A SIMPLIFIED METHOD OF ENUMERATING LATIN SQUARES BY MacMAHON'S DIFFERENTIAL OPERATORS

Part II. The 7 × 7 Latin Squares

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CONTENTS

§1. Résume

In Part I of this paper a modification of MacMahon's formula had been developed for the direct and exhaustive enumeration of the standard Latin squares of any order n, and this formula was applied to the enumeration of the 6×6 Latin squares. Although the 6×6 squares had already been exhaustively enumerated before, no such enumeration except that attempted by Norton (1939) so far existed for the 7×7 squares. In this paper the formula established in the earlier part has been applied to the complete and exhaustive enumeration of the 7×7 Latin squares. It is interesting to recall in this connection that Norton (1939) in his own words only presented, " an extensive —possibly an exhaustive—study of 7×7 Latin and higher squares." It would be shown that his figure 16,927,968 actually falls short of the true size of the universe of standard 7×7 Latin squares by 14,112.

Before beginning with the calculations, we shall first restate, without proof, the three main results established in Part I, as these would be used again and again in the sequel.

THEOREM I.—" The number of standard Latin squares of side 'n' is enumerated by the formula :

$$R_n = D_{(1345\ldots n)}\, D_{(1245\ldots n)}\, D_{(1235\ldots n)} \ldots D_{(1234\ldots \overline{n-1})}\, (1345\ldots n)\,(1245\ldots n)\,(1235\ldots n)\ldots(1234\ldots\overline{n-1}), \qquad (1.1)$$

where the $(n-1)$ partitions may be written down by striking out in turn the numbers 2, 3, 4, ...n from the series of integers 1, 2, 3, ...n; and where the operation $D_{(\lambda_1\lambda_2\ldots\lambda_k)}$ is performed upon the product $\phi_1.\phi_2\ldots\phi_k$ of symmetric functions denoted by partitions, by abstracting the partition $(\lambda_1\lambda_2\ldots\lambda_k)$ from the product in all possible ways, one part at most being taken from each factor, and adding the terms."

Of course, the totality of Latin squares of order n, whether standard or non-standard, is given, as is well known, by

$$I_n = n!\,(n-1)!\,R_n,$$

so that the direct evaluation of R_n is considerably simpler than the task of evaluating I_n from the usual MacMahon formula :

$$I_n = D^n{}_{(123\ldots n)}.(123\ldots n)^n.$$

THEOREM II.—" Let the universe of standard Latin squares of order n containing the letters $a, c, d, e, f, \ldots$ in the second-row second-column cell be denoted by

$$A_n,\ C_n,\ D_n,\ E_n,\ F_n,\ \ldots.$$

Then

$$C_n = D_n = E_n = F_n = \ldots,$$

so that the obvious relation

$$R_n = A_n + C_n + D_n + E_n + F_n + \ldots$$

becomes

$$R_n = A_n + (n-2)\,C_n. \qquad (1.2)$$

Also the analytical expressions for A_n and C_n are obtained from the formula (1·1), if in applying the *first* operation $D_{(1345\ldots n)}$, we restrict ourselves to the following terms alone:

(i) For A_n—where 1 is deleted *only* from the first factor $(1345\ldots n)$ of the operand, and 3, 4, 5, ...n from the remaining factors,

(ii) For C_n—where 3 is deleted *only* from the first factor $(1345\ldots n)$ of the operand, and 1, 4, 5, ...n from the remaining factors,

and the remaining operations are kept unaltered."

This reduces the problem of enumeration to the evaluation of A_n and C_n only.

THEOREM III.—" If two terms of a set, obtained after a D-operation, are connected by a permutation which leaves the remaining D-operators invariant, then the two terms are equinumerous, *i.e.*, lead to the same numerical result on performing the remaining D-operations."

To avoid confusion, the bracket notation of partitions shall be exclusively reserved for the symmetric functions of our formulæ, unless explicit reference is made to these connecting permutations.

§ 2. THE FORMULA FOR ENUMERATING 7×7 LATIN SQUARES

Formula (1.1) for the case $n = 7$, when written out in full is—

$$R_7 = D_{(134567)}\, D_{(124567)}\, D_{(123567)}\, D_{(123467)}\, D_{(123457)}\, D_{(123456)}\, [P],$$

where P is the following product of symmetric functions:

$$P = (134567)\,(124567)\,(123567)\,(123467)\,(123457)\,(123456). \tag{2.1}$$

Denoting the operators by δ_i and the factors of the operand by f_i we can write the above formula as follows:

$$R_7 = \delta_1 . \delta_2 . \delta_3 . \delta_4 . \delta_5 . \delta_6 . [f_1 \cdot f_2 \cdot f_3 \cdot f_4 \cdot f_5 \cdot f_6] \tag{2·2}$$

where,

$$\left.\begin{array}{lcl} \delta_1 \equiv D_{(134567)} & & f_1 = (134567) \\ \delta_2 \equiv D_{(124567)} & & f_2 = (124567) \\ \delta_3 \equiv D_{(123567)} & \text{and,} & f_3 = (123567) \\ \delta_4 \equiv D_{(123467)} & & f_4 = (123467) \\ \delta_5 \equiv D_{(123457)} & & f_5 = (123457) \\ \delta_6 \equiv D_{(123456)} & & f_6 = (123456) \end{array}\right\} \tag{2.3}$$

so that

$$P = f_1 \cdot f_2 \cdot f_3 \cdot f_4 \cdot f_5 \cdot f_6 .$$

§ 3. OUTLINE OF DIFFERENTIAL OPERATIONS WITH THE ATTENDANT REDUCTIONS

The following operator-wise descriptive summary provides a brief over-all picture of the entire calculations. The reductions mentioned at each stage are a consequence of using Theorems II and III as given above in the résumé.

Writing the formula (2.1) in the form:

$$R_7 = \delta_1 . \delta_2 . \delta_3 . \delta_4 . \delta_5 . \delta_6 [P] \tag{3.1}$$

where the δ's denote partition-deleting operators and P is the product of symmetric functions 'f_i' expressed in partitional notation as given in (2.3) above, we have shown in § 4 that $\delta_1 . P$ generates the sum of 309 terms, and have also reduced them to only eight terms, such that

$$\delta_1 . P = \sum_{i=1}^{8} a_i x_i, \tag{3.2}$$

where the a's are known coefficients and the x's denote symmetric-function products as given in Table I.

It follows from (3·2) that

$$\delta_2 . \delta_1 . P = \sum_{i=1}^{8} a_i (\delta_2 . x_i).$$

In § 5 are given the complete expressions for each of the $\delta_2 . x_i$ ($i = 1, 2, \ldots 8$), the second operation δ_2 generating 663 terms in all (*vide* Table II).

In § 6*a* we have reduced each $\delta_2 . x_i$ to the form

$$\sum_r w_{i(r)} . y_{i(r)}$$

the weights w_i and the symmetric-function products y_i being given in Table III. The y_i's are only 127 in number.

In § 6*b* we reduce the 127 y_j's above to only 40 representative y_j's, so that we may write

$$\delta_2 . \delta_1 . P = \sum_{i=1}^{8} a_i \left(\sum_r w_{i(r)} y_{i(r)}\right)$$

$$= \sum_{i=1}^{127} b_i y_i \qquad \text{say,}$$

$$= \sum_{p=1}^{40} c_{j_p} y_{j_p}, \qquad \text{say,} \tag{3.3}$$

where $j_1, j_2, \ldots$ are some of the numbers from $1, 2, \ldots 127$ (*vide* Table IV). Hence, from (3·1) and (3·3)

$$R_7 = \sum_{p=1}^{40} c_{j_p} (\delta_3 . \delta_4 . \delta_5 . \delta_6 . y_{j_p}). \tag{3.4}$$

This reduction of the 663 terms, generated by δ_2, to the 40 y's has been the main *tour de force* leading to the present enumeration.

The evaluation of the 40 quantities (*vide* Table V)

$$\delta_3 . \delta_4 . \delta_5 . \delta_6 . (y_j)$$

next confronts us with the heaviest, though routine, part of the problem, as it involves the application of $\delta_4 . \delta_5 . \delta_6$ to the 1159 terms generated by $\delta_3 y_j$. These calculations have been briefly illustrated in § 7 by means of an *example* for the case

$$\delta_3 . \delta_4 . \delta_5 . \delta_6 \, (y_1).$$

The remainder of the work, §§ 8 and 9, is pretty simple, being only the formation of a cumulative sum of products, leading directly to the value of R_7.

§ 4. The First Operation $\delta_1 \equiv D_{(134567)}$—Reduction to Eight Terms

We shall now apply the first operator $D_{(134567)}$ to the product $P = f_1 f_2 f_3 f_4 f_5 f_6$. The operation breaks up into two parts, according as,

(i) 1 is deleted from the first factor 'f_1',

or (ii) 3, 4, 5, 6 or 7 is deleted from the first factor 'f_1'.

Proceeding as in Theorem II, it follows that the five sets in (ii) obtained by deleting 3, 4, 5, 6 or 7 are equivalent, so that (Theorem II)

$$R_7 = A_7 + 5 . C_7, \tag{4.1}$$

where,

$$\left.\begin{aligned} A_7 &= \xi \; [(34567) . D_{(34567)} \cdot f_2 f_3 f_4 f_5 f_6] \\ C_7 &= \xi \; [(14567) . D_{(14567)} \cdot f_2 f_3 f_4 f_5 f_6]. \end{aligned}\right\} \tag{4.2}$$

and,

$$\xi \equiv \delta_2 . \delta_3 . \delta_4 . \delta_5 . \delta_6 .$$

Next, it follows from Theorem III, that equivalent terms are obtained through permutations which keep $\xi \equiv \delta_2 . \delta_3 . \delta_4 . \delta_5 . \delta_6$ invariant, *i.e.*, factors of the transformation

$$(12) . (34567).$$

Reduction of A_7

Applying $D_{(34567)}$ we can delete 4, 5, 6 or 7 from 'f_2'. This gives the following four sets of terms:

$$A_7 = \xi \; [(34567) \{(12567) . D_{(3567)} \cdot f_3 f_4 f_5 f_6 + \ldots \text{(Dropping 4 from } f_2) \tag{4.3}$$

$$+ (12467) . D_{(3467)} \cdot f_3 f_4 f_5 f_6 + \ldots (\quad ,, \quad 5 \quad ,, \quad) \tag{4.4}$$

$$+ (12457) . D_{(3457)} \cdot f_3 f_4 f_5 f_6 + \ldots (\quad ,, \quad 6 \quad ,, \quad) \tag{4.5}$$

$$+ (12456) . D_{(3456)} \cdot f_3 f_4 f_5 f_6 . \}] \ldots (\quad ,, \quad 7 \quad ,, \quad) \tag{4.6}$$

Using the transformations (45), (56), (67) which keep the product $f_3 . f_4 . f_5 . f_6$ and the operator ξ invariant, we notice that the four sets above are connected as follows:

Set	(4.3)	(4.4)	(4.5)	(4.6)
Connecting Permutation ..		(45)	(56)	(67)

and so, by Theorem III, are equivalent.

Hence, we obtain

$$A_7 = 4 . \xi \, [(34567)\,(12567) . D_{(3567)} . f_3 . f_4 . f_5 . f_6 .] \qquad (\alpha)$$

Performing the operations indicated in [ ] in (α), we obtain:

$$A_7 = 4 \cdot \xi \Big[(34567)\,(12567) . (12567)\,(12346)\,(12347)\,(12345) + \qquad (\alpha_1)$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad (12347)\,(12345)\,(12346) + \qquad (\alpha_2)$$

$$\text{,,} \quad \text{,,} \quad . (12467)\,(12367)\,(12345)\,(12345) + \qquad (\alpha_3)$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad (12356)\,(12347)\,(12345) + \qquad (\alpha_4)$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad (12357)\,(12345)\,(12346) + \qquad (\alpha_5)$$

$$\text{,,} \quad \text{,,} \quad . (12457)\,(12367)\,(12346)\,(12345) + \qquad (\alpha_6)$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad \frac{(12357)}{(12356)} \cdot \frac{(12346)}{(12347)}\,(12346) + \qquad (\alpha_7), (\alpha_8)^*$$

$$\text{,,} \quad \text{,,} \quad . (12456)\,(12367)\,(12347)\,(12345) + \qquad (\alpha_9)$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad \frac{(12357)}{(12356)} \cdot \frac{(12346)}{(12347)}\,(12347) . \Big] \qquad (\alpha_{10}), (\alpha_{11}).$$

Notice that the first three factors of each term are obtained by deleting in order 1, 4, 3.

The α's are connected by the following transformations:

$$\left.\begin{array}{l} \alpha_1 = \alpha_2, \quad \alpha_3 \quad \alpha_7 \quad \alpha_{11}, \\ \qquad\qquad\quad (56) \quad (67) \\[1ex] \alpha_{10} \quad \alpha_8 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_9 . \\ \quad (67) \quad (56) \quad (67) \quad (56) \quad (67) \end{array}\right\}$$

Thus under the operation ξ, we have the following equivalences:

$$\alpha_1 \sim \alpha_2, \quad \alpha_3 \sim \alpha_7 \sim \alpha_{11}, \quad \alpha_4 \sim \alpha_5 \sim \alpha_6 \sim \alpha_8 \sim \alpha_9 \sim \alpha_{10},$$

**N.B.*—Take either both upper or both lower factors.

Thus α_7 .. (34567) (12567).(12457) (12357) (12346) (12346)
and, α_8 .. (34567) (12567).(12457) (12356) (12347) (12346).

