SOME RATIO-TYPE ESTIMATORS*

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1. INTRODUCTION

One of the main objectives of a sample survey is the estimation of the population mean or total of a characteristic 'y' attached to the units in the population. Ratio estimators are among the most commonly used estimators of the population mean or total of 'y' utilizing an auxiliary characteristic 'x' that is positively correlated with 'y'. The precision of the regression estimator is usually higher than that of the ratio estimator but in large-scale sample surveys, the ratio estimator is frequently employed because of its simplicity. In this paper, we develop some ratio-type estimators which will be more efficient than the customary ratio estimator and/or the unbiased estimator and yet computationally comparable to the customary ratio estimator.

We shall, without loss of generality, confine ourselves to the estimation of $\bar{Y}$, the population mean of 'y'. Further, to simplify the discussion, we shall confine ourselves to simple random sampling and assume the population size is infinite. From a simple random sample of n pairs (y_i, x_i) we have the unbiased estimator of $\bar{Y}$, as

$$\bar{y}=\sum_{i=1}^{n} y_i/n. \tag{1.1}$$

The customary ratio estimator of $\bar{Y}$ is

$$\hat{\bar{y}}_r=(\bar{y}/\bar{x})\bar{X}=r\bar{X} \tag{1.2}$$

where $\bar{x}$ is the sample mean and $\bar{X}$ is the known population mean of x, and

$$r=\bar{y}/\bar{x} \tag{1.3}$$

is the ratio estimator of the ratio $R=\bar{Y}/\bar{X}$.

It is well known that the ratio estimator $\bar{y}_r$ is more efficient than the unbiased estimator $\bar{y}$ in large samples if $\rho>C_x/(2C_y)$

*Revised version of the paper presented at the annual meeting of the American Statistical Association held at New York in December, 1973 where ρ is the coefficient of correlation between y and x and C_y and C_x are coefficients of variation of y and x respectively. The question of choice between $\bar{y}$ and $\bar{y}_r$ arises when it is suspected that $\rho(\geqslant 0)$ is not high and/or $C_x > C_y$. The customary procedure in such situations is to use $\bar{y}_r$ when $\rho > C_x/(2C_y)$ otherwise use $\bar{y}$. It is, however, desirable to develop alternative ratio-type estimators which are more efficient than $\bar{y}_r$ as well as $\bar{y}$ and yet computationally comparable to $\bar{y}_r$. The two ratio-type estimators we propose are

$$t_1 = (1-W)\bar{y} + W\bar{y}_r \; ; \; W \geqslant 0 \tag{1.4}$$

and

$$t_2 = (1-W)\bar{y} + Wr^*\bar{X} \; ; \; W \geqslant 0 \tag{1.5}$$

where W is a constant weight to be determined and

$$r^* = 2r - \tfrac{1}{2}(r_1 + r_2) \tag{1.6}$$

is obtained by splitting the sample at random into two groups, each of size $n/2$ when n is even and $r_j = \bar{y}_j/\bar{x}_j$, $(j=1, 2)$, $\bar{y}_j$ and $\bar{x}_j$ are means of y and x respectively obtained from jth half-sample. The estimator t_1 reduces to $\bar{y}$ and $\bar{y}_r$ when $W=0$ and 1 respectively. The estimator t_2 reduces to $\bar{y}$ when $W=0$ and when $W=1$ it reduces to $r^*\bar{X}$ which is the 'Jack-knife' ratio estimator of $\bar{Y}$. It may be mentioned here that by dividing the sample at random into $g(\leqslant n)$ groups, each of size n/g, a more general form of the estimator t_2 could be obtained as

$$t_{2g} = (1-W)\bar{y} + W\left[gr - \frac{g-1}{g} \sum_{j=1}^{g} r_j' \right]\bar{X}$$

where r_j' is the customary ratio estimator calculated from the sample after omitting the jth group. However, in this paper we shall consider the special case of t_{2g} given in (1.5). Srivastava (1967) proposed the estimator

$$t_3 = \bar{y}(\bar{X}/\bar{x})^W \tag{1.7}$$

where W is a constant weight and obtained its asymptotic variance. The estimator t_1 was suggested earlier by Chakrabarty (1968). In this paper these estimators will be compared regarding the properties of bias and efficiency. In section 2, we discuss the asymptotic theory and in section 3 we give the *exact* biases and variances of these estimators under a regression model.

