	

N.B.—The arrows indicate that the permutation (456) is applied to the earlier terms, *i.e.*, to t_{10} and t_{13}.

We have thus reduced the enumeration of A_6 and C_6 to the following:

$$A_6 = D_{(12456)} D_{(12356)} D_{(12346)} D_{(12345)} \, [3t_1 + 6t_2] \tag{4.2}$$

and

$$C_6 = D_{(12456)} D_{(12356)} D_{(12346)} D_{(12345)} \, [2t_7 + 3t_8 + 6t_9]. \tag{4.3}$$

We shall now apply the operation $D_{(12456)}$ to the terms t_1, t_2, t_7, t_8, t_9 in (4.2) and (4.3).

$D_{(12456)} . t_1 = D_{(12456)} \, (1234)^2 . (1256)^2 . (3456)$ gives

$$\begin{array}{ll}
= 4\,[(123)\,(125)\,(134)\,(256)\,(346) + & 4\,[u_1 \\
+ (123)\,(126)\,(134)\,(256)\,(345) & u_2 \\
+ (125)\,(126)\,(134)\,(234)\,(356) & u_3 \\
+ (123)\,(125)\,(156)\,(234)\,(346) & u_4 \\
+ (123)\,(126)\,(156)\,(234)\,(345).] & u_5]
\end{array}$$

$D_{(12456)} . t_2 = D_{(12456)} \, (1234)\,(1236)\,(1245)\,(1256) . (3456)$ gives

$$\begin{array}{ll}
= [(125)^2 . (136)(234)(346) + (125)(126)(136)(234)(345) + & [u_6 + u_7 \\
+ (124)(125)(136)(234)(356) + (123)(126)(145)(234)(356) & u_8 + u_9 \\
+ (123)(125)(156)(234)(346) + (123)(124)(156)(234)(356) & u_{10} + u_{11} \\
+ (124)(125)(134)(236)(356) + (125)^2 . (134)(236)(346) & u_{12} + u_{13} \\
+ (125)(126)(134)(236)(345) + (123)(125)(145)(236)(346) & u_{14} + u_{15} \\
+ (123)(126)(145)(236)(345) + (123)(124)(156)(236)(345) & u_{16} + u_{17} \\
+ (123)^2 . (156)(245)(346) + (123)(125)(136)(245)(346) & u_{18} + u_{19} \\
+ (123)(126)(136)(245)(345) + (123)(126)(134)(245)(356) & u_{20} + u_{21} \\
+ (123)^2 . (145)(256)(346) + (123)(124)(136)(256)(345) & u_{22} + u_{23} \\
+ (123)(124)(134)(256)(356) + (123)(125)(134)(256)(346).] & u_{24} + u_{25}]
\end{array}$$

$D_{(12456)} . t_7 = D_{(12456)} \, (1234)\,(1235)\,(1236) . (1456)\,(2456)$ gives

$$\begin{array}{ll}
= [(123)^3 . (456)^2 + & [u_{26} \\
+ (123)^2 . (136)(245)(456) + (123)^2 . (134)(256)(456) & u_{27} + u_{28} \\
+ (123)^2 . (135)(246)(456) + (123)^2 . (145)(236)(456) & u_{29} + u_{30} \\
+ (123)^2 . (156)(234)(456) + (123)^2 . (146)(235)(456) & u_{31} + u_{32}
\end{array}$$

$$+ (123)(134)(156)(236)(245)+(123)(134)(145)(236)(256) \qquad u_{33} + u_{34}$$
$$+ (123)(135)(146)(236)(245)+(123)(135)(145)(236)(246) \qquad u_{35} + u_{36}$$
$$+ (123)(136)(156)(234)(245)+(123)(136)(145)(234)(256) \qquad u_{37} + u_{38}$$
$$+ (123)(135)(156)(234)(246)+(123)(135)(146)(234)(256) \qquad u_{39} + u_{40}$$
$$+ (123)(136)(146)(235)(245)+(123)(136)(145)(235)(246) \qquad u_{41} + u_{42}$$
$$+ (123)(134)(156)(235)(246)+(123)(134)(146)(235)(256).] \qquad u_{43} + u_{44}]$$