This convention is followed later on as well.

so that

$$A_7 = 4\xi\ [2\alpha_1 + 3\alpha_3 + 6\alpha_4].$$

Renaming α_1, α_3, α_4 as x_1, x_2, x_3 respectively, we may write this equation in the form:

$$A_7 = 4\xi\ [2x_1 + 3x_2 + 6x_3]$$
$$= \xi\ [8x_1 + 12x_2 + 24x_3] \quad (4.7)$$

where the explicit expressions for the x_i are:

x_1 .. (12345) (12346) (12347).(12567)^2.(34567),

x_2 .. (12345)^2.(12367) (12467) (12567).(34567),

x_3 .. (12345) (12347) (12356) (12467) (12567) (34567).

We have reduced A_7 to the evaluation of three terms.

Reduction of C_7

Applying $D_{(14567)}$ we can delete 1, 4, 5, 6 or 7 from 'f_2'. This gives the following five sets of terms:

$C_7 = \xi\cdot[(14567)\ \{(24567).D_{(4567)}\cdot f_3 f_4 f_5 f_6 +$ (Dropping 1 from f_2) (4.8)

$+ (12567).D_{(1567)}\cdot f_3 f_4 f_5 f_6 +$ (,, 4 ,,) (4.9)

$+ (12467).D_{(1467)}\cdot f_3 f_4 f_5 f_6 +$ (,, 5 ,,) (4.10)

$+ (12457).D_{(1457)}\cdot f_3 f_4 f_5 f_6 +$ (,, 6 ,,) (4.11)

$+ (12456).D_{(1456)}\cdot f_3 f_4 f_5 f_6\cdot\}]$ (,, 7 ,,) (4.12)

Using the transformations (45), (56), (67) which keep the product $f_3\cdot f_4\cdot f_5\cdot f_6$ and the operator ξ invariant, we notice that the last four sets above are connected as follows:

Set	(4.9)	(4.10)	(4.11)	(4.12)
Connecting Permutation ..		(45)	(56)	(67)

and so, by Theorem III, are equivalent.

Hence, we obtain,

$$C_7 = \xi.[(14567)\ (24567).D_{(4567)}\cdot f_3 f_4 f_5 f_6 + $$
$$+\ 4.(14567)\ (12567).D_{(1567)}\cdot f_3 f_4 f_5 f_6\cdot] \quad (\gamma)$$

Performing the operations indicated in [ ] in (γ), we obtain:

$$
\begin{array}{lll}
C_7 = \xi \cdot \Big[\Big\{ (14567)\,(24567).(12367)\,(12367)\,(12345)\,(12345) + & & (\gamma_1) \\
\quad ,, \quad ,, \quad . \quad ,, \quad (12356)\,(12347)\,(12345) + & & (\gamma_2) \\
\quad ,, \quad ,, \quad . \quad ,, \quad (12357)\,(12345)\,(12346) + & & (\gamma_3) \\
\quad ,, \quad ,, \quad .(12357)\,(12367)\,(12346)\,(12345) + & & (\gamma_4) \\
\quad ,, \quad ,, \quad . \quad ,, \quad \dfrac{(12357)}{(12356)} \cdot \dfrac{(12346)}{(12347)}\,(12346) + & & (\gamma_5), (\gamma_6) \\
\quad ,, \quad ,, \quad .(12356)\,(12367)\,(12347)\,(12345) + & & (\gamma_7) \\
\quad ,, \quad ,, \quad . \quad ,, \quad \dfrac{(12357)}{(12356)} \cdot \dfrac{(12346)}{(12347)}\,(12347) \Big\} + & & (\gamma_8), (\gamma_9) \\
+\, 4. \Big\{ (14567)\,(12567) \cdot (23567)\,(12346)\,(12347)\,(12345) + & & (\gamma_{10}) \\
\quad ,, \quad ,, \quad . \quad ,, \quad (12347)\,(12345)\,(12346) + & & (\gamma_{11}) \\
\quad ,, \quad ,, \quad .(23467)\,(12367)\,(12345)\,(12345) + & & (\gamma_{12}) \\
\quad ,, \quad ,, \quad . \quad ,, \quad (12356)\,(12347)\,(12345) + & & (\gamma_{13}) \\
\quad ,, \quad ,, \quad . \quad ,, \quad (12357)\,(12345)\,(12346) + & & (\gamma_{14}) \\
\quad ,, \quad ,, \quad .(23457)\,(12367)\,(12346)\,(12345) + & & (\gamma_{15}) \\
\quad ,, \quad ,, \quad . \quad ,, \quad \dfrac{(12357)}{(12356)}\,\dfrac{(12346)}{(12347)}\,(12346) + & & (\gamma_{16}), (\gamma_{17}) \\
\quad ,, \quad ,, \quad .(23456)\,(12367)\,(12347)\,(12345) + & & (\gamma_{18}) \\
\quad ,, \quad ,, \quad . \quad ,, \quad \dfrac{(12357)}{(12356)}\,\dfrac{(12346)}{(12347)}\,(12347). \Big\} \Big] & & (\gamma_{19}), (\gamma_{20})
\end{array}
$$

Notice that the first three factors of each of the terms γ_1–γ_9 are obtained by deleting in order the numbers 3, 1, 4; and the first three factors of each of the terms γ_{10}–γ_{20} are obtained by deleting in order the numbers 3, 4, 1.

The γ's are connected by the following transformations:

$$
\begin{array}{ccccccc}
\gamma_1 & \gamma_5 & \gamma_9, & \gamma_7 = \gamma_2 & & \gamma_4 = \gamma_3 & \gamma_8 = \gamma_6, \\
(56) & (67) & & & (67) & (57) &
\end{array}
$$

Also,

$$
\begin{array}{cccc}
\gamma_{10} = \gamma_{11}, & \gamma_{12} & \gamma_{16} & \gamma_{20}, \\
 & & (56) & (67)
\end{array}
$$

$$
\begin{array}{cccccc}
\gamma_{15} & \gamma_{14} & \gamma_{13} & \gamma_{18} & \gamma_{19} & \gamma_{17}. \\
(56) & (67) & (57) & (56) & (67) &
\end{array}
$$

Thus, under the operation ξ, we have the following equivalences:

$$\gamma_1 \sim \gamma_5 \sim \gamma_9, \ \gamma_2 \sim \gamma_3 \sim \gamma_4 \sim \gamma_6 \sim \gamma_7 \sim \gamma_8,$$

also,

$$\gamma_{10} \sim \gamma_{11}, \ \gamma_{12} \sim \gamma_{16} \sim \gamma_{20}, \ \gamma_{13} \sim \gamma_{14} \sim \gamma_{15} \sim \gamma_{17} \sim \gamma_{18} \sim \gamma_{19},$$

so that

$$C_7 = \xi \ [(3\gamma_1 + 6\gamma_2) + 4.(2\gamma_{10} + 3\gamma_{12} + 6\gamma_{13})].$$

Renaming γ_1, γ_2, γ_{10}, γ_{12}, γ_{13} as x_4, x_5, x_6, x_7, x_8 respectively, we may write this equation in the form

$$C_7 = \xi \ [(3x_4 + 6x_5) + 4\cdot(2x_6 + 3x_7 + 6x_8)]$$

$$= \xi \ [3x_4 + 6x_5 + 8x_6 + 12x_7 + 24x_8] \qquad (4.13)$$

where the explicit expressions for the x_i are:

x_4 .. $(12345)^2.(12367)^2.(14567)$ (24567),

x_5 .. (12345) (12347) (12356) (12367) (14567) (24567),

x_6 .. (12345) (12346) (12347) (12567) (14567) (23567),

x_7 .. $(12345)^2.(12367)$ (12567) (14567) (23467),

x_8 .. (12345) (12347) (12356) (12567) (14567) (23467).

We have reduced C_7 to the evaluation of five terms.

The results of this section may be summarised as follows:

Summary of Results

We have shown that

$$R_7 \doteq A_7 + 5.C_7,$$

where

$$A_7 = \xi.[8x_1 + 12x_2 + 24x_3]$$

$$C_7 = \xi.[3x_4 + 6x_5 + 8x_6 + 12x_7 + 24x_8]$$

$$\xi \ \equiv \delta_2.\delta_3.\delta_4.\delta_5.\delta_6$$

and the x's stand for the following products of symmetric functions:

TABLE I

$\delta_1.P$ *reduced to eight* x's

x_i	Term
x_1	(12345) (12346) (12347).(12567)^2.(34567)
x_2	(12345)^2.(12367) (12467) (12567) (34567)
x_3	(12345) (12347) (12356) (12467) (12567) (34567).
x_4	(12345)^2.(12367)^2.(14567) (24567)
x_5	(12345) (12347) (12356) (12367) (14567) (24567)
x_6	(12345) (12346) (12347) (12567) (14567) (23567)
x_7	(12345)^2.(12367) (12567) (14567) (23467)
x_8	(12345) (12347) (12356) (12567) (14567) (23467).
$\delta_1.P = (8x_1+12x_2+24x_3)+5.(3x_4+6x_5+8x_6+12x_7+24x_8)$	

It is evident that,

Number of terms in $A_7 = 8 + 12 + 24 = 44$,

Number of terms in $C_7 = 3 + 6 + 8 + 12 + 24 = 53$,

Number of terms in $R_7 = 44 + 5\times 53 = 309$

so that the problem has been reduced to the evaluation of eight $\xi.x$'s instead of 309.

It will also be noticed that for any term, say x_1, the digits within each factor as well as the factors appear in ascending order. This procedure has been found convenient for the sake of uniformity and identification, and will be adhered to throughout.

§5. THE SECOND OPERATION $\delta_2 \equiv D_{(124567)}$ APPLIED TO THE EIGHT x's

We shall now give the results of applying the second operator $D_{(124567)}$ to $x_1, x_2, \ldots x_8$ in sub-sections I, II, ... VIII. The terms obtained will be numbered $t_1, t_2, t_3, \cdots$ for each x *afresh*. It is also important to observe that δ_2 generates the *sum* of these t_i's, and the sign + connecting them has been omitted only for the sake of brevity.

I. δ_2 *applied to* x_1

To write down the terms generated by:

$D_{(124567)}$ [(12345) (12346) (12347). (12567). (12567). (34567)].

There are only five ways of deleting 1, and we divide the work into five stages corresponding to these five ways. Consider the first way or stage (i) where, deleting 1 from the first factor, we obtain

(2345).

Now in this stage, there are only four ways of deleting 2, giving the pairs:

(2345) (1346), (2345) (1347),

(2345) (1567), (2345) (1567).

Corresponding to these four ways of deleting 1 and 2, there are the following ten ways of deleting 4, giving the triplets:

(2345) (1346). (1237), (2345) (1346). (3567),

(2345) (1347). (1236), (2345) (1347). (3567),

(2345) (1567). (1236), (2345) (1567). (1237), (2345) (1567). (3567),

(2345) (1567). (1236), (2345) (1567). (1237), (2345) (1567). (3567).

A sufficient description of these triplets is that they have been obtained by deleting, in order, the numbers 1, 2, 4.

It is now easy to delete the remaining numbers 5, 6, 7 for each of these triplets in all possible ways. This gives all the terms in stage (i).

In stage (ii), we delete 1 from the second factor, and proceed as above to form triplets. The remaining stages are also completed in a similar manner, and the sum of all the terms thus generated clearly gives $\delta_2 \cdot x_1$.

This explanation of method may then be summarised in the form: '*Delete, in order, the numbers* 1, 2, 4 *and next* 5, 6, 7 *in all possible ways.*'

We shall use this language again later.

Of course, we could delete 1, 2, 4, 5, 6 and 7 in any manner and order we may choose, and the result would be independent of the method chosen. We have given here the procedure detailed above only for describing the particular order in which the terms t_i of this section have been obtained. In addition, there are no chances of missing a term, and uniformity is maintained.

A similar explanation holds for δ_2 applied to the other x's—except x_3 where we delete, in order, the numbers 7, 6, 5 and next 1, 2, 4 in all possible ways.

