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SOME BOUNDS ON THE NUMBER OF BLOCKS IN BIB DESIGNS

D. V. S. SASTRY*

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SUMMARY

The purpose of this paper is to obtain some bounds on the number of blocks in a BIB design under some parametric restrictions.

Keywords ; bounds on blocks, BIB designs.

Introduction

Kageyama [1] has shown that for BIB designs with $v = nk$, b, r, k and λ

$$b > v + r - 1 \Leftrightarrow r \geqslant \lambda + 2k.$$

Using such relations, Kageyama *et al.* [3] obtained bounds on b. In this paper the above inequality is generalised to a wider class of designs and used for obtaining bounds on b, the number of blocks.

2. Main Results

THEOREM 2.1 : *For a BIB design with $v = nk$, b, r, k and λ and some $\alpha > 0$,*

*Present address : A-63, Jalada, RBI Officers Quarters, Prabhadevi, Bombay-25.

$$b > \alpha v + r - \alpha \Leftrightarrow r \geqslant \lambda + (\alpha + 1) k.$$

Proof : From the basic relation $vr = bk$ we can write

$$b = \frac{vr}{k} = \frac{1}{k}\left\{ vr + k(\alpha v + r - \alpha) - k(\alpha v + r - \alpha) - r + r \right\}$$

We can get after some algebra

$$b - (\alpha v + r - \alpha) = \frac{v-1}{k}\left\{ r - \alpha k - \lambda \right\}.$$

As $v = nk$, we have $(v - 1, k) = 1$. Since $b - (\alpha v + r - \alpha)$ is a positive quantity we have

$$\frac{(r - \alpha k - \lambda)}{k} \geqslant 1$$

or $$r \geqslant \lambda + (\alpha + 1) k.$$

The converse is easy to show.

COROLLARY 2.1 : $r = \lambda + \alpha k \Rightarrow b = \alpha v + r - \alpha.$

THEOREM 2.2 : *A BIB design with parameters* $v, b, r = p\lambda + q$ *where* p *and* q $(q \lhd \lambda)$ *are integers, is a symmetric design if either* $q = 1$ *or* $q = 0$. *If* $q = 1$, *the symmetric design has parameters*

$$v = b = p(p\lambda + 1) + 1; \qquad r = k = (p\lambda + 1), \lambda,$$

while if $q = 0$, *the symmetric design has parameters*

$$v = b = p(p\lambda - 1) + 1, \qquad r = k = p\lambda, \lambda.$$

Proof: From the basic relation $\lambda(v - 1) = r(k - 1)$ and $r = p\lambda + q$ we get

$$v \equiv \frac{(p\lambda + q)(k - 1)}{\lambda} + 1,$$

Since $(p\lambda + q, \lambda) = 1$, we have $k - 1 = l\lambda$ where l is an integer. As $r > k$ we have

$$p\lambda + q \geqslant l\lambda + 1.$$

Equality is attained when $q = 1$ and $p = 1$. Thus $r = k = p\lambda + 1$ and in such a case $v = b = p(p\lambda + 1) + 1$.

If $q = 0$ then $r = k = p\lambda$ and $v = b = p(p\lambda - 1) + 1$.

COROLLARY 2.2 : *If $r \neq p\lambda + q$, then the BIB design will have the parameters*

$$v = l(p\lambda + q) + 1,\ b,\ r = p\lambda + q,\ k = l\lambda + 1.$$

THEOREM 2.3 : *In a BIB design with $v, b, r \neq p\lambda + 1$ k and*

$$\lambda,\ b \leqslant \frac{r^2}{\lambda} - \left\{(p - 1) + 2/\lambda\right\}.$$

Proof : From $r \neq p\lambda + 1$ we have $r \geqslant p\lambda + 2$. Also,

$$rk - \lambda v = r - \lambda \geqslant (p - 1)\lambda + 2.$$

Multiplying both sides by b and simplifying (using $r > k$) we get,

$$b \leqslant r^2/\lambda - \{(p - 1) + 2/\lambda\}.$$

When $p = 1$ this bound is same as that in Theorem 2.1 of Kageyama *et al.* [3]; for all other values of p this bound is an improvement over the Kageyama [1] bound.