2. Asymptotic Theory

2.1. *Biases of the estimators*

It is obvious that the estimators t_1, t_2, and t_3 are consistent but in general biased, like the ratio estimator $\bar{y}_r$. Now, as it is customary in the asymptotic theory of ratio method of estimation, we shall assume that the sample size n is sufficiently large so that

$$|\delta_{\bar{x}}| = \left|\frac{\bar{x}-\bar{X}}{\bar{X}}\right| << 1 \tag{2.1}$$

Under the above assumption, the expected value of r is given by

$$E(r)=R+\frac{R}{n}(C_x^2-\rho C_y C_x)+0(n^{-2})$$

Now, since r_1 and r_2 are independent

$$E(r^*)=R+0(n^{-2})$$

Consequently, the biases of t_1 and t_2 are

$$\begin{aligned}\text{Bias}(t_1)&=W\,\text{Bias}(\bar{y}_r)\\ &=\frac{W\bar{Y}}{n}(C_x^2-\rho C_y C_x)+0(n^{-2})\end{aligned} \tag{2.2}$$

and

$$\text{Bias}(t_2)=0+0(n^{-2}) \tag{2.3}$$

respectively. From Srivastava (1967), the bias t_3 is given by

$$\text{Bias}(t_3)=\frac{W\bar{Y}}{n}\left[\frac{(W+1)}{2}C_x^2-\rho C_y C_x\right]+0(n^{-2}) \tag{2.4}$$

Thus, the asymptotic bias of t_2 is of order n^{-2} and hence smaller than that of $\bar{y}_r$, t_1 and t_3 whose biases are order n^{-1}. The bias of t_1 is smaller than that of $\bar{y}_r$ for $0<W<1$. We note that $C_x^2-\rho C_y C_x=0$ when the regression of y on x passes through the origin. Consequently, for the important case of regression through the origin the estimators $\bar{y}_r$ and t_1 are unbiased to terms of order n^{-1} but the bias of t_3 is still of order n^{-1}. Further, substituting the formula for exact bias of $\bar{y}_r$ from Hartley and Ross (1954) we get the exact bias of t_1 as

$$\text{Bias}(t_1)=-W\,\text{Cov}(r,\bar{x})$$

and

$$\frac{|\text{Bias}(t_1)|}{\sigma_{\bar{y}_r}}\leqslant\frac{WC_x}{\sqrt{n}} \tag{2.5}$$

Thus if $\frac{WC_x}{\sqrt{n}}\leqslant 0.1$, the bias of t_1 is negligible in relation to the standard error of $\bar{y}_r$. No such upper bound to the bias of t_3 relative to its standard error could be obtained.

2.2. *Variances of the estimators*

In deriving the variances of estimators t_1, t_2 and t_3 we consider up to terms of n^{-1} only and biases which are of order n^{-1} are neglected. Expanding r and r_j by Tylor's series in terms of $\delta_{\bar{x}}, \delta_{\bar{y}}$ and $\delta_{\bar{x}_j}, \delta_{\bar{y}_j}$ $(j=1, 2)$ it can be shown that to terms of order n^{-1} the variances of t_1, t_2 and t_3 are identical and are given by

$$V(t_1)=V(t_2)=V(t_3)=\frac{S_y^2}{n}[1+WK(WK-2\rho)] \tag{2.6}$$

where $$K=C_x/C_y. \tag{2.7}$$

The value of W which minimizes this variance is

$$W_{opt}=\rho/K \tag{2.8}$$

The minimum variance is given by

$$V_{min}=\frac{S_y^2}{n}(1-\rho^2) \tag{2.9}$$

which is equal to the variance of the linear regression estimator up to terms of order n^{-1}. Substituting $W=1$ in (2.6) we get the variance of $\bar{y}_r$ as