$D_{(12456)} . t_8 = D_{(12456)}$ $(1234)^2.(1256).(1456)(2356)$ gives

$$=2\ [(123)(126)(134)(235)(456)+(123)(125)(134)(236)(456)+ \qquad 2[u_{45}+ u_{46}$$
$$+ (123)(134)(146)(235)(256)+(123)(134)(145)(236)(256) \qquad u_{47} + u_{48}$$
$$+ (123)(146)(156)(234)(235)+(126)(134)(156)(234)(235) \qquad u_{49} + u_{50}$$
$$+ (123)(145)(156)(234)(236)+(125)(134)(156)(234)(236) \qquad u_{51} + u_{52}$$
$$+ (123)(125)(146)(234)(356)+(123)(126)(145)(234)(356).] \qquad u_{53} + u_{54}]$$

$D_{(12456)} . t_9 = D_{(12456)}$ $(1234).(1235)(1246).(1456)(2356)$ gives

$$=[(123)^2.(146)(235)(456)+(123)^2.(124)(356)(456)+ \qquad [u_{55} + u_{56}$$
$$+ (123)(126)(134)(235)(456)+(123)(124)(135)(236)(456) \qquad u_{57} + u_{58}$$
$$+ (123)^2.(145)(246)(356)+(123)(124)(156)(234)(356) \qquad u_{59} + u_{60}$$
$$+ (123)(126)(145)(234)(356)+(123)(124)(146)(235)(356 \qquad u_{61} + u_{62}$$
$$+ (123)(134)(156)(235)(246)+(123)(135)(146)(235)(246) \qquad u_{63} + u_{64}$$
$$+ (123)(135)(145)(236)(246)+(123)(146)(156)(234)(235) \qquad u_{65} + u_{66}$$
$$+ (126)(135)(146)(234)(235)+(123).(146)^2.(235)^2 \qquad u_{67} + u_{68}$$
$$+ (126)(134)(146).(235)^2+(126)(135)(145)(234)(236) \qquad u_{69} + u_{70}$$
$$+ (124)(135)(156)(234)(236)+(123)(145)(146)(235)(236) \qquad u_{71} + u_{72}$$
$$+ (126)(134)(145)(235)(236)+(124)(134)(156)(235)(236).] \qquad u_{73} + u_{74}]$$

The reduction of the u_i $(i = 1, 2, \ldots 74)$

We shall now use Theorem 3, to detect equinumerous terms among the set of u's. Notice that the connecting permutations which can be used for the purpose must now keep the operator set

$$\eta \equiv D_{(12356)} D_{(12346)} D_{(12345)}$$

invariant.

We notice the following equinumerous subsets ;

TABLE II. *Equinumerous subsets with η invariant*

N.B.—Figures in brackets denote connecting permutations

Subset No.	Terms in the Subset	Representative Term
1	$u_{25}=u_1$ (56) u_2 (12)(56) u_4 (56) u_5 (56) u_{10} (46) u_{15} (12) u_{19} $\xrightarrow{(132)(45)}$ $u_{48}=u_{34}$ (45) $u_{65}=u_{36}$ $\xrightarrow{(456)}$ u_{37} (56) u_{39} $\xrightarrow{(456)}$ u_{41} (46) $u_{47}=u_{44}$ (23) u_{62}.	u_1
2	u_3 (13) u_{74}.	u_3
3	u_6 (46) u_{13} (13) u_{69}.	u_6
4	u_7 (12)(46) u_8 (12) u_{12} (12)(46) u_{14}.	u_7
5	$u_{61}=u_{54}=u_9$ (46) u_{17} (12)(46) u_{21} (46) u_{23} $\xrightarrow{(132)}$ u_{33} (45) u_{35} $\xrightarrow{(456)}$ u_{38} (56) u_{40} $\xrightarrow{(456)}$ u_{42} (46) $u_{63}=u_{43}$ (23)(45) u_{53}.	u_9
6	$u_{60}=u_{11}$ (46) u_{16} (12) u_{20} (46) u_{24} $\xrightarrow{(132)(45)}$ u_{64}.	u_{11}
7	u_{18} (46) u_{22} (45) u_{59}.	u_{18}
8	u_{26}.	u_{26}
9	u_{27} (46) u_{28} (45) u_{29} (12)(56) u_{30} (46) u_{31} (45) $u_{55}=u_{32}$ (13)(45) u_{56}.	u_{27}
10	$u_{57}=u_{45}$ (56) u_{46} (45) u_{58}.	u_{45}
11	$u_{66}=u_{49}$ (56) u_{51} (45) u_{72}.	u_{49}
12	u_{50} (56) u_{52} $\xrightarrow{(456)}$ u_{67} (23)(56) u_{70} (46) u_{71} $\xrightarrow{(465)}$ u_{73}.	u_{50}
13	u_{68}.	u_{68}