Evidently, a square term will give a common multiplier 2 !. Hence $\delta_2 \cdot x_1$ generates the *sum* of the following terms:

2×(i)

(2345) (1346).(1237) (1257) (1256) (3467)	t_1
,, ,, . ,, (1267) (1256) (3457)	t_2
,, ,, . ,, (1267) (1257) (3456)	t_3
,, ,, .(3567) (1234) (1267) (1257)	t_4
(2345) (1347).(1236) (1257) (1256) (3467)	t_5
,, ,, . ,, (1267) (1256) (3457)	t_6
,, ,, . ,, (1267) (1257) (3456)	t_7
,, ,, .(3567) (1234) (1267) (1256)	t_8
(2345) (1567).(1236) (1234) $\frac{(1267)(3457)}{(1257)(3467)}$	t_9, t_{10}
,, ,, .(1237) (1234) $\frac{(1267)(3456)}{(1256)(3467)}$	t_{11}, t_{12}
,, ,, .(3567) (1234) (1234) (1267)	t_{13}

2×(ii)

(2346) (1345).(1237) (1257) (1256) (3467)	t_{14}
,, ,, . ,, (1267) (1256) (3457)	t_{15}
,, ,, . ,, (1267) (1257) (3456)	t_{16}
,, ,, .(3567) (1234) (1267) (1257)	t_{17}
(2346) (1347).(1235) (1257) (1256) (3467)	t_{18}
,, ,, . ,, (1267) (1256) (3457)	t_{19}
,, ,, . ,, (1267) (1257) (3456)	t_{20}
,, ,, .(3567) (1234) (1257) (1256)	t_{21}
(2346) (1567).(1235) (1234) $\frac{(1267)(3457)}{(1257)(3467)}$	t_{22}, t_{23}
,, ,, .(1237) (1234) $\frac{(1257)(3456)}{(1256)(3457)}$	t_{24}, t_{25}
,, ,, .(3567) (1234) (1234) (1257)	t_{26}

$\delta_1 \cdot x_2$—(*Contd.*)

2×(iii)

(2347) (1345).(1236) (1257) (1256) (3467)	t_{27}
,, ,, . ,, (1267) (1256) (3457)	t_{28}
,, ,, . ,, (1267) (1257) (3456)	t_{29}
,, ,, .(3567) (1234) (1267) (1256)	t_{30}
(2347) (1346).(1235) (1257) (1256) (3467)	t_{31}
,, ,, . ,, (1267) (1256) (3457)	t_{32}
,, ,, . ,, (1267) (1257) (3456)	t_{33}
,, ,, .(3567) (1234) (1257) (1256)	t_{34}
(2347) (1567).(1235) (1234) $\frac{(1267)}{(1256)}\frac{(3456)}{(3467)}$	t_{35}, t_{36}
,, ,, .(1236) (1234) $\frac{(1257)}{(1256)}\frac{(3456)}{(3457)}$	t_{37}, t_{38}
,, ,, .(3567) (1234) (1234) (1256)	t_{39}

2×(iv)

(2567) (1345).(1236) (1234) $\frac{(1267)}{(1257)}\frac{(3457)}{(3467)}$	t_{40}, t_{41}
,, ,, .(1237) (1234) $\frac{(1267)}{(1256)}\frac{(3456)}{(3467)}$	t_{42}, t_{43}
,, ,, .(3567) (1234) (1234) (1267)	t_{44}
(2567) (1346).(1235) (1234) $\frac{(1267)}{(1257)}\frac{(3457)}{(3467)}$	t_{45}, t_{46}
,, ,, .(1237) (1234) $\frac{(1257)}{(1256)}\frac{(3456)}{(3457)}$	t_{47}, t_{48}
,, ,, .(3567) (1234) (1234) (1257)	t_{49}
(2567) (1347).(1235) (1234) $\frac{(1267)}{(1256)}\frac{(3456)}{(3467)}$	t_{50}, t_{51}
,, ,, .(1236) (1234) $\frac{(1267)}{(1256)}\frac{(3456)}{(3457)}$	t_{52}, t_{53}
,, ,, .(3567) (1234) (1234) (1256)	t_{54}
(2567) (1567).(1235) (1234) (1234) (3467)	t_{55}
,, ,, .(1236) (1234) (1234) (3457)	t_{56}
,, ,, .(1237) (1234) (1234) (3456)	t_{57}
,, ,, .(3567) (1234) (1234) (1234)	t_{58}

Hence

$$\delta_2 . x_1 = 2.(t_1 + t_2 + \ldots + t_{58}).$$

II. δ_2 *applied to* x_2

To write down the terms generated by

$D_{(124567)}$ [(12345).(12345).(12367) (12467) (12567) (34567)]

We delete, in order, the numbers 1, 2, 4 and next 5, 6, 7 in all possible ways. Hence $\delta_2 . x_2$ generates the *sum* of the following terms:

$2\times$(i)

$$(2345)\ (1345).(1267)\ \frac{(1237)}{(1236)}\ (1267)\ \frac{(3456)}{(3457)} \qquad t_1,\ t_2$$

$$\text{,,} \quad \text{,,} \ . \ \text{,,} \quad \frac{(1237)}{(1236)}\frac{(1256)}{(1257)}\ (3467) \qquad t_3,\ t_4$$

$$\text{,,} \quad \text{,,} \ .(3567)\ \frac{(1237)}{(1236)}\frac{(1246)}{(1247)}\ (1267) \qquad t_5,\ t_6$$

$$(2345)\ (1367).(1235)\ \frac{(1247)}{(1246)}\ (1267)\ \frac{(3456)}{(3457)} \qquad t_7,\ t_8$$

$$\text{,,} \quad \text{,,} \ . \ \text{,,} \quad \frac{(1247)}{(1246)}\frac{(1256)}{(1257)}\ (3467) \qquad t_9,\ t_{10}$$

$$\text{,,} \quad \text{,,} \ .(1267)\ (1234)\ \frac{(1257)}{(1256)}\frac{(3456)}{(3457)} \qquad t_{11},\ t_{12}$$

$$\text{,,} \quad \text{,,} \ .(3567)\ (1234)\ \frac{(1247)}{(1246)}\frac{(1256)}{(1257)} \qquad t_{13},\ t_{14}$$

$$(2345)\ (1467).(1235)\ \frac{(1237)}{(1236)}\ (1267)\ \frac{(3456)}{(3457)} \qquad t_{15},\ t_{16}$$

$$\text{,,} \quad \text{,,} \ . \ \text{,,} \quad \frac{(1237)}{(1236)}\frac{(1256)}{(1257)}\ (3467) \qquad t_{17},\ t_{18}$$

$$\text{,,} \quad \text{,,} \ .(3567)\ (1234)\ \frac{(1237)}{(1236)}\frac{(1256)}{(1257)} \qquad t_{19},\ t_{20}$$

$$(2345)\ (1567).(1235)\ \frac{(1237)}{(1236)}\frac{(1246)}{(1247)}\ (3467) \qquad t_{21},\ t_{22}$$

$$\text{,,} \quad \text{,,} \ .(1267)\ (1234)\ \frac{(1237)}{(1236)}\frac{(3456)}{(3457)} \qquad t_{23},\ t_{21}$$

$$\text{,,} \quad \text{,,} \ .(3567)\ (1234)\ \frac{(1237)}{(1236)}\frac{(1246)}{(1247)} \qquad t_{25},\ t_{26}$$

$\delta_2 \cdot x_2$—(*Contd.*)

$2\times$(ii)

$(2367)\,(1345).(1235)\,\frac{(1247)}{(1246)}\,(1267)\,\frac{(3456)}{(3457)}$ t_{27}, t_{28}

$,, \quad ,, \quad . \quad ,, \quad \frac{(1247)}{(1246)}\frac{(1256)}{(1257)}\,(3467)$ t_{29}, t_{30}

$,, \quad ,, \quad .(1267)\,(1234)\,\frac{(1257)}{(1256)}\frac{(3456)}{(3457)}$ t_{31}, t_{32}

$,, \quad ,, \quad .(3567)\,(1234)\,\frac{(1247)}{(1246)}\frac{(1256)}{(1257)}$ t_{33}, t_{34}

$(2367)\,(1467).(1235)\,(1234)\,\frac{(1257)}{(1256)}\frac{(3456)}{(3457)}$ t_{35}, t_{36}

$(2367)\,(1567).(1235)\,(1234)\,\frac{(1247)}{(1246)}\frac{(3456)}{(3457)}$ t_{37}, t_{38}

$2\times$(iii)

$(2467)\,(1345).(1235)\,\frac{(1237)}{(1236)}\,(1267)\,\frac{(3456)}{(3457)}$ t_{39}, t_{40}

$,, \quad ,, \quad . \quad ,, \quad \frac{(1237)}{(1236)}\frac{(1256)}{(1257)}\,(3467)$ t_{41}, t_{42}

$,, \quad ,, \quad .(3567)\,(1234)\,\frac{(1237)}{(1236)}\frac{(1256)}{(1257)}$ t_{43}, t_{44}

$(2467)\,(1367).(1235)\,(1234).\frac{(1257)}{(1256)}\frac{(3456)}{(3457)}$ t_{45}, t_{46}

$(2467)\,(1567).(1235)\,(1234)\,\frac{(1237)}{(1236)}\frac{(3456)}{(3457)}$ t_{47}, t_{48}

$2\times$(iv)

$(2567)\,(1345).(1235)\,\frac{(1237)}{(1236)}\frac{(1246)}{(1247)}\,(3467)$ t_{49}, t_{50}

$,, \quad ,, \quad .(1267)\,(1234)\,\frac{(1237)}{(1236)}\frac{(3456)}{(3457)}$ t_{51}, t_{52}

$,, \quad ,, \quad .(3567)\,(1234)\,\frac{(1237)}{(1236)}\frac{(1246)}{(1247)}$ t_{53}, t_{54}

$(2567)\,(1367).(1235)\,(1234)\,\frac{(1247)}{(1246)}\frac{(3456)}{(3457)}$ t_{55}, t_{56}

$(2567)\,(1467).(1235)\,(1234)\,\frac{(1237)}{(1236)}\frac{(3456)}{(3457)}$ t_{57}, t_{58}

Hence

$$\delta_2 . x_2 = 2.(t_1 + t_2 + \ldots + t_{58}).$$

III. δ_2 *applied to* x_3

To write down the terms generated by

$$D_{(124567)}\,[(12345)\,(12347)\,(12356)\,(12467)\,(12567)\,(34567)].$$

To facilitate identification of equivalent terms, this differential operation is carried out in a slightly different order. Here, we delete in order the numbers 7, 6, 5 and next 1, 2, 4 in all posible ways. Hence $\delta_2 . x_3$ generates the *sum* of the following terms:

(i)

$$(1234)\,(1235).(1234)\,\frac{(2467)}{(1467)}\,\frac{(1567)}{(2567)}\,(3567) \qquad t_1,\ t_2$$

$$\text{,,} \quad \text{,,} \quad .(1267)\,\frac{(2345)}{(1345)}\,\frac{(1467)}{(2467)}\,(3567) \qquad t_3,\ t_4$$

$$\text{,,} \quad \text{,,} \quad .(3467)\,(1235)\,\frac{(2467)}{(1467)}\,\frac{(1567)}{(2567)} \qquad t_5,\ t_6$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad \frac{(2345)}{(1345)}\,(1267)\,\frac{(1567)}{(2567)} \qquad t_7,\ t_8$$

$$(1234)\,(1247).(1234)\,\frac{(2356)}{(1356)}\,\frac{(1567)}{(2567)}\,(3567) \qquad t_9,\ t_{10}$$

$$\text{,,} \quad \text{,,} \quad .(1236)\,\frac{(2345)}{(1345)}\,\frac{(1567)}{(2567)}\,(3567) \qquad t_{11},\ t_{12}$$

$$\text{,,} \quad \text{,,} \quad .(1267)\,\frac{(2345)}{(1345)}\,\frac{(1356)}{(2356)}\,(3567) \qquad t_{13},\ t_{14}$$

$$\text{,,} \quad \text{,,} \quad .(3467)\,(1235)\,\frac{(2356)}{(1356)}\,\frac{(1567)}{(2567)} \qquad t_{15},\ t_{16}$$

$$(1234)\,(1257).(1234)\,\frac{(2356)}{(1356)}\,\frac{(1467)}{(2467)}\,(3567) \qquad t_{17},\ t_{18}$$

$$\text{,,} \quad \text{,,} \quad .(1236)\,\frac{(2345)}{(1345)}\,\frac{(1467)}{(2467)}\,(3567) \qquad t_{19},\ t_{20}$$

$$\text{,,} \quad \text{,,} \quad .(3467)\,(1235)\,\frac{(2356)}{(1356)}\,\frac{(1467)}{(2467)} \qquad t_{21},\ t_{22}$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad \frac{(2345)}{(1345)}\,\frac{(1356)}{(2356)}\,(1267) \qquad t_{23},\ t_{24}$$

$\delta_2 \cdot x_3$—(*Contd.*)

$$(1234)\,(3457).(1234)\,\frac{(2356)}{(1356)}\,(1267)\,\frac{(1567)}{(2567)} \qquad t_{25},\ t_{26}$$

$$\text{,,} \quad \text{,,} \quad .(1236)\,(1235)\,\frac{(2467)}{(1467)}\,\frac{(1567)}{(2567)} \qquad t_{27},\ t_{28}$$

$$\text{,,} \quad \text{,,} \quad . \text{ ,,} \quad \frac{(2345)}{(1345)}\,(1267)\,\frac{(1567)}{(2567)} \qquad t_{29},\ t_{30}$$

$$\text{,,} \quad \text{,,} \quad .(1267)\,(1235)\,\frac{(2356)}{(1356)}\,\frac{(1467)}{(2467)} \qquad t_{31},\ t_{32}$$

$$\text{,,} \quad \text{,,} \quad . \text{ ,,} \quad \frac{(2345)}{(1345)}\,\frac{(1356)}{(2356)}\,(1267) \qquad t_{33},\ t_{34}$$