$$V(\bar{y}_r)=\frac{S_y^2}{n}[1+K(K-2\rho)] \tag{2.10}$$

The asymptotic efficiencies of t_1 (t_2 and t_3) over $\bar{y}$ and $\bar{y}_r$ are given by

$$E_1=\frac{V(\bar{y})}{V(t_1)}=\frac{1}{[1+WK(WK-2\rho)]} \tag{2.11}$$

and

$$E_2=\frac{V(\bar{y}_r)}{V(t_1)}=\frac{[1+K(K-2\rho)]}{[1+WK(WK-2\rho)]} \tag{2.12}$$

respectively. From (2.11) and (2.12) we get

$$E_1 \geqslant 1 \quad \text{if} \quad W \leqslant 2\rho/K$$

and

$$E_2 \geqslant 1 \quad \text{if} \quad (2\rho-K)/K \leqslant W \leqslant 1 \tag{2.13}$$

Thus the estimators t_1, t_2 and t_3 are better than $\bar{y}$ and $\bar{y}_r$ for a wide range of W-values. For example, if $\rho=.6$, $K=1$ and W is between 0.2 and 1 estimators t_1, t_2 and t_3 are asymptotically more efficient than $\bar{y}$ and $\bar{y}_r$. The efficiencies E_1 & E_2 of the estimators t_1, t_2 and t_3 over $\bar{y}$ and $\bar{y}_r$ will depend on ρ, K and the weight W. The numerical values of E_1 and E_2 for different values of ρ, K and for

$W=\frac{1}{4}$ and $W=\frac{1}{2}$ are given as percentages in Tables 1 and 2 respectively. Comparing the results in the two tables we may conclude that if a good guess of ρ/K is not available from a pilot sample survey, past data or experience (1) $W=\frac{1}{4}$ appears to be a good overall choice for t_1, t_2 and t_3 for low correlation $(.2<\rho<.4)$ and/or $K>1$. (2) $W=\frac{1}{2}$ appears to be a good choice for moderate to high correlation $(\rho>.4)$ and $K>1$. (3) In cases where $\rho>.8$ and $K\leqslant 1$ it is preferable to use $\bar{y}_r$. The asymptotic variance given in (2.9) of the estimators t_1, t_2 and t_3 with optimum value of $W=\rho/K$ is equal to the asymptotic variance of the linear regression estimator

$$\bar{y}_{lr}=\bar{y}+b(\bar{X}-\bar{x}) \tag{2.14}$$

where b is the sample regression coefficient. Thus these estimators with constant weights ($W=\frac{1}{4}$ or $\frac{1}{2}$) are asymptotically no more efficient than $\bar{y}_{lr}$. However, if the regression of y on x is not linear, Cochran (1963) has shown that the bias in $\bar{y}_{lr}$ is of order n^{-1} and hence it is more biased than t_2 whose bias is of order n^{-2}. Thus t_2 may be preferable to $\bar{y}_{lr}$ in situations where freedom from bias is important. Moreover, computationally t_2 is simpler than $\bar{y}_{lr}$.

3. The Exact Theory

We assume the following model for the comparison of estimators :

$$y_i=\alpha+\beta x_i+u_i\,;\ \beta>0$$
$$E(u_i \mid x_i)=0,\ E(u_i, u_j \mid x_i, x_j)=0$$
$$V(u_i \mid x_i)=n\delta\ \ (\delta \text{ is a constant of order } n^{-1})\text{(I)}$$

where the variates x_i/n have the gamma distribution with parameter h so that $\bar{x}=\Sigma x_i/n$ has the gamma distribution with the parameter $m=nh$. This model was used by Durbin (1959), and Rao and Webster (1966) to investigate the bias in estimation of ratios, and Chakrabarty and Rao (1967) to investigate the stability of the 'Jack-Knife' variance estimator in ratio estimation. Chakrabarty (1973) has used this model to investigate the exact efficiency of the ratio estimator $\bar{y}_r$ and stability of the variance estimator of $\bar{y}_r$ relative to that of $\bar{y}$. He has shown that for $\rho\geqslant.4$ and $K<2\rho$ the ratio estimator is generally more efficient than the unbiased estimator $\bar{y}$ even in small samples, and that the variance estimator of the ratio estimator is generally more stable than the variance estimator of $\bar{y}$. It may be noted that all our results under this model are exact for any sample size, n.