Note.—The arrows indicate that the permutations are applied to the earlier terms.

We have thus reduced the terms $u_1, u_2, \ldots u_{74}$ to the following thirteen representative terms:

$$u_1,\ u_3,\ u_6,\ u_7,\ u_9,\ u_{11},\ u_{18},\ u_{26},\ u_{27},\ u_{45},\ u_{49},\ u_{50},\ u_{68}. \qquad (4.4)$$

The evaluation of the ηu_i *in the series* (4.4)

We shall now evaluate in succession the result of operating with

$$\eta \equiv D_{(12356)} D_{(12346)} D_{(12345)}$$

on the terms in (4.4).

(i) *Term* $\eta u_1 = \eta.(123)\,(125)\,(134)\,(256)\,(346)$

$D_{(12356)}$ gives	Last two operations give
$[(23)\,(15)\,(14)\,(26)\,(34)\,+$	1
$+\,(23)\,(12)\,(14)\,(56)\,(34)$	$+2$
$+\,(13)\,(25)\,(14)\,(26)\,(34)$	$+2$
$+\,(13)\,(12)\,(34)\,(25)\,(46)$	$+1$
$+\,(12)\,(15)\,(34)\,(26)\,(34)$	$+2$
$+\,(12)\,(12)\,(34)\,(56)\,(34)]$	$+4$
	$\therefore\ \eta u_1 = 12$

(ii) *Term* $\eta u_3 = \eta.(125)\,(126)\,(134)\,(234)\,(356)$

$D_{(12356)}$ gives	
$[(25)\,(12)\,(14)\,(34)\,(36)\,+$	1
$+\,(12)\,(26)\,(14)\,(34)\,(35)$	$+1$
$+\,(15)\,(12)\,(34)\,(24)\,(36)$	$+1$
$+\,(12)\,(16)\,(34)\,(24)\,(35)$	$+1$
$+\,(12)\,(12)\,(34)\,(34)\,(56)]$	$+4$
	$\therefore\ \eta u_3 = 8$

(iii) *Term* $\eta u_6 = \eta.(125)^2.(136)\,(234)\,(346)$

$D_{(12356)}$ gives	
$2\,[(25)\,(12)\,(16)\,(34)\,(34)\,+$	$2\,[2$
$+\,(25)\,(12)\,(13)\,(34)\,(46)$	$+1$
$+\,(15)\,(12)\,(36)\,(24)\,(34)]$	$+1]$
	$\therefore\ \eta u_6 = 8$

(iv) *Term* $\eta u_7 = \eta.(125)\,(126)\,(136)\,(234)\,(345)$

$D_{(12356)}$ gives

[(25) (16) (13) (24) (34) +	1
+ (25) (12) (16) (34) (34)	+2
+ (15) (26) (13) (24) (34)	+1
+ (12) (26) (13) (34) (45)	+1
+ (15) (12) (36) (24) (34)	+1
+ (12) (12) (36) (34) (45)]	+2

$\therefore \quad \eta u_7 = 8$

(v) *Term* $\eta u_9 = \eta.(123)\,(126)\,(145)\,(234)\,(356)$

$D_{(12356)}$ gives

[(23) (16) (14) (24) (35) +	1
+ (23) (12) (14) (34) (56)	+2
+ (13) (26) (14) (24) (35)	+1
+ (12) (26) (14) (34) (35)	+1
+ (13) (12) (45) (24) (36)	+1
+ (12) (12) (45) (34) (36)]	+2