(ii)

$$(1246)\,(1235).(1234)\,\frac{(2347)}{(1347)}\,\frac{(1567)}{(2567)}\,(3567) \qquad t_{35},\ t_{36}$$

$$\text{,,} \quad \text{,,} \quad .(1267)\,\frac{(2345)}{(1345)}\,\frac{(1347)}{(2347)}\,(3567) \qquad t_{37},\ t_{38}$$

$$\text{,,} \quad \text{,,} \quad .(3467)\,(1235)\,\frac{(2347)}{(1347)}\,\frac{(1567)}{(2567)} \qquad t_{39},\ t_{40}$$

$$\text{,,} \quad \text{,,} \quad . \text{ ,,} \quad \frac{(2345)}{(1345)}\,(1237)\,\frac{(1567)}{(2567)} \qquad t_{41},\ t_{42}$$

$$(1246)\,(1257).(1234)\,\frac{(2347)}{(1347)}\,\frac{(1356)}{(2356)}\,(3567) \qquad t_{43},\ t_{44}$$

$$\text{,,} \quad \text{,,} \quad .(1236)\,\frac{(2345)}{(1345)}\,\frac{(1347)}{(2347)}\,(3567) \qquad t_{45},\ t_{46}$$

$$\text{,,} \quad \text{,,} \quad .(3467)\,(1235)\,\frac{(2347)}{(1347)}\,\frac{(1356)}{(2356)} \qquad t_{47},\ t_{48}$$

$$\text{,,} \quad \text{,,} \quad . \text{ ,,} \quad \frac{(2345)}{(1345)}\,(1237)\,\frac{(1356)}{(2356)} \qquad t_{49},\ t_{50}$$

$$(1246)\,(3457).(1234)\,(1237)\,\frac{(2356)}{(1356)}\,\frac{(1567)}{(2567)} \qquad t_{51},\ t_{52}$$

$$\text{,,} \quad \text{,,} \quad .(1236)\,(1235)\,\frac{(2347)}{(1347)}\,\frac{(1567)}{(2567)} \qquad t_{53},\ t_{54}$$

$$\text{,,} \quad \text{,,} \quad . \text{ ,,} \quad \frac{(2345)}{(1345)}\,(1237)\,\frac{(1567)}{(2567)} \qquad t_{55},\ t_{56}$$

$$\text{,,} \quad \text{,,} \quad .(1267)\,(1235)\,\frac{(2347)}{(1347)}\,\frac{(1356)}{(2356)} \qquad t_{57},\ t_{58}$$

$$\text{,,} \quad \text{,,} \quad . \text{ ,,} \quad \frac{(2345)}{(1345)}\,(1237)\,\frac{(1356)}{(2356)} \qquad t_{59},\ t_{60}$$

(iii)

$(1256)\ (1235).(1234)\ \frac{(2347)}{(1347)}\frac{(1467)}{(2467)}\ (3567)$ t_{61}, t_{62}

,, ,, $.(3467)\ (1235)\ \frac{(2347)}{(1347)}\frac{(1467)}{(2467)}$ t_{63}, t_{64}

,, ,, . ,, $\frac{(2345)}{(1345)}\ (1237)\ \frac{(1467)}{(2467)}$ t_{65}, t_{66}

,, ,, . ,, $\frac{(2345)}{(1345)}\frac{(1347)}{(2347)}\ (1267)$ t_{67}, t_{68}

$(1256)\ (1247).(1234)\ \frac{(2347)}{(1347)}\frac{(1356)}{(2356)}\ (3567)$ t_{69}, t_{70}

,, ,, $.(1236)\ \frac{(2345)}{(1345)}\frac{(1347)}{(2347)}\ (3567)$ t_{71}, t_{72}

,, ,, $.(3467)\ (1235)\ \frac{(2347)}{(1347)}\frac{(1356)}{(2356)}$ t_{73}, t_{74}

,, ,, . ,, $\frac{(2345)}{(1345)}\ (1237)\ \frac{(1356)}{(2356)}$ t_{75}, t_{76}

$(1256)\ (3457).(1234)\ (1237)\ \frac{(2356)}{(1356)}\frac{(1467)}{(2467)}$ t_{77}, t_{78}

,, ,, . ,, $\frac{(2347)}{(1347)}\frac{(1356)}{(2356)}\ (1267)$ t_{79}, t_{80}

,, ,, $.(1236)\ (1235)\ \frac{(2347)}{(1347)}\frac{(1467)}{(2467)}$ t_{81}, t_{82}

,, ,, . ,, $\frac{(2345)}{(1345)}\ (1237)\ \frac{(1467)}{(2467)}$ t_{83}, t_{84}

,, ,, . ,, $\frac{(2345)}{(1345)}\frac{(1347)}{(2347)}\ (1267)$ t_{85}, t_{86}

(iv)

$(3456)\ (1235).(1234)\ (1237)\ \frac{(2467)}{(1467)}\frac{(1567)}{(2567)}$ t_{87}, t_{88}

,, ,, . ,, $\frac{(2347)}{(1347)}\ (1267)\ \frac{(1567)}{(2567)}$ t_{89}, t_{90}

,, ,, $.(1267)\ (1235)\ \frac{(2347)}{(1347)}\frac{(1467)}{(2467)}$ t_{91}, t_{92}

,, ,, . ,, $\frac{(2345)}{(1345)}\ (1237)\ \frac{(1467)}{(2467)}$ t_{93}, t_{94}

,, ,, . ,, $\frac{(2345)}{(1345)}\frac{(1347)}{(2347)}\ (1267)$ t_{95}, t_{96}

$\delta_2 \cdot x_3$—(*Contd.*)

$$(3456)\,(1247).(1234)\,(1237)\,\frac{(2356)}{(1356)}\,\frac{(1567)}{(2567)} \qquad t_{97},\ t_{98}$$

$$,, \quad ,, \quad .(1236)\,(1235)\,\frac{(2347)}{(1347)}\,\frac{(1567)}{(2567)} \qquad t_{99},\ t_{100}$$

$$,, \quad ,, \quad . \quad ,, \quad \frac{(2345)}{(1345)}\,(1237)\,\frac{(1567)}{(2567)} \qquad t_{101},\ t_{102}$$

$$,, \quad ,, \quad .(1267)\,(1235)\,\frac{(2347)}{(1347)}\,\frac{(1356)}{(2356)} \qquad t_{103},\ t_{104}$$

$$,, \quad ,, \quad . \quad ,, \quad \frac{(2345)}{(1345)}\,(1237)\,\frac{(1356)}{(2356)} \qquad t_{105},\ t_{106}$$

$$(3456)\,(1257).(1234)\,(1237)\,\frac{(2356)}{(1356)}\,\frac{(1467)}{(2467)} \qquad t_{107},\ t_{108}$$

$$,, \quad ,, \quad . \quad ,, \quad \frac{(2347)}{(1347)}\,\frac{(1356)}{(2356)}\,(1267) \qquad t_{109},\ t_{110}$$

$$,, \quad ,, \quad .(1236)\,(1235)\,\frac{(2347)}{(1347)}\,\frac{(1467)}{(2467)} \qquad t_{111},\ t_{112}$$

$$,, \quad ,, \quad . \quad ,, \quad \frac{(2345)}{(1345)}\,(1237)\,\frac{(1467)}{(2467)} \qquad t_{113},\ t_{114}$$

$$,, \quad ,, \quad . \quad ,, \quad \frac{(2345)}{(1345)}\,\frac{(1347)}{(2347)}\,(1267) \qquad t_{115},\ t_{116}$$

Hence

$$\delta_2 . x_3 = (t_1 + t_2 + \ldots + t_{116}).$$

IV. δ_2 *applied to* x_4

To write down the terms generated by

$$D_{(1245667)}\,[(12345).(12345).(12367).(12367).(14567).(24567)].$$

We delete, in order, the numbers 1, 2, 4 and next 5, 6, 7 in all possible ways. Hence $\delta_2 . x_4$ generates the *sum* of the following terms:

$4 \times$ (i)

$$(2345)\,(1345).(1567)\,(1237)\,(1236)\,(2467) \qquad t_1$$

$$,, \quad ,, \quad .(2567)\,(1237)\,(1236)\,(1467) \qquad t_2$$

$$(2345)\,(1367).(1235)\,\frac{(1237)}{(1236)}\,(1467)\,\frac{(2456)}{(2457)} \qquad t_3,\ t_4$$

$$,, \quad ,, \quad . \quad ,, \quad \frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)}\,(2467) \qquad t_5,\ t_6$$

$$(2345)\,(1367)\,.\,(1567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(2456)}{(2457)} \qquad t_7,\ t_8$$

$$\text{,,} \qquad \text{,,} \quad .\,(2567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)} \qquad t_9,\ t_{10}$$

$$(2345)\,(4567)\,.\,(1235)\,(1237)\,(1236)\,(1467) \qquad t_{11}$$

$$\text{,,} \qquad \text{,,} \quad .\,(1567)\,(1234)\,(1237)\,(1236) \qquad t_{12}$$

$4\times$(ii)

$$(2367)\,(1345)\,.\,(1235)\,\frac{(1237)}{(1236)}\,(1467)\,\frac{(2456)}{(2457)} \qquad t_{13},\ t_{14}$$

$$\text{,,} \qquad \text{,,} \quad . \quad \text{,,} \quad \frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)}\,(2467) \qquad t_{15},\ t_{16}$$

$$\text{,,} \qquad \text{,,} \quad .\,(1567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(2456)}{(2457)} \qquad t_{17},\ t_{18}$$

$$\text{,,} \qquad \text{,,} \quad .\,(2567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)} \qquad t_{19},\ t_{20}$$

$$(2367)\,(1367)\,.\,(1235)\,(1234)\,\frac{(1457)}{(1456)}\,\frac{(2456)}{(2457)} \qquad t_{21},\ t_{22}$$

$$(2367)\,(4567)\,.\,(1235)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)} \qquad t_{23},\ t_{24}$$

$4\times$(iii)

$$(4567)\,(1345)\,.\,(1235)\,(1237)\,(1236)\,(2467) \qquad t_{25}$$

$$\text{,,} \qquad \text{,,} \quad .\,(2567)\,(1234)\,(1237)\,(1236) \qquad t_{26}$$

$$(4567)\,(1367)\,.\,(1235)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(2456)}{(2457)} \qquad t_{27},\ t_{28}$$

$$(4567)\,(4567)\,.\,(1235)\,(1234)\,(1237)\,(1236) \qquad t_{29}$$

Hence

$$\delta_2 . x_4 = 4 . (t_1 + t_2 + \ldots + t_{29}).$$

V. δ_2 *applied to* x_5

To write down the terms generated by

$$D_{(124567)}\ [(12345)\,(12347)\,(12356)\,(12367)\,(14567)\,(24567)].$$

We delete, in order, the numbers 1, 2, 4 and next 5, 6, 7 in all possible ways. Hence $\delta_2 . x_5$ generates the *sum* of the following terms:

$\delta_2 \cdot x_5$—(*Contd.*)

(i)

$(2345)\ (1347).(1567)\ (1236)\ \frac{(1237)}{(1236)}\ \frac{(2456)}{(2457)}$	$t_1,\ t_2$
,, ,, . ,, $(1235)\ (1236)\ (2467)$	t_3
,, ,, $.(2567)\ (1236)\ \frac{(1237)}{(1236)}\ \frac{(1456)}{(1457)}$	$t_4,\ t_5$
,, ,, . ,, $(1235)\ (1236)\ (1467)$	t_6
$(2345)\ (1356).(1237)\ \frac{(1237)}{(1236)}\ (1467)\ \frac{(2456)}{(2457)}$	$t_7,\ t_8$
,, ,, . ,, $\frac{(1237)}{(1236)}\ \frac{(1456)}{(1457)}\ (2467)$	$t_9,\ t_{10}$
,, ,, $.(1567)\ (1234)\ (1237)\ (2467)$	t_{11}
,, ,, $.(2567)\ (1234)\ (1237)\ (1467)$	t_{12}
$(2345)\ (1367).(1237)\ (1236)\ \frac{(1457)}{(1456)}\ \frac{(2456)}{(2457)}$	$t_{13},\ t_{14}$
,, ,, . ,, $(1235)\ (1467)\ (2456)$	t_{15}
,, ,, . ,, $(1235)\ (1456)\ (2467)$	t_{16}
,, ,, $.(1567)\ (1234)\ \frac{(1236)}{(1235)}\ \frac{(2457)}{(2467)}$	$t_{17},\ t_{18}$
,, ,, $.(2567)\ (1234)\ \frac{(1236)}{(1235)}\ \frac{(1457)}{(1467)}$	$t_{19},\ t_{20}$
$(2345)\ (4567).(1237)\ (1236)\ \frac{(1237)}{(1236)}\ \frac{(1456)}{(1457)}$	$t_{21},\ t_{22}$
,, ,, . ,, $(1235)\ (1236)\ (1467)$	t_{23}
,, ,, $.(1567)\ (1234)\ (1236)\ (1237)$	t_{24}

(ii)

$(2347)\ (1345).(1567)\ (1236)\ \frac{(1237)}{(1236)}\ \frac{(2456)}{(2457)}$	$t_{25},\ t_{26}$
,, ,, . ,, $(1235)\ (1236)\ (2467)$	t_{27}
,, ,, $.(2567)\ (1236)\ \frac{(1237)}{(1236)}\ \frac{(1456)}{(1457)}$	$t_{28},\ t_{29}$
,, ,, . ,, $(1235)\ (1236)\ (1467)$	t_{30}

$$(2347)\,(1356).(1235)\,\frac{(1237)}{(1236)}\,(1467)\,\frac{(2456)}{(2457)} \qquad t_{31},\, t_{32}$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad \frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)}\,(2467) \qquad t_{33},\, t_{34}$$

$$\text{,,} \quad \text{,,} \quad .(1567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(2456)}{(2457)} \qquad t_{35},\, t_{36}$$

$$\text{,,} \quad \text{,,} \quad .(2567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)} \qquad t_{37},\, t_{38}$$

$$(2347)\,(1367).(1235)\,(1236)\,\frac{(1457)}{(1456)}\,\frac{(2456)}{(2457)} \qquad t_{39},\, t_{40}$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad (1235)\,(1467)\,(2456) \qquad t_{41}$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad (1235)\,(1456)\,(2467) \qquad t_{42}$$

$$\text{,,} \quad \text{,,} \quad .(1567)\,(1234)\,(1235)\,(2456) \qquad t_{43}$$

$$\text{,,} \quad \text{,,} \quad .(2567)\,(1234)\,(1235)\,(1456) \qquad t_{44}$$

$$(2347)\,(4567).(1235)\,(1236)\,\frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)} \qquad t_{45},\, t_{46}$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad (1235)\,(1236)\,(1467) \qquad t_{47}$$

$$\text{,,} \quad \text{,,} \quad .(1567)\,(1234)\,(1235)\,(1236) \qquad t_{48}$$