3.1 *The exact biases of the estimators*

In terms of the model (I) we have

$$\bar{y}=\alpha+\beta\bar{x}+\bar{u}$$

$$E(\bar{y})=\alpha+\beta m=\bar{Y}$$

$$t_1=\alpha\left(1-W+\frac{Wm}{\bar{x}}\right)+\beta[(1-W)\bar{x}+Wm]$$

$$+\bar{u}\left\{(1-W)+\frac{Wm}{\bar{x}}\right\} \tag{3.1}$$

Consequently, the bias of t_1 is

$$\text{Bias }(t_1)=E(t_1)-(\alpha+\beta m)$$

$$=\alpha W/(m-1) \tag{3.2}$$

$$t_2=\alpha\left[(1-W)+Wm\left(\frac{2}{\bar{x}}-\frac{1}{2\bar{x}_1}-\frac{1}{2\bar{x}_2}\right)\right]$$

$$+\beta[(1-\text{W})\bar{x}-Wm]-\frac{Wm}{2}\left(\frac{\bar{u}_1}{\bar{x}_1}+\frac{\bar{u}_2}{\bar{x}_2}\right)$$

$$+\bar{u}\left[(1-W)+\frac{2Wm}{\bar{x}}\right]$$

$$E(t_2)=\beta m+\alpha[1-2W/(m-1)(m-2)]$$

Thus the bias of t_2 is

$$\text{Bias }(t_2)=-2W\alpha/\{m-1)(m-2) \tag{3.3}$$

$$t_3=(\alpha+\beta\bar{x}+\bar{u})m^W\bar{x}^{-W}$$

$$E(t_3)=\frac{m^W}{\Gamma(m)}[\alpha\Gamma(m-W)+\beta\Gamma(m-W+1)]$$

Consequently, the bias of t_3 is

$$\text{Bias }(t_3)=\alpha\left[\frac{m^W\Gamma(m-W)}{\Gamma(m)}-1\right]$$

$$+\beta\left[\frac{m^W\Gamma(m-W+1)}{\Gamma(m)}-m\right] \tag{3.4}$$

Now. putting, $W=1$ in either (3.2) or (3.4) we get the bias of $\bar{y}_r$ as

$$\text{Bias }(y_r)=\alpha/(m-1) \tag{3.5}$$

From (3.2) through (3.5) it can be seen that the bias of t_2 is of order n^{-2} while those of $\bar{y}_r$, t_1 and t_3 are of order n^{-1} since $m=nh$ in our model. Also, the bias of t_1 is less than the bias of $\bar{y}_r$ if $W<1$. Further, for the special case of the linear regression through the origin (*i.e.* $\alpha=0$ in model I) the estimators $\bar{y}_r$, t_1 and t_2 are unbiased but t_3 is still biased. A numerical evaluation of the biases of these estimators is given in the next section.

3.2 *The exact variances of the estimators*

The method of obtaining exact expressions for the variances of these estimators under model I is similar to that of Rao and Webster (1966). The details of evaluating these variances, which involve some algebra, are omitted and only the final results are given here. The variance of t_1 can be shown to be

$$V(t_1)=\frac{W^2m^2}{(m-1)^2(m-2)}\alpha^2+(1-W)^2m\beta^2$$
$$+\left[\frac{W^2m^2}{(m-1)(m-2)}+\frac{W(1-W)(m+1)}{(m-1)}+(1-W)\right]\delta$$
$$-\frac{2W(1-W)m}{(m-1)}\alpha\beta \qquad (3.6)$$

