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An Improved Preliminary Test Estimator for the Variance of a Normal Distribution

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Summary

A preliminary test estimator (PTE) for the variance of a normal population has been proposed when a prior information about the variance is available. Empirical results of bias and relative efficiency reveal that the proposed estimator is better than the similar estimator constructed by Srivastava [2].

Key Words : Minimum Mean Square Error Estimator, Preliminary Test Estimator, Relative Bias, Relative Efficiency.

Introduction

Let $x_1, x_2, \ldots, x_n$ be a random, sample drawn from a normal population $N(\mu, \sigma^2)$, where μ and σ^2 both are not known. It is desired to estimate σ^2. Goodman [1] showed that the minimum mean square error estimator among the class of estimators of the form Cs^2 is $T = (n-1)s^2/(n+1)$, where s^2 is the usual unbiased estimator of σ^2.

Let σ_0^2 be available as a prior information on σ^2 besides the sample information in the form of T. These informations are used in the construction of the estimator $\hat{\sigma}^2$ proposed here. This estimator is obtained as a consequence of the preliminary test of the hypothesis $\sigma^2 = \sigma_0^2$ and is called Preliminary test estimator (PTE). It is defined as follows :

$$\hat{\sigma}^2 = \begin{cases} \sigma_0^2 & \text{if } H_0 : \sigma^2 = \sigma_0^2 \text{ is accepted,} \\ T & \text{otherwise.} \end{cases}$$

The expressions of bias and mean square error of $\hat{\sigma}^2$ are derived. Srivastava [2] has proposed a similar PTE of σ^2 as defined by

$$\hat{\sigma}^2_{PT} = \begin{cases} \sigma_0^2 & \text{if } H_0 : \sigma^2 = \sigma_0^2 \text{ is accepted,} \\ s^2 & \text{otherwise.} \end{cases}$$

We have investigated gain in the relative efficiency of $\hat{\sigma}^2$ over $\hat{\sigma}^2_{PT}$.

2. *Bias and mean square error of* $\hat{\sigma}^2$

We know that ms^2/σ^2 is distributed as χ^2 with m = (n−1) degrees of freedom. Hence the hypothesis $H_o : \sigma^2 = \sigma_0^2$ is tested using the statistic $w = ms^2/\sigma_0^2$ which has the density given by

$$f(w, k) = \frac{k^{m/2}}{2^{m/2}\Gamma(m/2)} e^{-kw/2} w^{(m/2)-1} . \quad (k = \sigma_0^2/\sigma^2, \ k > 0) \tag{2.1}$$

If we denote by χ^2 (m, α) the upper 100α % point of Chi-square distribution with m degree of freedom, then the estimator $\hat{\sigma}^2$ may be written as

$$\hat{\sigma}^2 = \begin{cases} \sigma_0^2 & \text{if } w \le X^2 (m, \alpha), \\ T & \text{otherwise.} \end{cases} \tag{2.2}$$

The expected value of $\hat{\sigma}^2$ is given by

$$E(\hat{\sigma}^2) = E\left[\sigma_0^2 \mid w \le \chi^2 (m, \alpha)\right] P\left[w \le \chi^2 (m, \alpha)\right] + E\left[T \mid w \ge \chi^2 (m, \alpha)\right] P\left[w \ge \chi^2 (m, \alpha)\right] \tag{2.3}$$

$$= \frac{\sigma_0^2 k^{m/2}}{2^{m/2}\Gamma(m/2)} \left[\int_0^{\chi^2(m, \alpha)} e^{-k w/2} w^{(m/2)-1} dw + \frac{1}{n+1} \int_{\chi^2(m, \alpha)}^{\infty} e^{-kw/2} w^{m/2} dw \right] \tag{2.4}$$

On evaluating the integrals in (2.4) and simplifying we get

$$E(\hat{\sigma}^2) = \sigma_0^2 I(C, m/2) + \frac{m\sigma_0^2}{k(n+1)} \left\{ 1 - I\left(C, \frac{m}{2} + 1\right)\right\}, \tag{2.5}$$

where $C = \frac{1}{2} k\chi^2 (m, \alpha)$ and $I(y, n) = \int_0^y e^{-x} x^{n-1} \frac{dx}{\Gamma n}$

Therefore

$$\frac{E(\hat{\sigma}^2)}{\sigma^2} = k\, I\left(C, \frac{m}{2}\right) + \frac{m}{(n+1)} \left\{1 - I\left(C, \frac{m}{2} + 1\right)\right\}. \tag{2.6}$$

The relative bias (Bias/σ^2) can be easily obtained from (2.6) and is given by

$$\text{Relative Bias} = k\, I\left(C, \frac{m}{2}\right) - \frac{m}{(n+1)}\, I\left(C, \frac{m}{2} + 1\right) - \frac{2}{(n+1)}. \tag{2.7}$$