(iii)

$$(2356)\,(1345).(1237)\,\frac{(1237)}{(1236)}\,(1467)\,\frac{(2456)}{(2457)} \qquad t_{49},\, t_{50}$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad \frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)}\,(2467) \qquad t_{51},\, t_{52}$$

$$\text{,,} \quad \text{,,} \quad .(1567)\,(1234)\,(1237)\,(2467) \qquad t_{53}$$

$$\text{,,} \quad \text{,,} \quad .(2567)\,(1234)\,(1237)\,(1467) \qquad t_{54}$$

$$(2356)\,(1347).(1235)\,\frac{(1237)}{(1236)}\,(1467)\,\frac{(2456)}{(2457)} \qquad t_{55},\, t_{56}$$

$$\text{,,} \quad \text{,,} \quad . \quad \text{,,} \quad \frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)}\,(2467) \qquad t_{57},\, t_{58}$$

$$\text{,,} \quad \text{,,} \quad .(1567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(2456)}{(2457)} \qquad t_{59},\, t_{60}$$

$$\text{,,} \quad \text{,,} \quad .(2567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)} \qquad t_{61},\, t_{62}$$

$\delta_2 \cdot x_5$—(*Contd.*)

$(2356)\ (1367).(1235)\ (1234)\ \dfrac{(1467)}{(1457)}\ \dfrac{(2457)}{(2467)}$	t_{63}, t_{64}
,, ,, $.(1237)\ (1234)\ \dfrac{(1457)}{(1456)}\ \dfrac{(2456)}{(2457)}$	t_{65}, t_{66}
,, ,, $.(1567)\ (1234)\ (1234)\ (2457)$	t_{67}
,, ,, $.(2567)\ (1234)\ (1234)\ (1457)$	t_{68}
$(2356)\ (4567).(1235)\ (1234)\ (1237)\ (1467)$	t_{69}
,, ,, $.(1237)\ (1234)\ \dfrac{(1237)}{(1236)}\ \dfrac{(1456)}{(1457)}$	t_{70}, t_{71}
,, ,, $.(1567)\ (1234)\ (1234)\ (1237)$	t_{72}

(iv)

$(2367)\ (1345).(1237)\ (1236)\ \dfrac{(1457)}{(1456)}\ \dfrac{(2456)}{(2457)}$	t_{73}, t_{74}
,, ,, . ,, $(1235)\ (1467)\ (2456)$	t_{75}
,, ,, . ,, $(1235)\ (1456)\ (2467)$	t_{76}
,, ,, $.(1567)\ (1234)\ \dfrac{(1236)}{(1235)}\ \dfrac{(2457)}{(2467)}$	t_{77}, t_{78}
,, ,, $.(2567)\ (1234)\ \dfrac{(1236)}{(1235)}\ \dfrac{(1457)}{(1467)}$	t_{79}, t_{80}
$(2367)\ (1347).(1235)\ (1236)\ \dfrac{(1457)}{(1456)}\ \dfrac{(2456)}{(2457)}$	t_{81}, t_{82}
,, ,, . ,, $(1235)\ (1467)\ (2456)$	t_{83}
,, ,, . ,, $(1235)\ (1456)\ (2467)$	t_{84}
,, ,, $.(1567)\ (1234)\ (1235)\ (2456)$	t_{85}
,, ,, $.(2567)\ (1234)\ (1235)\ (1456)$	t_{86}
$(2367)\ (1356).(1235)\ (1234)\ \dfrac{(1467)}{(1457)}\ \dfrac{(2457)}{(2467)}$	t_{87}, t_{88}
,, ,, $.(1237)\ (1234)\ \dfrac{(1457)}{(1456)}\ \dfrac{(2456)}{(2457)}$	t_{89}, t_{90}
,, ,, $.(1567)\ (1234)\ (1234)\ (2457)$	t_{91}
,, ,, $.(2567)\ (1234)\ (1234)\ (1457)$	t_{92}
$(2367)\ (4567).(1235)\ (1234)\ \dfrac{(1236)}{(1235)}\ \dfrac{(1457)}{(1467)}$	t_{93}, t_{94}
,, ,, $.(1237)\ (1234)\ (1235)\ (1456)$	t_{95}
,, ,, $.(1567)\ (1234)\ (1234)\ (1235)$	t_{96}

(v)

$(4567)\,(1345).(1237)\,(1236)\,\frac{(1237)}{(1236)}\,\frac{(2456)}{(2457)}$ $t_{97},\ t_{98}$

,, ,, . ,, $(1235)\,(1236)\,(2467)$ t_{99}

,, ,, $.(2567)\,(1234)\,(1236)\,(1237)$ t_{100}

$(4567)\,(1347).(1235)\,(1236)\,\frac{(1237)}{(1236)}\,\frac{(2456)}{(2457)}$ $t_{101},\ t_{102}$

,, ,, . ,, $(1235)\,(1236)\,(2467)$ t_{103}

,, ,, $.(2567)\,(1234)\,(1235)\,(1236)$ t_{104}

$(4567)\,(1356).(1235)\,(1234)\,(1237)\,(2467)$ t_{105}

,, ,, $.(1237)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(2456)}{(2457)}$ $t_{106},\ t_{107}$

,, ,, $.(2567)\,(1234)\,(1234)\,(1237)$ t_{108}

$(4567)\,(1367).(1235)\,(1234)\,\frac{(1236)}{(1235)}\,\frac{(2457)}{(2467)}$ $t_{109},\ t_{110}$

,, ,, $.(1237)\,(1234)\,(1235)\,(2456)$ t_{111}

,, ,, $.(2567)\,(1234)\,(1234)\,(1235)$ t_{112}

$(4567)\,(4567).(1235)\,(1234)\,(1236)\,(1237)$ t_{113}

,, ,, $.(1237)\,(1234)\,(1235)\,(1236)$ t_{114}

Hence

$$\delta_2 . x_5 = (t_1 + t_2 + \ldots + t_{114}).$$

VI. δ_2 *applied to* x_6

To write down the terms generated by

$$D_{(124567)}\,[(12345)\,(12346)\,(12347)\,(12567)\,(14567)\,(23567)].$$

We delete, in order, the numbers 1, 2, 4 and next 5, 6, 7 in all possible ways. Hence $\delta_2 . x_6$ generates the *sum* of the following terms:

(i)

$(2345)\,(1346).(1237)\,(1267)\,\frac{(1457)}{(1456)}\,\frac{(2356)}{(2357)}$ $t_1,\ t_2$

,, ,, . ,, $\frac{(1257)}{(1256)}\,(1467)\,\frac{(2356)}{(2357)}$ $t_3,\ t_4$

,, ,, . ,, $\frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)}\,(2367)$ $t_5,\ t_6$

,, ,, $.(1567)\,(1234)\,\frac{(1267)}{(1257)}\,\frac{(2357)}{(2367)}$ $t_7,\ t_8$

$\delta_2 \cdot x_6$—(*Contd.*)

$(2345)\ (1347).(1236)\ (1267)\ \frac{(1457)}{(1456)}\ \frac{(2356)}{(2357)}$	$t_9,\ t_{10}$
,, ,, . ,, $\frac{(1257)}{(1256)}\ (1467)\ \frac{(2356)}{(2357)}$	$t_{11},\ t_{12}$
,, ,, . ,, $\frac{(1257)}{(1256)}\ \frac{(1456)}{(1457)}\ (2367)$	$t_{13},\ t_{14}$
,, ,, $.(1567)\ (1234)\ \frac{(1267)}{(1256)}\ \frac{(2356)}{(2367)}$	$t_{15},\ t_{16}$
$(2345)\ (1567).(1236)\ (1234)\ \frac{(1467)}{(1457)}\ \frac{(2357)}{(2367)}$	$t_{17},\ t_{18}$
,, ,, $.(1237)\ (1234)\ \frac{(1467)}{(1456)}\ \frac{(2356)}{(2367)}$	$t_{19},\ t_{20}$
,, ,, $.(1567)\ (1234)\ (1234)\ (2367)$	t_{21}
$(2345)\ (3567).(1236)\ (1234)\ \frac{(1267)}{(1257)}\ \frac{(1457)}{(1467)}$	$t_{22},\ t_{23}$
,, ,, $.(1237)\ (1234)\ \frac{(1267)}{(1256)}\ \frac{(1456)}{(1467)}$	$t_{24},\ t_{25}$
,, ,, $.(1567)\ (1234)\ (1234)\ (1267)$	t_{26}

(ii)

$(2346)\ (1345).(1237)\ (1267)\ \frac{(1457)}{(1456)}\ \frac{(2356)}{(2357)}$	$t_{27},\ t_{28}$
,, ,, . ,, $\frac{(1257)}{(1256)}\ (1467)\ \frac{(2356)}{(2357)}$	$t_{29},\ t_{30}$
,, ,, . ,, $\frac{(1257)}{(1256)}\ \frac{(1456)}{(1457)}\ (2367)$	$t_{31},\ t_{32}$
,, ,, $.(1567)\ (1234)\ \frac{(1267)}{(1257)}\ \frac{(2357)}{(2367)}$	$t_{33},\ t_{34}$
$(2346)\ (1347).(1235)\ (1267)\ \frac{(1457)}{(1456)}\ \frac{(2356)}{(2357)}$	$t_{35},\ t_{36}$
,, ,, . ,, $\frac{(1257)}{(1256)}\ (1467)\ \frac{(2356)}{(2357)}$	$t_{37},\ t_{38}$
,, ,, . ,, $\frac{(1257)}{(1256)}\ \frac{(1456)}{(1457)}\ (2367)$	$t_{39},\ t_{40}$
,, ,, $.(1567)\ (1234)\ \frac{(1257)}{(1256)}\ \frac{(2356)}{(2357)}$	$t_{41},\ t_{42}$

$(2346)\,(1567).(1235)\,(1234)\,\frac{(1467)}{(1457)}\,\frac{(2357)}{(2367)}$ $t_{43},\ t_{44}$

$,, \quad ,, \quad .(1237)\,(1234)\,\frac{(1457)}{(1456)}\,\frac{(2356)}{(2357)}$ $t_{45},\ t_{46}$

$,, \quad ,, \quad .(1567)\,(1234)\,(1234)\,(2357)$ t_{47}

$(2346)\,(3567).(1235)\,(1234)\,\frac{(1267)}{(1257)}\,\frac{(1457)}{(1467)}$ $t_{48},\ t_{49}$

$,, \quad ,, \quad .(1237)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)}$ $t_{50},\ t_{51}$

$,, \quad ,, \quad .(1567)\,(1234)\,(1234)\,(1257)$ t_{52}

(iii)

$(2347)\,(1345).(1236)\,(1267)\,\frac{(1457)}{(1456)}\,\frac{(2356)}{(2357)}$ $t_{53},\ t_{54}$

$,, \quad ,, \quad . \quad ,, \quad \frac{(1257)}{(1256)}\,(1467)\,\frac{(2356)}{(2357)}$ $t_{55},\ t_{56}$

$,, \quad ,, \quad . \quad ,, \quad \frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)}\,(2367)$ $t_{57},\ t_{58}$

$,, \quad ,, \quad .(1567)\,(1234)\,\frac{(1267)}{(1256)}\,\frac{(2356)}{(2367)}$ $t_{59},\ t_{60}$

$(2347)\,(1346).(1235)\,(1267)\,\frac{(1457)}{(1456)}\,\frac{(2356)}{(2357)}$ $t_{61},\ t_{62}$

$,, \quad ,, \quad . \quad ,, \quad \frac{(1257)}{(1256)}\,(1467)\,\frac{(2356)}{(2357)}$ $t_{63},\ t_{64}$

$,, \quad ,, \quad . \quad ,, \quad \frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)}\,(2367)$ $t_{65},\ t_{66}$

$,, \quad ,, \quad .(1567)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(2356)}{(2357)}$ $t_{67},\ t_{68}$

$(2347)\,(1567).(1235)\,(1234)\,\frac{(1467)}{(1456)}\,\frac{(2356)}{(2367)}$ $t_{69},\ t_{70}$

$,, \quad ,, \quad .(1236)\,(1234)\,\frac{(1457)}{(1456)}\,\frac{(2356)}{(2357)}$ $t_{71},\ t_{72}$

$,, \quad ,, \quad .(1567)\,(1234)\,(1234)\,(2356)$ t_{73}

$(2347)\,(3567).(1235)\,(1234)\,\frac{(1267)}{(1256)}\,\frac{(1456)}{(1467)}$ $t_{74},\ t_{75}$

$,, \quad ,, \quad .(1236)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)}$ $t_{76},\ t_{77}$

$,, \quad ,, \quad .(1567)\,(1234)\,(1234)\,(1256)$ t_{78}

$\delta_2 \cdot x_6$—(*Contd.*)

(iv)