Putting $W=1$ and $W=0$ in (3.6) the variance of $\bar{y}_r$ and $\bar{y}$ are obtained as :

$$V(\bar{y}_r)=\frac{m^2\alpha^2}{(m-1)^2(m-2)}+\frac{m^2\delta}{(m-1)(m-2)} \qquad (3.7)$$

and

$$V(\bar{y})=\delta+\beta^2 m \qquad (3.8)$$

respectively. The variance of t_2 is obtained as

$$V(t_2)=\frac{W^2m^2(m^2-6m+17)}{(m-1)^2(m-2)^2(m-4)}\alpha^2$$
$$-\frac{2W(1-W)m(m-3)}{(m-1)(m-2)}\alpha\beta+(1-W)^2m\beta^2$$
$$+\left[(1-W)^2+\frac{W^2(m^2-7m+18)m^2}{(m-1)(m-2)^2(m-4)}\right.$$
$$\left.+\frac{2W(1-W)m(m-3)}{(m-1)(m-2)}\right]\delta \qquad (3.9)$$

Finally, the variance of t_3 is given by

$$[m^{-2W}\Gamma^2(m)]V(t_3)=[\Gamma(m-2W)\Gamma(m)-\Gamma^2(m-W)]\alpha^2$$
$$+[\Gamma(m+2-2W)\Gamma(m)-\Gamma^2(m+1-W)]\beta^2$$
$$+2[\Gamma(m+1-2W)\Gamma(m)-\Gamma(m-1-W)\Gamma(m-W)]\alpha\beta$$
$$+[\Gamma(m-2W)\Gamma(m)]\delta \qquad (3.10)$$

We note that in terms of the model I

$$\alpha=\bar{Y}[(K-\rho)/K]$$
$$\beta=\bar{Y}[\rho/(Km)]$$
$$\delta=\bar{Y}^2[(1-\rho^2)/(K^2m)] \qquad (3.11)$$

and $K=C_x/C_y$

The exact efficiencies of $\bar{y}_r$ and t_i ($i=1$, 2 and 3), relative to that of $\bar{y}$ are given by

$$E'_r=V(\bar{y})/\text{MSE}(\bar{y}_r)$$

$$E'_i=V(\bar{y})/\text{MSE}(t_i) \quad i=1,\ 2\ \&\ 3 \qquad (3.12)$$

Now, using (3.2) through (3.10) and substituting the values of α, β and δ given by (3.11) efficiencies E'_r and E'_i ($i=1$, 2 & 3) can be expressed explicitly as functions of $K=C_x/C_y$, $m=nh$, ρ and weight W. However, it is difficult to investigate analytically the efficiencies of the estimators from the resulting expressions. Therefore, we have evaluated the values of E'_r and E'_i (percentages) for selected values of ρ, K and m and for $W=\frac{1}{4}$ and $\frac{1}{2}$. The results are given in Tables 3 and 4 respectively. The results of Table 3 may be summarized as follows : (1) The ratio estimator $\bar{y}_r$ is less efficient than $\bar{y}$ for low correlation ($\rho\leqslant.4$) except when $\rho=.4$, $K<1$ and $m\geqslant 20$. (2) The estimators t_1, t_2 and t_3 with $W=\frac{1}{4}$ are more efficient than both $\bar{y}$ and $\bar{y}_r$ for the following values of ρ, K and m, (a) $.2\leqslant\rho\leqslant.4$, $K\leqslant 1$, $m\geqslant 16$. (b) $.2<\rho<.4$, $K>1$, $m\geqslant 32$. Noting that in our model $C_x=h^{-1/2}C\bar{x}=m^{-1/2}$ and $n\leqslant m$ if $h\geqslant 1$ we may conclude that for low correlation ($.2\leqslant\rho\leqslant.4$), $W=\frac{1}{4}$ appears to be a good choice for estimators t_1, t_2 & t_3 even in small samples if $K\leqslant 1$ and in large samples only when $K>1$. Further, the exact efficiencies of these estimators with $W=\frac{1}{4}$ are of the same order as judged by their mean square errors.