The mean square error (MSE) of $\hat{\sigma}^2$ is defined as

$$\text{MSE}\,(\hat{\sigma}^2) = E[(\hat{\sigma}^2)]^2 - 2\sigma^2\, E(\hat{\sigma}^2) + \sigma^4, \tag{2.8}$$

where

$$E\,(\hat{\sigma}^2)^2 = E\,[\,\sigma_0^4 \mid w \le \chi^2\,(m, \alpha)]\ P\,[\,w \le \chi^2\,(m, \alpha)]$$
$$+ E \mid T^2 \mid w \ge \chi^2\,(m, \alpha)]\ P\,[\,w \ge \chi^2\,(m, \alpha)] \tag{2.9}$$

and E $(\hat{\sigma}^2)$ is given by (2.5). It is easy to evaluate (2.9) on substituting the values of $E(\hat{\sigma}^2)^2$ and $E(\hat{\sigma}^2)$ in (2.8) we obtain

$$\frac{\text{MSE}\,(\hat{\sigma}^2)}{\sigma^4} = \frac{2}{(n+1)} + k\,(k-2)\, I\left(C, \frac{m}{2}\right) + \frac{2m}{(n+1)}\, I\left(C, \frac{m}{2} + 1\right)$$
$$- \frac{m}{(n+1)}\, I\left(C, \frac{m}{2} + 2\right). \tag{2.10}$$

3. *Relative efficiency of* $\hat{\sigma}^2$

It is known that the variance of the unbiased estimator s^2 is $2\sigma^4/m$. Hence the relative efficiency of $\hat{\sigma}^2$ with respect to s^2 will be given by the ratio,

$$\text{RE}\,(\hat{\sigma}^2) = \frac{V(s^2)}{\text{MSE}}\,(\hat{\sigma}^2)$$
$$= \left[\frac{m}{(n+1)} + \frac{m\,k(k-2)}{2}\, I\left(C, \frac{m}{2}\right) + \frac{m^2}{(n+1)}\, I\left(C, \frac{m}{2} + 1\right) - \frac{m^2}{2\,(n+1)}\, I\left(C, \frac{m}{2} + 2\right)\right]^{-1} \tag{3.1}$$

which is less than 1 for $k \ge 2$.

Table 1. Relative Bias : Bias / σ^2

n	k 0.4	0.6	0.8	1.0	1.2	1.6
5	-0.304 -0.286	-0.228 -0.202	-0.106 -0.081	0.049 0.070	0.224 0.240	0.606 0.612
7	-0.250 -0.232	-0.205 -0.176	-0.102 -0.074	0.045 0.067	0.220 0.234	0.602 0.608
9	-0.212 -0.194	-0.188 -0.157	-0.100 -0.070	0.042 0.063	0.216 0.229	0.602 0.605

Table 2. Relative Efficiency of σ^2 with Respect to s^2

n	k 0.4	0.6	0.8	1.0	1.2	1.6
5	1.771 1.896	2.623 2.826	4.988 5.048	9.154 7.822	6.306 5.345	1.334 1.292
7	1.383 1.500	1.894 2.057	3.607 3.648	7.324 6.016	4.820 4.039	0.906 0.886
9	1.254 1.315	1.540 1.711	2.925 2.971	6.497 5.208	4.012 3.374	0.686 0.676

Table 3 : Gain in Relative Efficiency

n	k 0.4	0.6	0.8	1.0	1.2	1.6
5	0.849 0.908	1.519 1.712	3.404 3.629	6.748 5.989	3.498 3.373	0.110 0.210
7	0.513 0.550	0.881 0.999	2.070 2.240	4.684 4.0.44	1.935 1.956	0.036 0.136
9	0.345 0.389	0.598 0.703	1.439 1.546	3.701 3.126	1.220 1.299	0.008 0.012

4. *Discussion of the numerical results*

We have calculated values of the relative bias and relative efficiency RE $(\hat{\sigma}^2)$ for n = 5,7,9, level of significance α = .05, 0.10 and k=0.4, 0.6, 0.8, 1.0, 1.2 and 1.6. These results are assembled in Table 1 and 2. Results of the gain in relative efficiency [RE $(\hat{\sigma}^2)$ –RE $(\hat{\sigma}^2_{PT})$] are assembled in Table 3. In these tables the values in the first row correspond to the level of significance α = .05 and those in the second row refer to α = .10.

From Table 1, we observe that the relative bias of the proposed estimator increases as we increase the level of significance α. The estimator is positively biased for $k \geq 1$ and negatively biased for $k < 1$.

From Table 2, we observe that the proposed estimator $\hat{\sigma}^2$ is more efficient than the unbiased estimator s^2 for all values of k lying between 0.4 to 1.2. We also observe that the relative efficiency increases with k and reaches its maximum at k=1 and then decreases.

We have compared the relative efficiency of the estimator $\hat{\sigma}^2$ defined by (2.2) with the relative efficiency of a similar estimator $\hat{\sigma}^2_{PT}$ proposed by Srivastava [2]. From Table 3 it is observed that the gain increases as k increases and reaches its maximum at k = 1. For k > 1 gain decreases rapidly. The values of relative bias of $\hat{\sigma}^2$ shown in Table 1 can be easily compared with the relative bias of σ^2_{PT} obtained by Srivastava [2]. It is found that $\hat{\sigma}^2$ is always less biased. Therefore, the estimator $\hat{\sigma}^2$ is not only less biased but more efficient than σ^2_{PT}.

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