$(2567)\,(1345).(1236)\,(1234)\,\dfrac{(1467)\,(2357)}{(1457)\,(2367)}$	t_{79}, t_{80}
,, ,, $.(1237)\,(1234)\,\dfrac{(1467)\,(2356)}{(1456)\,(2367)}$	t_{81}, t_{82}
,, ,, $.(1567)\,(1234)\,(1234)\,(2367)$	t_{83}
$(2567)\,(1346).(1235)\,(1234)\,\dfrac{(1467)\,(2357)}{(1457)\,(2367)}$	t_{84}, t_{85}
,, ,, $.(1237)\,(1234)\,\dfrac{(1457)\,(2356)}{(1456)\,(2357)}$	t_{86}, t_{87}
,, ,, $.(1567)\,(1234)\,(1234)\,(2357)$	t_{88}
$(2567)\,(1347).(1235)\,(1234)\,\dfrac{(1467)\,(2356)}{(1456)\,(2367)}$	t_{89}, t_{90}
,, ,, $.(1236)\,(1234)\,\dfrac{(1457)\,(2356)}{(1456)\,(2357)}$	t_{91}, t_{92}
,, ,, $.(1567)\,(1234)\,(1234)\,(2356)$	t_{93}
$(2567)\,(3567).(1235)\,(1234)\,(1234)\,(1467)$	t_{94}
,, ,, $.(1236)\,(1234)\,(1234)\,(1457)$	t_{95}
,, ,, $.(1237)\,(1234)\,(1234)\,(1456)$	t_{96}
,, ,, $.(1567)\,(1234)\,(1234)\,(1234)$	t_{97}

(v)

$(4567)\,(1345).(1236)\,(1234)\,\dfrac{(1267)\,(2357)}{(1257)\,(2367)}$	t_{93}, t_{99}
,, ,, $.(1237)\,(1234)\,\dfrac{(1267)\,(2356)}{(1256)\,(2367)}$	t_{100}, t_{101}
$(4567)\,(1346).(1235)\,(1234)\,\dfrac{(1267)\,(2357)}{(1257)\,(2367)}$	t_{102}, t_{103}
,, ,, $.(1237)\,(1234)\,\dfrac{(1257)\,(2356)}{(1256)\,(2357)}$	t_{101}, t_{105}
$(4567)\,(1347).(1235)\,(1234)\,\dfrac{(1267)\,(2356)}{(1256)\,(2367)}$	t_{106}, t_{107}
,, ,, $.(1236)\,(1234)\,\dfrac{(1257)\,(2356)}{(1256)\,(2357)}$	t_{108}, t_{109}
$(4567)\,(1567).(1235)\,(1234)\,(1234)\,(2367)$	t_{110}
,, ,, $.(1236)\,(1234)\,(1234)\,(2357)$	t_{111}
,, ,, $.(1237)\,(1234)\,(1234)\,(2356)$	t_{112}

$(4567)\,(3567).(1235)\,(1234)\,(1234)\,(1267)$ t_{113}

,, ,, $.(1236)\,(1234)\,(1234)\,(1257)$ t_{114}

,, ,, $.(1237)\,(1234)\,(1234)\,(1256)$ t_{115}

Hence,

$$\delta_2 . x_6 = (t_1 + t_2 + \ldots + t_{115}).$$

VII. δ_2 *applied to* x_7

To write down the terms generated by

$$D_{(124567)}\,[(12345).(12345).(12367)\,(12567)\,(14567)\,(23467)].$$

We delete, in order, the numbers 1, 2, 4 and next 5, 6, 7 in all possible ways. Hence $\delta_2 . x_7$ generates the *sum* of the following terms:

$2\times$(i)

$(2345)\,(1345).(1567)\,\frac{(1237)}{(1236)}\,(1267)\,\frac{(2346)}{(2347)}$ $t_1,\ t_2$

,, ,, $.(2367)\,\frac{(1237)}{(1236)}\,(1267)\,\frac{(1456)}{(1457)}$ $t_3,\ t_4$

,, ,, ,, $\frac{(1237)}{(1236)}\,\frac{(1256)}{(1257)}\,(1467)$ $t_5,\ t_6$

$(2345)\,(1367).(1235)\,(1267)\,\frac{(1457)}{(1456)}\,\frac{(2346)}{(2347)}$ $t_7,\ t_8$

,, ,, ,, $\frac{(1257)}{(1256)}\,(1467)\,\frac{(2346)}{(2347)}$ $t_9,\ t_{10}$

,, ,, $.(1567)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(2346)}{(2347)}$ $t_{11},\ t_{12}$

,, ,, $.(2367)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)}$ $t_{13},\ t_{14}$

$(2345)\,(1567).(1235)\,\frac{(1237)}{(1236)}\,(1467)\,\frac{(2346)}{(2347)}$ $t_{15},\ t_{16}$

,, ,, $.(1567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(2346)}{(2347)}$ $t_{17},\ t_{18}$

,, ,, $.(2367)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)}$ $t_{19},\ t_{20}$

$(2345)\,(3467).(1235)\,\frac{(1237)}{(1236)}\,(1267)\,\frac{(1456)}{(1457)}$ $t_{21},\ t_{22}$

,, ,, ,, $\frac{(1237)}{(1236)}\,\frac{(1256)}{(1257)}\,(1467)$ $t_{23},\ t_{24}$

,, ,, $.(1567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1256)}{(1257)}$ $t_{25},\ t_{26}$

$\delta_2 \cdot x_7$—(*Contd.*)

$2\times$(ii)

$$(2367)\,(1345).(1235)\,(1267)\,\frac{(1457)}{(1456)}\,\frac{(2346)}{(2347)} \qquad t_{27},\ t_{28}$$

$$\text{,,}\quad \text{,,}\quad .\ \text{,,}\quad \frac{(1257)}{(1256)}\,(1467)\,\frac{(2346)}{(2347)} \qquad t_{29},\ t_{30}$$

$$\text{,,}\quad \text{,,}\quad .(1567)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(2346)}{(2347)} \qquad t_{31},\ t_{32}$$

$$\text{,,}\quad \text{,,}\quad .(2357)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)} \qquad t_{33},\ t_{34}$$

$$(2367)\,(1567).(1235)\,(1234)\,\frac{(1457)}{(1456)}\,\frac{(2346)}{(2347)} \qquad t_{35},\ t_{36}$$

$$(2367)\,(3467).(1235)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)} \qquad t_{37},\ t_{36}$$

$2\times$(iii)

$$(2567)\,(1345).(1235)\,\frac{(1237)}{(1236)}\,(1467)\,\frac{(2346)}{(2347)} \qquad t_{39},\ t_{40}$$

$$\text{,,}\quad \text{,,}\quad .(1567)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(2346)}{(2347)} \qquad t_{41},\ t_{42}$$

$$\text{,,}\quad \text{,,}\quad .(2367)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)} \qquad t_{43},\ t_{44}$$

$$(2567)\,(1367).(1235)\,(1234)\,\frac{(1457)}{(1456)}\,\frac{(2346)}{(2347)} \qquad t_{45},\ t_{46}$$

$$(2567)\,(3467).(1235)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1456)}{(1457)} \qquad t_{47},\ t_{48}$$

$2\times$(iv)

$$(4567)\,(1345).(1235)\,\frac{(1237)}{(1236)}\,(1267)\,\frac{(2346)}{(2347)} \qquad t_{49},\ t_{50}$$

$$\text{,,}\quad \text{,,}\quad .(2367)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1256)}{(1257)} \qquad t_{51},\ t_{52}$$

$$(4567)\,(1367).(1235)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(2346)}{(2347)} \qquad t_{53},\ t_{54}$$

$$(4567)\,(1567).(1235)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(2346)}{(2347)} \qquad t_{55},\ t_{56}$$

$$(4567)\,(3467).(1235)\,(1234)\,\frac{(1237)}{(1236)}\,\frac{(1256)}{(1257)} \qquad t_{57},\ t_{58}$$

Hence,

$$\delta_2 . x_7 = 2\,(t_1 + t_2 + \ldots + t_{58}).$$

VIII. δ_2 *applied to* x_8

To write down the terms generated by

$$D_{(124567)} [(12345)\ (12347)\ (12356)\ (12567)\ (14567)\ (23467)].$$

We delete, in order, the numbers 1, 2, 4 and next 5, 6, 7 in all possible ways. Hence $\delta_2 . x_8$ generates the *sum* of the following terms:

(i)

(2345) (1347).(1567) (1236) $\dfrac{(1257)}{(1256)}\dfrac{(2346)}{(2347)}$ — $t_1,\ t_2$

,, ,, . ,, (1235) (1267) (2346) — t_3

,, ,, .(2367) (1236) $\dfrac{(1257)}{(1256)}\dfrac{(1456)}{(1457)}$ — $t_4,\ t_5$

,, ,, . ,, (1235) (1267) (1456) — t_6

,, ,, . ,, (1235) (1256) (1467) — t_7

(2345) (1356).(1237) (1267) $\dfrac{(1457)}{(1456)}\dfrac{(2346)}{(2347)}$ — $t_8,\ t_9$

,, ,, . ,, $\dfrac{(1257)}{(1256)}$ (1467) $\dfrac{(2346)}{(2347)}$ — $t_{10},\ t_{11}$

,, ,, .(1567) (1234) (1267) (2347) — t_{12}

,, ,, .(2367) (1234) $\dfrac{(1267)}{(1257)}\dfrac{(1457)}{(1467)}$ — $t_{13},\ t_{14}$

(2345) (1567).(1237) (1236) $\dfrac{(1457)}{(1456)}\dfrac{(2346)}{(2347)}$ — $t_{15},\ t_{16}$

,, ,, . ,, (1235) (1467) (2346) — t_{17}

,, ,, .(1567) (1234) (1236) (2347) — t_{18}

,, ,, .(2367) (1234) $\dfrac{(1236)}{(1235)}\dfrac{(1457)}{(1467)}$ — $t_{19},\ t_{20}$

(2345) (3467).(1237) (1236) $\dfrac{(1257)}{(1256)}\dfrac{(1456)}{(1457)}$ — $t_{21},\ t_{22}$

,, ,, . ,, (1235) (1267) (1456) — t_{23}

,, ,, . ,, (1235) (1256) (1467) — t_{24}

,, ,, .(1567) (1234) $\dfrac{(1236)}{(1235)}\dfrac{(1257)}{(1267)}$ — $t_{25},\ t_{26}$

$\delta_2 \cdot x_8$—(*Contd.*)

(ii)

$(2347)\,(1345).(1567)\,(1236)\,\frac{(1257)}{(1256)}\,\frac{(2346)}{(2347)}$	$t_{27},\ t_{28}$
,, ,, . ,, $(1235)\,(1267)\,(2346)$	t_{29}
,, ,, $.(2367)\,(1236)\,\frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)}$	$t_{30},\ t_{31}$
,, ,, . ,, $(1235)\,(1267)\,(1456)$	t_{32}
,, ,, . ,, $(1235)\,(1256)\,(1467)$	t_{33}
$(2347)\,(1356).(1235)\,(1267)\,\frac{(1457)}{(1456)}\,\frac{(2346)}{(2347)}$	$t_{34},\ t_{35}$
,, ,, . ,, $\frac{(1257)}{(1256)}\,(1467)\,\frac{(2346)}{(2347)}$	$t_{36},\ t_{37}$
,, ,, $.(1567)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(2346)}{(2347)}$	$t_{38},\ t_{39}$
,, ,, $.(2367)\,(1234)\,\frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)}$	$t_{40},\ t_{41}$
$(2347)\,(1567).(1235)\,(1236)\,\frac{(1457)}{(1456)}\,\frac{(2346)}{(2347)}$	$t_{42},\ t_{43}$
,, ,, . ,, $(1235)\,(1467)\,(2346)$	t_{44}
,, ,, $.(1567)\,(1234)\,(1235)\,(2346)$	t_{45}
,, ,, $.(2367)\,(1234)\,(1235)\,(1456)$	t_{46}
$(2347)\,(3467).(1235)\,(1236)\,\frac{(1257)}{(1256)}\,\frac{(1456)}{(1457)}$	$t_{47},\ t_{48}$
,, ,, . ,, $(1235)\,(1267)\,(1456)$	t_{49}
,, ,, . ,, $(1235)\,(1256)\,(1467)$	t_{50}
,, ,, $.(1567)\,(1234)\,(1235)\,(1256)$	t_{51}

(iii)

$(2356)\,(1345).(1237)\,(1267)\,\frac{(1457)}{(1456)}\,\frac{(2346)}{(2347)}$	$t_{52},\ t_{53}$
,, ,, . ,, $\frac{(1257)}{(1256)}\,(1467)\,\frac{(2346)}{(2347)}$	$t_{54},\ t_{55}$
,, ,, $.(1567)\,(1234)\,(1267)\,(2347)$	t_{56}
,, ,, $.(2367)\,(1234)\,\frac{(1267)}{(1257)}\,\frac{(1457)}{(1467)}$	$t_{57},\ t_{58}$

(2356) (1347).(1235) (1267) $\frac{(1457)}{(1456)}\frac{(2346)}{(2347)}$ t_{59}, t_{60}