From table 4, it can be seen that the estimators t_1, t_2 and t_3 with $W=\frac{1}{2}$ are more efficient than both $\bar{y}$ and $\bar{y}_r$ for $\rho\geqslant.5$, $.25\leqslant K\leqslant 1.50$ and $m\geqslant 16$. However, the ratio estimator $\bar{y}_r$ is most efficient when $\rho=.9$ and $.5\leqslant K\leqslant 1$. Thus, $W=\frac{1}{2}$ appears to be a good choice for estimators t_1, t_2 and t_3 for moderate to high correlation ($\rho>.4$), except when $\rho=.9$ and $.5\leqslant K\leqslant 1$. The exact efficiencies of t_1, t_2 and t_3 with $W=\frac{1}{2}$ are again generally of the same order. It is interesting to note that under model I the exact efficiencies of the estimators t_1, t_2 and t_3 approach the asymptotic efficiency when $m=nh\geqslant 32$. For example when $\rho=.4$ & $K=1.0$, $E_1=116$ (table 1) & $E'_1=114$, $E'_2=E'_3=115$ for $m=32$ (table 3).

We note from tables 3 and 4 that it is difficult to choose among the estimators t_1, t_2 and t_3 on the basis of their exact mean square errors. The absolute biases of estimators $\bar{y}_r$ and t_i relative to their mean square errors are given by

$$B_r=\mid \text{Bias}(\bar{y}_r)\mid /[\text{MSE}(\bar{y}_r)]^{1/2}$$

and

$$B_i=\mid \text{Bias}(t_i)\mid /[\text{MSE}(t_i)]^{1/2},\ i=1,2\ \&\ 3 \tag{3.13}$$

respectively. The numerical values of B_r and B_i ($i=1$, 2 & 3) for $W=\frac{1}{4}$ and $W=\frac{1}{2}$ are given in tables 5 and 6 respectively for selected values of m, K & ρ. From table 5, it can be seen that B_2 is generally less than 1% ; B_1 is slightly greater than B_3 but B_1 is still less than

10% for $m=nh \geqslant 16$. The ratio estimator $\bar{y}_r$ is generally badly biased ($B_r > 10\%$ for $K \geqslant 1$). From table 6, we find that $B_2 < 1\%$ for $K \leqslant 1$ and for $K > 1$, $B_2 < 2.5\%$ when $m \geqslant 16$. Turning to the relative biases of t_1 and t_3 we find that $B_1 < B_3$ for $K < 1$ and $B_1 > B_3$ for $K > 1$. It is also interesting to note that although $\text{MSE}(\bar{y}_r) < \text{MSE}(t_i)$ for $\rho = .9$ and $.5 \leqslant K \leqslant 1$ (table 4), B_r in this case exceeds 10% and is considerably higher than B_i. Thus, for $\rho = .9$ and $.5 \leqslant K \leqslant 1$, although $\text{MSE}(\bar{y}_r) < \text{MSE}(t_i)$, the estimators t_i's may be preferable in situations where the freedom from bias is desirable.

It may be noted that in surveys with many strata and small samples within strata the bias of the ratio estimator relative to its standard error may be considerable if it is appropriate to use 'separate' ratio estimators [see Cochran (1963)]. In such situations it may be of great advantage to use the proposed estimators $t_i (i=1, 2 \text{ and } 3)$. These estimators not only reduce the bias but also increase the precision.

In light of the above results we conclude that the three ratio-type estimators t_1, t_2 and t_3 are preferable to both $\bar{y}$ and $\bar{y}_r$. The efficiencies of these estimators are the same in large samples and are practically of the same order in small samples. Computationally t_1 is simplest and the bias of t_2 is least.