In the light of Theorem 2.1, we have the following

THEOREM 2.4 : *In a BIB design with parameters $v = nk$, b, r, k and λ,*

$$b > \alpha v + r - \alpha \Leftrightarrow b \leqslant \frac{r(r - \overline{\alpha + 1})}{\lambda}.$$

The equality is attained if $r = \lambda + (\alpha + 1)k$.

Proof : We have $rk - \lambda v = r - \lambda \geqslant (\alpha + 1)k$, or

$$k\,(r - \overline{\alpha + 1}) \geqslant \lambda\, v$$

$$r - (\lambda + 1) \geqslant \lambda\, n$$

Multiplying by r we have $b \leqslant \dfrac{r\,(r - \overline{\alpha + 1})}{\lambda}$.

The converse is obvious and equality is attained when

$$r = \lambda + (\alpha + 1)\, k.$$

The above bound is superior to the one given by Theorem 2.5 of Kageyama [3] as can be seen by the following example.

EXAMPLE 2.1 : Consider the BIB design with $v = 18$, $b = 102$, $r = 17$, $k = 3$ and $\lambda = 2$. Applying Kageyama's bound gives $b \leqslant 127$ whereas applying the above Theorem with $\alpha = 4$ yields $b \leqslant 102$ which is actually attained.

COROLLARY 2.3 : *In a resolvable design which is not affine,*

$$b \leqslant \frac{r\,(r - \overline{\alpha + 1})}{\lambda},$$

the equality holding if and only if $r = \lambda + (\alpha + 1)\, k$.

COROLLARY 2.4.2 : *In a resolvable design which is not affine resolvable, if* $r = \lambda + (\alpha + 1)\, k$ *then*

$$v = \frac{k\,(r - \overline{\alpha + 1})}{\lambda}, \quad b = \frac{r\,(r - \overline{\alpha + 1})}{\lambda}; \quad r, k, \lambda.$$

For designs with special parameters v, $b = mt$, $r = \mu t$, k, and λ in the light of Theorem 2.1 we have the following

THEOREM 2.5 : *In a BIB design with parameters* v, $b = mt$, $r = \mu t$, k *and* λ,

$$b \gtreqless \alpha v + r - \alpha \Leftrightarrow b \gtreqless \{m^2 \lambda + \mu m (\alpha + 1)\}/\mu^2$$

respectively.

Proof :

$$b \gtreqless \alpha v + r - \alpha \Leftrightarrow \mu r m \gtreqless m^2\lambda + \mu m (\alpha + 1)$$

or $$b \gtreqless \{m^2 \lambda + \mu m (\alpha + 1)\}/\mu^2$$

This bound is meaningful and better if we compare with Example 2.2 of Kageyama *et al.* [3].

EXAMPLE 2.2 : Consider the BIB design $v = 6, b = 20, r = 10, k = 3$ and $\lambda = 4$ having $\mu = 2, m = 4$ and $t = 5$.

In this case $\{m^2 \lambda + \mu (\alpha + 1)/m\}/\mu^2 = 20$ which is actually attained. Shah [5] has proved that the necessary and sufficient condition for the inequality $b \gtreqless v + r - 1$ to hold for any BIB design is that $r \gtreqless \lambda + k$. However to cover a wider class of designs we have the following

THEOREM 2.6 : *A necessary and sufficient condition for the inequality* $b \gtreqless \alpha v + r - \alpha$ *to hold for any BIB design is that* $r \gtreqless \lambda + \alpha k$.

Proof : From the proof of Theorem 2.1 we have

$$b - (\alpha v + r - \alpha) = (v - 1) k^{-1} (r - \lambda - \alpha k)$$

from which the above theorem follows.

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