,, ,, . ,, $\frac{(1257)}{(1256)}$ (1467) $\frac{(2346)}{(2347)}$ t_{61}, t_{62}

,, ,, .(1567) (1234) $\frac{(1257)}{(1256)}\frac{(2346)}{(2347)}$ t_{63}, t_{64}

,, ,, .(2367) (1234) $\frac{(1257)}{(1256)}\frac{(1456)}{(1457)}$ t_{65}, t_{66}

(2356) (1567).(1235) (1234) (1467) (2347) t_{67}

,, ,, .(1237) (1234) $\frac{(1457)}{(1456)}\frac{(2346)}{(2347)}$ t_{68}, t_{69}

,, ,, .(1567) (1234) (1234) (2347) t_{70}

,, ,, .(2367) (1234) (1234) (1457) t_{71}

(2356) (3467).(1235) (1234) $\frac{(1267)}{(1257)}\frac{(1457)}{(1467)}$ t_{72}, t_{73}

,, ,, .(1237) (1234) $\frac{(1257)}{(1256)}\frac{(1456)}{(1457)}$ t_{74}, t_{75}

,, ,, .(1567) (1234) (1234) (1257) t_{76}

(iv)

(2567) (1345).(1237) (1236) $\frac{(1457)}{(1456)}\frac{(2346)}{(2347)}$ t_{77}, t_{78}

,, ,, . ,, (1235) (1467) (2346) t_{79}

,, ,, .(1567) (1234) (1236) (2347) t_{80}

,, ,, .(2367) (1234) $\frac{(1236)}{(1235)}\frac{(1457)}{(1467)}$ t_{81}, t_{82}

(2567) (1347).(1235) (1236) $\frac{(1457)}{(1456)}\frac{(2346)}{(2347)}$ t_{83}, t_{84}

,, ,, . ,, (1235) (1467) (2346) t_{85}

,, ,, .(1567) (1234) (1235) (2346) t_{86}

,, ,, .(2367) (1234) (1235) (1456) t_{87}

(2567) (1356).(1235) (1234) (1467) (2347) t_{88}

,, ,, .(1237) (1234) $\frac{(1457)}{(1456)}\frac{(2346)}{(2347)}$ t_{89}, t_{90}

,, ,, .(1567) (1234) (1234) (2347) t_{91}

,, ,, .(2367) (1234) (1234) (1457) t_{92}

$\delta_2 \cdot x_8$—(*Contd.*)

$$(2567)\,(3467).(1235)\,(1234)\,\frac{(1236)\,(1457)}{(1235)\,(1467)} \qquad t_{93},\ t_{94}$$

$$,, \quad ,, \quad .(1237)\,(1234)\,(1235)\,(1456) \qquad t_{95}$$

$$,, \quad ,, \quad .(1567)\,(1234)\,(1234)\,(1235) \qquad t_{96}$$

(v)

$$(4567)\,(1345).(1237)\,(1236)\,\frac{(1257)\,(2346)}{(1256)\,(2347)} \qquad t_{97},\ t_{98}$$

$$,, \quad ,, \quad . \quad ,, \quad (1235)\,(1267)\,(2346) \qquad t_{99}$$

$$,, \quad ,, \quad .(2367)\,(1234)\,\frac{(1236)\,(1257)}{(1235)\,(1267)} \qquad t_{100},\ t_{101}$$

$$(4567)\,(1347).(1235)\,(1236)\,\frac{(1257)\,(2346)}{(1256)\,(2347)} \qquad t_{102},\ t_{103}$$

$$,, \quad ,, \quad . \quad ,, \quad (1235)\,(1267)\,(2346) \qquad t_{104}$$

$$,, \quad ,, \quad .(2367)\,(1234)\,(1235)\,(1256) \qquad t_{105}$$

$$(4567)\,(1356).(1235)\,(1234)\,(1267)\,(2347) \qquad t_{106}$$

$$,, \quad ,, \quad .(1237)\,(1234)\,\frac{(1257)\,(2346)}{(1256)\,(2347)} \qquad t_{107},\ t_{108}$$

$$,, \quad ,, \quad .(2367)\,(1234)\,(1234)\,(1257) \qquad t_{109}$$

$$(4567)\,(1567).(1235)\,(1234)\,(1236)\,(2347) \qquad t_{110}$$

$$,, \quad ,, \quad .(1237)\,(1234)\,(1235)\,(2346) \qquad t_{111}$$

$$,, \quad ,, \quad .(2367)\,(1234)\,(1234)\,(1235) \qquad t_{112}$$

$$(4567)\,(3467).(1235)\,(1234)\,\frac{(1236)\,(1257)}{(1235)\,(1267)} \qquad t_{113},\ t_{114}$$

$$,, \quad ,, \quad .(1237)\,(1234)\,(1235)\,(1256) \qquad t_{115}$$

Hence,

$$\delta_2 . x_8 = (t_1 + t_2 + \ldots + t_{115}).$$

To sum up, it will be noticed that δ_2, operating on the eight x's, has generated a total of 663 terms (excluding multiples occurring due to squared factors in certain x's), as follows:

TABLE II

Number of terms generated by δ_2

i	1	2	3	4	5	6	7	8	Total
No. of terms in $\delta_2 . x_i$	58	58	116	29	114	115	58	115	663

The numbering starts with $t_1, t_2, t_3, \ldots$ for each $\delta_2.x$ *afresh*, and no confusion need arise, as in the reductions which follow, we shall first consider the case of each $\delta_2.x$ separately, and assign new serial numbers when combining them all.

§6. The Reduction of the Terms Generated by δ_2

We have now to apply on the 663 terms, generated by operating with δ_2 on the eight x's, the operator

$$\eta \equiv \delta_3 . \delta_4 . \delta_5 . \delta_6 .$$

But our task will be much lighter if before doing so, we recognise that by Theorem III, a large number of equivalent terms among them can be obtained through permutations which keep η invariant, *i.e.*, factors of the transformation:

$$(123).(4567).$$

This reduction of the problem will be carried out in two stages:

(i) *Intra-Class Reduction*

Equalities, under operation η, within each $\delta_2 x_i$ separately will be given here. Note that the suffixes of the 't_i' refer here only to the particular $\delta_2.x$ under consideration.

(ii) *Inter-Class Reduction*

The reduced terms obtained after intra-class reduction of each $\delta_2.x$ are assigned new serial numbers and combined into a single set, renamed as the set of y_i's. Equalities, under operation η, within this set of y_i's, will be given here.

It will then be found that the 663 terms of Table II, reduce by (i) to 127 y_i's, which by (ii) reduce to only 40 y_j's, so that we have to evaluate $\eta . y_j$ only in these forty cases instead of the original six-hundred-and-sixty-three.

§6 *a*. The Intra-Class Reduction of the $\delta_2.x_i$

We shall carry out this reduction in eight sub-stages, considering each $\delta_2.x_i$ ($i = 1, 2, \ldots 8$) separately. It is also important to remark, that throughout this section, figures in brackets denote *connecting permutations*.

Observe that when the permutation connecting two terms involves three or more numbers, it is essential to point out the term to which it applies. In this sub-section and the next, we have used the convention that the start of the arrow indicates the term to which the permutation applies.

Thus in sub-stage (i), the notation

$$\overrightarrow{t_5} \qquad t_{15}$$
$$(567)$$

implies that the permutation (567) when applied to 't_5' transforms it into 't_{15}'.

In what follows, the equinumerous subsets have been separated by commas.

We obtain the following reductions:

(i) *Reduction of* $\delta_2 . x_1 = 2.(t_1 + t_2 + \ldots + t_{58})$

We notice that the t's are connected in the following manner:

$$t_{32} \quad \overrightarrow{t_{27}} \quad t_{20} \quad \overrightarrow{t_{14}} \quad t_7 \quad t_2 \qquad t_1 \quad \overrightarrow{t_5} \quad t_{15} \quad \overrightarrow{t_{19}} \quad t_{29} \quad t_{33},$$
$$(56)\ (576)\ (57)\ (576)\ (67)\ (12)\ (56)\ (67)\ (567)\ (57)\ (567)\ (56)$$

$$t_3 \quad \overrightarrow{t_6} \quad t_{16} \quad \overrightarrow{t_{18}} \quad t_{28} \quad t_{31}, \qquad t_4 \quad \overrightarrow{t_8} \quad t_{17} \quad \overrightarrow{t_{21}} \quad t_{30} \quad t_{34},$$
$$(67)\ (567)\ (57)\ (567)\ (56) \qquad (67)\ (567)\ (57)\ (567)\ (56)$$

$$t_{42} \quad t_{47} \quad t_{53} \quad t_{38} \quad t_{24} \quad t_{11} \quad t_9 \quad t_{23} \quad t_{36} \quad t_{51} \quad t_{46} \quad t_{40},$$
$$(56)\ (67)\ (12)\ (67)\ (56)\ (67)\ (56)\ (67)\ (12)\ (67)\ (56)$$

$$t_{43} \quad t_{48} \quad t_{52} \quad t_{37} \quad t_{25} \quad t_{12} \quad t_{10} \quad t_{22} \quad t_{35} \quad t_{50} \quad t_{45} \quad t_{41},$$
$$(56)\ (67)\ (12)\ (67)\ (56)\ (67)\ (56)\ (67)\ (12)\ (67)\ (56)$$

$$t_{13} \quad t_{26} \quad t_{39} \quad t_{54} \quad t_{49} \quad t_{44}, \qquad t_{55} \quad t_{56} \quad t_{57}, \quad \text{and} \quad t_{58}.$$
$$(56)\ (67)\ (12)\ (67)\ (56) \qquad (56)\ (67)$$

Taking the t_i's with the least suffix in each group as representative of the group, we have thus shown that

$$\delta_2 . x_1 = 2.[12t_1 + 6t_3 + 6t_4 + 12t_9 + 12t_{10} + 6t_{13} + 3t_{55} + t_{58}] \quad (6.1)$$

We shall rename these terms as $y_1, y_2, \ldots y_8$ respectively.

(ii) *Reduction of* $\delta_2 . x_2 = 2.(t_1 + t_2 + \ldots + t_{58})$

It is clear that members of each pair of terms are connected by the transformation (67)—so that we can immediately write down:

$$\delta_2 . x_2 = 2.(t_1 + t_2 + t_3 + \ldots + t_{58})$$
$$= 4.(t_1 + t_3 + t_5 + \ldots + t_{57}),$$

only the 29 terms with odd suffixes appearing. Further equalities will be given for these terms alone.

We notice that these t's are connected in the following manner:

t_1, t_3 t_5, t_{27} t_7 t_{11} t_{31}, t_{29} t_9 t_{13} t_{33},
(45) (12) (45) (12) (12) (45) (67) (12)

t_{51} t_{39} t_{15} t_{23}, t_{53} t_{41} t_{17} t_{25},
(45) (12) (45) (45) (12) (45)

t_{37} t_{35} t_{45} $\overrightarrow{t_{55}}$ t_{49} t_{43} t_{19} t_{21}, t_{47} t_{57}.
(45) (12) (45) (23) (457) (45) (12) (45) (45)

This shows that

$$\delta_2 . x_2 = 4 . [t_1 + 2t_3 + 4t_7 + 4t_9 + 4t_{15} + 4t_{17} + 8t_{19} + 2t_{47}]. \quad (6.2)$$

We shall rename these terms, in order, as $y_9, y_{10}, \ldots y_{16}$.

(iii) *Reduction of* $\delta_2 . x_3 = (t_1 + t_2 + \ldots + t_{116})$

It is clear that members of each pair of terms are connected by the transformation (12)—so that we may write down

$$\delta_2 . x_3 = (t_1 + t_2 + \ldots + t_{116})$$
$$= 2 . (t_1 + t_3 + t_5 + \ldots + t_{115}),$$

only the 58 terms with odd suffixes appearing. Further equalities shall be obtained for these terms alone.

We notice that these t's are connected in the following manner:

t_1 t_5, t_3 t_7 t_{81} t_{97}, t_9 t_{63},
(12) (45) (45) (13) (46) (45) (67) (45) (67)

$\overrightarrow{t_{11}}$ t_{21} $\overrightarrow{t_{35}}$ t_{65}, t_{13} t_{67}, t_{15} t_{61},
(456) (45) (67) (457) (45) (67) (45) (67)

$\overrightarrow{t_{17}}$ $\overrightarrow{t_{25}}$ $\overrightarrow{t_{39}}$ t_{91},
(13) (567) (13) (456) (13) (476)

t_{19} $\overrightarrow{t_{41}}$ t_{51} t_{111}, t_{23} t_{37},
(45) (67) (13) (465) (45) (67) (45) (67)

t_{27} t_{87}, t_{29} t_{93},
(67) (45) (67)

$\overrightarrow{t_{31}}$ t_{53} t_{55} t_{83} t_{89} $\overrightarrow{t_{101}}$ t_{107} t_{113},
(13) (456) (57) (45) (13) (46) (57) (13) (47) (56) (456) (46)

t_{33} t_{95}, t_{43} t_{47}, t_{45} t_{49},
(45) (67) (12) (45) (67) (45) (67)

t_{57} (12) (45) (67) t_{109}, t_{59} (45) (67) t_{115},

t_{69} (12) (45) (67) t_{73}, t_{71} (45) (67) t_{75}, t_{77} (45) (67) t_{89},

t_{79} (12) (45) (67) t_{103}, t_{85} (45) (67) t_{105}.

This shows that

$$\delta_2 . x_3 = 2.[2t_1+4t_3+2t_9+4t_{11}+2t_{13}+2t_{15}+4t_{17}+4t_{19}+2t_{23}+ 2t_{27}+2t_{29}+8t_{31}+2t_{33}+2t_{43}+2t_{45}+2t_{57}+2t_{59}+2t_{69}+ 2t_{71}+2t_{77}+2t_{79}+2t_{85}.] \quad (6.3)$$

We shall rename these terms, in order, as $y_{17}, y_{18}, \ldots y_{38}$.