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TABLE 1

Efficiencies, E_1 and E_2, of t_1, (t_2 and t_3) over $\bar{y}$ and $\bar{y}_r$ for selected values of ρ and K and $W=1/4$

ρ	$K=0.5$		$K=1.0$		$K=1.5$		$K=2.0$	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
.1	101	116	99	178	94	277	89	400
.2	104	109	104	166	101	268	95	400
.3	106	101	110	153	109	257	105	400
.4	109	93	116	139	119	244	118	400
.5	112	84	123	123	131	229	133	400
.6	116	75	131	105	145	210	154	400
.7	119	65	140	84	162	187	182	400
.8	123	55	150	63	185	157	222	400
.9	126	44	163	33	215	118	285	400

TABLE 2

Efficiencies, E_1 and E_2, of t_1, (t_2 and t_3) over $\bar{y}$ and $\bar{y}_r$ for selected values of ρ and K and $W=1/2$

ρ	$K=0.5$		$K=1.0$		$K=1.5$		$K=2.0$	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
.1	99	114	87	157	71	209	56	256
.2	104	109	95	152	79	210	62	262
.3	110	104	105	147	90	211	71	271
.4	116	99	117	141	104	213	83	283
.5	123	92	133	133	123	215	100	300
.6	131	85	153	123	151	219	125	325
.7	140	77	182	109	195	224	167	367
.8	150	68	222	89	276	234	250	450
.9	163	57	286	57	471	259	500	700

TABLE 3

The exact efficiencies, E'_r and E'_i, of $\bar{y}_r$ and t_i (i=1, 2 & 3) with W=1/4, for selected values of m, K & ρ

m	K	ρ=.2				ρ=.3				ρ=.4			
		E'_r	E'_1	E'_2	E'_3	E'_r	E'_1	E'_2	E'_3	E'_r	E'_1	E'_2	E'_3
8	.50	61	95	100	99	68	98	104	102	77	101	107	105
	1.00	37	93	96	98	43	99	102	104	51	106	109	111
	1.50	21	87	88	95	24	95	93	103	28	104	103	112
16	.50	77	100	103	101	86	103	105	104	96	106	108	107
	1.00	49	99	102	101	56	105	107	107	66	111	114	113
	1.50	29	94	96	98	32	102	104	106	37	112	114	116
20	.50	81	100	103	102	90	103	106	104	100	106	109	107
	1.00	51	100	102	102	59	106	108	107	69	112	114	114
	1.50	30	96	98	98	34	104	106	107	39	114	115	117
32	.50	86	102	103	102	95	104	106	105	107	107	109	108
	1.00	55	102	103	108	63	107	108	108	74	114	115	115
	1.50	33	98	99	99	37	106	107	108	43	116	117	117

TABLE 4

The exact efficiencies, E'_r and E'_i, of y_r, and t_i (i=1, 2 & 3) with $W=1/2$, for selected values of m, K & ρ

m	K	$\rho=.5$				$\rho=.7$				$\rho=.9$			
		E'_r	E'_1	E'_2	E'_3	E'_r	E'_1	E'_2	E'_3	E'_r	E'_1	E'_2	E'_3
8	.25	79	94	99	99	86	99	103	105	91	103	105	113
	.50	87	104	109	109	117	120	126	126	168	140	146	149
	1.00	62	106	105	116	105	152	152	162	324	260	262	269
	1.50	33	87	78	102	50	139	122	163	103	340	264	415
16	.25	100	103	108	106	111	109	114	112	123	115	120	120
	.50	109	114	119	116	147	130	136	113	222	152	157	156
	1.00	80	120	124	124	134	168	173	172	408	274	279	278
	1.50	44	105	107	112	67	167	168	179	139	409	391	444
20	.25	104	105	109	107	117	111	115	114	130	118	121	121
	.50	114	116	120	117	154	133	137	135	234	155	159	158
	1.00	83	123	127	126	140	171	175	174	425	277	280	279
	1.50	46	108	111	114	70	173	175	182	146	422	410	450
32	.25	111	108	110	109	125	114	117	116	142	121	124	123
	.50	121	119	121	120	164	136	138	138	252	158	161	160
	1.00	89	127	129	129	150	175	177	177	453	280	283	282
	1.50	50	114	116	118	76	181	183	187	159	441	435	458