$\therefore \quad \eta u_9 = 8$

(vi) *Term* $\eta u_{11} = \eta.(123)\,(124)\,(156)\,(234)\,(356)$

$D_{(12356)}$ gives

[(23) (14) (16) (24) (35) +	1
+ (23) (14) (15) (24) (36)	+1
+ (13) (24) (16) (24) (35)	+2
+ (13) (24) (15) (24) (36)	+2
+ (12) (24) (16) (34) (35)	+1
+ (12) (24) (15) (34) (36)	+1

$\therefore \quad \eta u_{11} = 8$

(vii) *Term* $\eta u_{18} = \eta.(123)^2.(156)\,(245)\,(346)$

$D_{(12356)}$ gives

2 [(23) (13) (15) (24) (46) +	2 [1
+ (23) (12) (16) (45) (34)	+1
+ (13) (12) (56) (24) (34)]	+2]

$\therefore \quad \eta u_{18} = 8$

(viii) *Term* $\eta u_{26} = \eta.(123)^3.(456)^2$

$D_{(12356)}$ gives

12.[(23) (13) (12).(46) (45)] 12.[2]

$\therefore \quad \eta u_{26} = 24$

(ix) *Term* $\eta u_{27} = \eta.(123)^2.(136)(245)(456)$

$D_{(12356)}$ gives

2 [(23) (13) (16) (24) (45) + 2 [1

+ (23) (12) (13) (45) (46) + 2

+ (13) (12) (36) (24) (45)] +1]

$\therefore \quad \eta u_{27} = 8$

(x) *Term* $\eta u_{45} = \eta.(123)(126)(134)(235)(456)$

$D_{(12356)}$ gives

[(23) (16) (14) (23) (45) + 2

+ (23) (12) (14) (35) (46) +1

+ (13) (26) (14) (23) (45) +1

+ (13) (12) (34) (25) (46) +1

+ (12) (16) (34) (23) (45) +1

+ (12) (12) (34) (35) (46)] +2

$\therefore \quad \eta u_{45} = 8$

(xi) *Term* $\eta u_{49} = \eta.(123)(146)(156)(234)(235)$

$D_{(12356)}$ gives

[(23) (14) (16) (34) (25) + 1

+ (23) (14) (16) (24) (35) +1

+ (13) (46) (15) (24) (23) +1

+ (12) (46) (15) (34) (23) +1

+ (13) (14) (56) (24) (23) +2

+ (12) (14) (56) (34) (23)] +2

$\therefore \quad \eta u_{49} = 8$

(xii) *Term* $\eta u_{50} = \eta.(126)(134)(156)(234)(235)$

$D_{(12356)}$ gives

$$\begin{aligned}&[(26)(14)(15)(34)(23)+ && 1\\ &+(16)(34)(15)(24)(23) && +2\\ &+(12)(34)(16)(34)(25) && +2\\ &+(12)(34)(16)(24)(35) && +1\\ &+(12)(14)(56)(34)(23)] && +2\\ & && \therefore\ \eta u_{50} = 8\end{aligned}$$

(xiii) *Term* $\eta u_{68} = \eta.(123).(146)^2.(235)^2$

$D_{(12356)}$ gives

$$\begin{aligned}&4\,[(13)(46)(14)(25)(23)+ && 4\,[1\\ &+(12)(46)(14)(35)(23)] && +1]\\ & && \therefore\ \eta u_{68} = 8\end{aligned}$$

The evaluation of the above thirteen representative terms is summarised in the following equation:

$$\left.\begin{aligned}\eta u_1 = 12, \qquad \eta u_{26} = 24.\\ \eta u_3=\eta u_6=\eta u_7=\eta u_9=\eta u_{11}=\eta u_{18}=\eta u_{27}=\eta u_{45}=\eta u_{49}=\eta u_{50}=\eta u_{68}=8\end{aligned}\right\} \quad (4.5)$$

Using the equinumerous subsets in Table II, and equation (4.5) we then obtain the following table of values for the $\eta.u_i$ $(i=1, 2, \ldots 74)$:

TABLE III

The values of $\eta.u_i$ $(i = 1, 2, \ldots 74)$

u_i	ηu_i
$u_1, u_2, u_4, u_5, u_{10}, u_{15}, u_{19}, u_{25}, u_{34}, u_{36}, u_{37}, u_{39}, u_{41}, u_{44}, u_{47}, u_{48}, u_{62}, u_{65}$ ..	12
u_{26} ..	24
Remaining u_i ..	8

Evaluation of the ξt_i *in* (4.2) *and* (4.3)

Denoting by ξ the operator

$$\xi \equiv D_{(12456)} D_{(12356)} D_{(12346)} D_{(12345)}$$

we can now write down the following equations:

$$\xi t_1 = 4.\eta\,(u_1 + u_2 + u_3 + u_4 + u_5)$$
$$\xi t_2 = \eta\,(u_6 + u_7 + u_8 + \ldots + u_{25})$$
$$\xi t_7 = \eta\,(u_{26} + u_{27} + u_{28} + \ldots + u_{44})$$
$$\xi t_8 = 2.\eta\,(u_{45} + u_{46} + u_{47} + \ldots + u_{54})$$
$$\xi t_9 = \eta\,(u_{55} + u_{56} + u_{57} + \ldots + u_{74}).$$

Using Table III, it is now easy to calculate the ξt_i, since we can easily pick out the terms contributing a 12 or 24. Every other term contributes an 8.

Thus,

$$\left.\begin{aligned} \xi t_1 &= 4\,[4 \times 12 + 1 \times 8] = 4.[56] = 224 \\ \xi t_2 &= [4 \times 12 + 16 \times 8] = 48 + 128 = 176 \\ \xi t_7 &= [1 \times 24 + 6 \times 12 + 12 \times 8] = 24 + 72 + 96 = 192 \\ \xi t_8 &= 2\,[2 \times 12 + 8 \times 8] = 2.[88] = 176 \\ \xi t_9 &= [2 \times 12 + 18 \times 8] = 24 + 144 = 168 \end{aligned}\right\} \quad (4.6)$$

The value of R_6

Substituting from (4.6) into (4.2) and (4.3) we obtain

$$A_6 = \xi\,\{3t_1 + 6t_2\} = 3 \times 224 + 6 \times 176 = 672 + 1056 = 1728$$

and

$$C_6 = \xi\,\{2t_7 + 3t_8 + 6t_9\} = 2 \times 192 + 3 \times 176 + 6 \times 168$$
$$= 384 + 528 + 1008 = 1920$$

Substitution in (4.1) then gives

$$R_6 = A_6 + 4.C_6$$
$$= 1728 + 4 \times 1920$$
$$= 1728 + 7680$$
$$= 9408.$$

§5. SUMMARY

MacMahon's method of differential operators acting on symmetric function operands has been simplified and applied to the enumeration of 6 × 6 Latin squares. A modified formula giving R_n—the number of standard Latin squares—directly and exhaustively, has been derived, The enumeration of R_n, in its turn, has been reduced to that of A_n and C_n, which are the numbers of standard Latin squares containing the letters '*a*' and '*c*' in the second row-second column-cell. The enumeration of A_n and C_n has been further simplified by the use of certain permutations which keep the operators invariant.

The 6×6 squares have been exhaustively enumerated before, and so the present enumeration could only be expected to give the same number, *viz.*, 9408 standard Latin squares.

The value of this approach, however, lies in providing a neat and exhaustive method for the enumeration of 7×7 Latin squares. The real strength inherent in the theorems established in the present paper would be evident from the simplifications they introduce therein. It is hoped to present this solution, as the second part of this paper, in a subsequent communication.

§6. Acknowledgements

In the end, I have gratefully to thank Dr. P. V. Sukhatme, Statistical Adviser, I.C.A.R., who suggested the problem to me, encouraged me to complete the solution, and gave helpful advice during the revision.

References

1. MacMahon, P. A. .. *Combinatory Analysis*, 1915, Vol. 1, Camb. Univ. Press.
2. Fisher, R. A., and Yates, F. "The 6×6 Latin Squares," *Proc. Camb. Phil. Soc.*, 1934, **30**, 492–507.
3. Norton, H. W. .. "The 7×7 Squares," *Ann. Eug.*, 1939, **9**, Part III, 269–307.