TABLE 5

The absolute values of % Bias/$(MSE)^{1/2}$, B_r and B_i of $\bar{y}_r$ and t_i (i=1, 2 & 3) with W=1/4, for selected values of m, K & ρ

m	K	ρ=.2				ρ=.3				ρ=.4			
		B_r	B_1	B_2	B_3	B_r	B_1	B_2	B_3	B_r	B_1	B_2	B_3
8	.50	9.48	2.96	1.01	1.07	6.68	2.00	.69	.18	3.55	1.02	.35	.74
	1.00	19.74	7.80	2.64	3.92	18.57	7.05	2.38	3.12	17.29	6.24	2.11	2.26
	1.50	25.02	12.31	4.06	6.65	24.57	11.88	3.91	6.01	24.15	11.42	3.78	5.33
16	.50	7.03	2.00	.29	.74	4.94	1.35	.20	.10	2.62	.68	.10	.55
	1.00	14.90	5.31	.77	2.74	13.99	4.78	.69	2.16	12.99	4.22	.61	1.56
	1.50	18.56	8.42	1.22	4.67	18.22	8.10	1.17	4.21	17.91	7.77	1.12	3.72
20	.50	6.34	1.77	.20	.65	4.46	1.20	.13	.09	2.40	.61	.07	.50
	1.00	13.50	4.71	.53	2.45	12.66	4.42	.48	1.93	11.77	3.74	.42	1.39
	1.50	16.85	7.48	.84	4.17	16.54	7.12	.81	3.76	16.25	6.90	.77	3.32
32	.50	5.08	1.38	.09	.51	3.56	.93	.06	.06	1.88	.47	.03	.40
	1 00	10.86	3.68	.25	1.93	10.18	3.31	.22	1.52	9 43	2.92	.20	1.90
	1.50	13.62	5.86	.39	3.29	13.36	5.64	.38	2.96	13.12	5.40	.36	2 61

TABLE 6

The absolute values of % Bias/$(MSE)^{1/2}$, B_r and B_i of $\bar{y}_r$ and t_i (i=1, 2, & 3) W=1/2, for selected values of m, K and ρ

m	K	ρ=.5				ρ=.7				ρ=.9			
		B_r	B_1	B_2	B_3	B_r	B_1	B_2	B_3	B_r	B_1	B_2	B_3
8	.25	8.99	4.89	1.67	5.71	16.86	9.04	3.09	9.71	25.11	13.36	4.49	13.98
	.50	0.00	0.00	0.00	2.29	8.75	4.42	1.51	6.62	20.97	9.57	3.25	11.73
	1.00	15.89	10.42	3.46	5.27	12.45	7.47	2.49	1.51	7.27	3.26	1.09	4.14
	1.50	23.15	18.80	5.95	12.10	22.88	19.04	5.94	10.58	24.55	22.36	6.56	9.29
16	.25	6.67	3.39	.50	4.09	12.65	6.27	.91	6.92	19.25	9.31	1.36	9.96
	.50	0.00	0.00	0.00	1.68	6.48	3.05	.44	4.75	15.88	6.58	.96	8.34
	1.00	11.89	7.31	1.06	3.67	9.26	5.18	.75	.96	5.38	2.21	.32	3.05
	1.50	17.64	13.65	1.96	8.62	17.42	13.79	1.98	7.46	18.79	16.17	2.26	6.35
20	.25	6.01	3.02	.34	3.67	11.44	5.59	.63	6.21	17.48	8.30	.94	8.92
	.50	0.00	0.00	0.00	1.51	5.85	2.71	.31	4.26	14.39	5.85	.66	7.47
	1.00	10.75	6.52	.74	3.27	8.37	4.61	.52	.83	4.85	1.96	.22	2.75
	1.50	16.00	12.25	1.38	7.71	15.80	12.38	1.38	6.67	17.06	14.50	1.59	5.64
32	.25	4.81	2.37	.16	2.91	9.19	4.39	.30	4.92	14.14	6.53	.44	7.06
	.50	0.00	0.00	0.00	1.21	4.68	2.12	.14	3.38	11.59	4.59	.31	5.92
	1.00	8.63	5.14	.35	2.57	6.70	3.62	.24	.64	3.88	1.53	.10	2.20
	1.50	12.92	9.74	.65	6.11	12.75	9.83	.66	5.26	13.79	11.49	.76	4.40