(iv) *Reduction of* $\delta_2 . x_4 = 4.(t_1 + t_2 + \ldots + t_{29})$.

We notice that the t's are connected in the following manner:

t_{22} (67) t_{21} (46) (57) t_1 (12) t_2,

t_{20} (67) t_{19} (12) t_7 (67) t_8 (12) (45) t_{16} (67) t_{15} (12) t_3 (67) t_4,

t_{10} (67) t_9 (12) t_{17} (67) t_{18} (45) t_{14} (67) t_{13} (12) t_5 (67) t_6,

t_{24} (67) t_{23} (12) t_{27} (67) t_{28} (46) (57) t_{26} (45) t_{25} (12) t_{11} (45) t_{12}, and t_{29}.

This shows that

$$\delta_2 . x_4 = 4 \; [4t_1 + 8t_3 + 8t_5 + 8t_{11} + t_{29}]. \quad (6.4)$$

We shall rename these terms, in order, as $y_{39}, y_{40}, \ldots y_{43}$.

(v) *Reduction of* $\delta_2 . x_5 = (t_1 + t_2 + \ldots + t_{114})$

We notice that the t's are connected in the following manner

t_{81} (12) t_{40} (57) t_8 (12) t_{52} (45) (67) t_{28} (12) t_1 (46) t_{65} (12) t_{90}

t_{83} (12) t_{42} (57) t_7 (12) t_{51} (45) (67) t_{29} (12) t_2 (46) t_{68} (12) t_{91}

t_{85} (12) t_{44} (57) t_{12} (12) t_{53} (45) (67) t_{30} (12) t_3 (46) t_{64} (12) t_{87}

t_{82} (12) t_{39} (57) t_{10} (12) t_{50} (45) (67) t_{25} (12) t_4 (46) t_{66} (12) t_{89}.

t_{84} (12) t_{41} (57) t_{9} (12) t_{49} (45) (67) t_{26} (12) t_{5} (46) t_{67} (12) t_{92}

t_{86} (12) t_{43} (57) t_{11} (12) t_{54} (45) (67) t_{27} (12) t_{6} (46) t_{63} (12) t_{88}

t_{58} (12) t_{32} (57) t_{14} (12) t_{73} (67) t_{74} (12) t_{13} (57) t_{34} (12) t_{56}

t_{62} (12) t_{36} (57) t_{17} (12) t_{79} (45) (67) t_{76} (12) t_{15} (57) t_{33} (12) t_{55}

t_{60} (12) t_{38} (57) t_{19} (12) t_{77} (45) (67) t_{75} (12) t_{16} (57) t_{31} (12) t_{57}

t_{59} (12) t_{37} (57) t_{20} (12) t_{78} (45) t_{80} (12) t_{18} (57) t_{35} (12) t_{61}

t_{110} (12) t_{94} (57) t_{70} (12) t_{106} (46) t_{97} (12) t_{21} (57) t_{47} (12) t_{103}

t_{112} (12) t_{96} (57) t_{72} (12) t_{108} (46) t_{98} (12) t_{22} (57) t_{46} (12) t_{102}

t_{111} (12) t_{95} (57) t_{69} (12) t_{105} (46) t_{99} (12) t_{23} (57) t_{45} (12) t_{101}

t_{109} (12) t_{93} (57) t_{71} (12) t_{107} (46) t_{100} (12) t_{24} (57) t_{48} (12) t_{104}

and $t_{113} = t_{114}$.

We also notice the following further equivalences:

t_1 (12) (57) t_6, t_2 (12) (57) t_5, t_3 (12) (57) t_4;

t_{21} (67) t_{22}, t_{23} (45) t_{24}, t_{13} (12) (47) (56) t_{18}.

This shows that

$$\delta_2 . x_5 = [16\,(t_1+t_2+t_3+t_{13})+8\,(t_{15}+t_{16})+16\,(t_{21}+t_{23})+2t_{113}]. \quad (6.5)$$

We shall rename these terms, in order, as $y_{44}, y_{45}, \ldots y_{52}$.

(vi) *Reduction of* $\delta_2 . x_6 = (t_1 + t_2 + \ldots + t_{115})$

We notice that the t's are connected in the following manner:

t_{10} (67) t_1 (56) t_{29} (57) $\overrightarrow{t}_{39}$ (567) t_{56} (56) t_{66},

t_{64} (56) $\overrightarrow{t}_{58}$ (576) t_{37} (57) t_{31} (56) t_2 (67) t_9 (23) (46) t_{15} (67) t_7 (56) t_{34} (57) $\overrightarrow{t}_{41}$ (567) t_{60} (56) t_{68},

$$t_{12}\quad t_{3}\quad t_{27}\quad \vec{t}_{40}\quad t_{54}\quad t_{65},\qquad t_{11}\quad t_{4}\quad t_{32}\quad \vec{t}_{35}\quad t_{57}\quad t_{62},$$
$$(67)\ (56)\ (57)\ (567)\ (56)\qquad\qquad (67)\ (56)\ (57)\ (567)\ (56)$$

$$t_{63}\quad \vec{t}_{53}\quad t_{38}\quad t_{28}\quad t_{5}\quad t_{14},\qquad t_{16}\quad t_{8}\quad t_{33}\quad \vec{t}_{42}\quad t_{59}\quad t_{67},$$
$$(56)\ (576)\ (57)\ (56)\ (67)\ (23)\ (46)\ (57)\ (67)\ (56)\ (57)\ (567)\ (56)$$

$$t_{13}\quad t_{6}\quad t_{30}\quad \vec{t}_{36}\quad t_{55}\quad t_{61},\qquad t_{70}\quad t_{44}\quad t_{17}\quad t_{19}\quad t_{45}\quad t_{72},$$
$$(67)\ (56)\ (57)\ (567)\ (56)\qquad\qquad (67)\ (56)\ (67)\ (56)\ (67)$$

$$t_{69}\quad t_{43}\quad t_{18}\quad t_{20}\quad t_{46}\quad t_{71},\qquad t_{21}\quad t_{47}\quad t_{73},$$
$$(67)\ (56)\ (67)\ (56)\ (67)\qquad\qquad (56)\ (67)$$

$$t_{75}\quad t_{49}\quad t_{22}\quad t_{24}\quad t_{50}\quad t_{77},\qquad t_{74}\quad t_{48}\quad t_{23}\quad t_{25}\quad t_{51}\quad t_{76},$$
$$(67)\ (56)\ (67)\ (56)\ (67)\qquad\qquad (67)\ (56)\ (67)\ (56)\ (67)$$

$$t_{78}\quad t_{52}\quad \vec{t}_{26}\quad t_{83}\quad t_{88}\quad t_{93},\qquad t_{90}\quad t_{85}\quad t_{79}\quad t_{81}\quad t_{86}\quad t_{92},$$
$$(67)\ (56)\ (123)\ (56)\ (67)\qquad\qquad (67)\ (56)\ (67)\ (56)\ (67)$$

$$t_{89}\quad t_{84}\quad t_{80}\quad t_{82}\quad t_{87}\quad t_{91},\qquad t_{94}\quad t_{95}\quad t_{96},\qquad t_{97},$$
$$(67)\ (56)\ (67)\ (56)\ (67)\qquad\qquad (56)\ (67)$$

$$t_{109}\quad t_{104}\quad t_{100}\quad t_{98}\quad t_{103}\quad t_{107}\quad t_{106}\quad t_{102}\quad t_{99}\quad t_{101}\quad t_{105}\quad t_{108},$$
$$(67)\ (56)\ (67)\ (56)\ (67)\ (13)\ (67)\ (56)\ (67)\ (56)\ (67)$$

$$t_{112}\quad t_{111}\quad t_{110}\quad t_{113}\quad t_{114}\quad t_{115}.$$
$$(67)\ (56)\ (13)\ (56)\ (67).$$

This shows that

$$\delta_2 . x_6 = [6t_1+12t_2+6t_3+6t_4+12t_5+6t_6+6t_{17}+6t_{18}+3t_{21}+ +6t_{22}+6t_{23}+6t_{26}+6t_{79}+6t_{80}+3t_{94}+t_{97}+12t_{98}+6t_{110}] \qquad (6.6)$$

We shall rename these terms, in order, as $y_{53}, y_{54}, \ldots y_{70}$.

(vii) *Reduction of* $\delta_2 . x_7 = 2\,(t_1 + t_2 + \ldots + t_{58})$

It is evident that the members of each pair of terms are connected by the transformation (67)—so that we may write down:

$$\delta_2 . x_7 = 2\ (t_1 + t_2 + \ldots + t_{58})$$
$$= 4.(t_1 + t_3 + t_5 + \ldots + t_{57}),$$

only the 29 terms with odd suffixes appearing. Further equatities will be given for these terms alone.

We notice that these t's are connected in the following manner:

t_1 t_{27}, t_3, $\vec{t}_5$ t_{13}, t_7, $\vec{t}_9$ t_{31}, t_{11},
(23) (46) (57) (4657) (23) (4675)

$\overset{\downarrow}{t}_{15}$ t_{35}, t_{17}, t_{19}, t_{21}, $\vec{t}_{23}$ t_{41}, $\vec{t}_{25}$ t_{37}, t_{29}, t_{33},
(4675) (23) (465) (13) (457)

$\vec{t}_{39}$ t_{45}, t_{43}, t_{47}, t_{49} $\vec{t}_{51}$ t_{53}, t_{55} t_{57}.
(12) (4675) (13)(45) (132) (567) (13) (45)

This shows that

$$\delta_2 . x_7 = 4\,[2t_1+t_3+2t_5+t_7+2t_9+t_{11}+2t_{15}+t_{17}+t_{19}+t_{21}+ +2t_{23}+2t_{25}+t_{29}+t_{33}+2t_{39}+t_{43}+t_{47}+3t_{49}+2t_{55}]. \quad (6.7)$$

We shall rename these terms, in order, as $y_{71}, y_{72}, \ldots y_{89}$.

(viii) *Reduction of* $\delta_2 . x_8 = (t_1 + t_2 + \ldots + t_{115})$

We notice that the t's are connected in the following manner:

$\vec{t}_{59}$ $\vec{t}_{30}$ t_1 $\vec{t}_3$ $\vec{t}_{27}$ t_{40}, $\vec{t}_2$ t_{32} t_{57} t_{66},
(567) (23) (457) (56) (567) (23) (476) (23) (4765) (45) (67) (57)

t_4 t_{14}, t_5 t_7 t_{53} t_{56}, $\vec{t}_{55}$ t_{13} t_6 t_{60},
(46) (57) (47) (56) (57) (23) (47) (56) (4567) (45) (67) (23) (46)

t_{65} $\vec{t}_{58}$ t_{34} t_8 t_{10} t_{54}, $\vec{t}_9$ $\vec{t}_{31}$ t_{61} t_{63}, t_{11} t_{41},
(57) (456) (23) (57) (56) (46) (4765) (576) (23) (45) (47) (56)

t_{12} t_{38}, $\vec{t}_{46}$ t_{15} $\vec{t}_{16}$ $\vec{t}_{17}$ t_{42} t_{68}, t_{18} t_{45}, $\vec{t}_{19}$ t_{67},
(56) (4657) (67) (567) (567) (46) (57) (56) (576)

t_{20} t_{69}, $\vec{t}_{89}$ $\vec{t}_{79}$ $\vec{t}_{72}$ t_{21} $\vec{t}_{22}$ t_{74},
(57) (12) (456) (132) (467) (13) (4657) (67) (467)

$\vec{t}_{23}$ t_{88}, $\vec{t}_{86}$ $\vec{t}_{80}$ $\vec{t}_{24}$ t_{73},
(23) (475) (567) (23) (4567) (467)

$\vec{t}_{25}$ $\vec{t}_{47}$ t_{77} $\vec{t}_{78}$ $\vec{t}_{83}$ t_{87},
(13) (457) (123) (4567) (67) (576) (12) (476)

$\vec{t}_{26}$ $\vec{t}_{48}$ $\vec{t}_{82}$ t_{90}, t_{28} t_{35}, t_{29} t_{52},
(13) (456) (23) (465) (12) (475) (23) (47) (56) (23) (46) (57)

t_{33} t_{36}, t_{37}, t_{39}, t_{43}, $\vec{t}_{44}$ t_{71}, $\vec{t}_{49}$ t_{76} $\vec{t}_{85}$ t_{92},
(23) (47) (465) (13) (475) (23) (45) (12) (465)

$\vec{t}_{50}$ t_{91}, $\vec{t}_{51}$ t_{81}, t_{62} t_{64}, t_{70}, t_{75} t_{84},
(123) (45) (67) (23) (465) (23) (45) (67) (23) (45) (67)

t_{93} t_{95}, t_{94} t_{96},
(67) (23) (45)

t_{106} $\vec{t}_{105}$ t_{100} t_{99} t_{97} $\vec{t}_{98}$ t_{102} t_{107},
(13) (45) (67) (13) (567) (13) (45) (67) (12) (56) (67) (576) (46) (57)

t_{108} $\vec{t}_{101}$ t_{103}, $\vec{t}_{104}$ t_{109},
(12) (57) (23) (4567) (4675)

t_{115} $\vec{t}_{113}$ t_{110} t_{111}, t_{112} t_{114}.
(67) (13) (465) (67) (13) (45)

This shows that

$$\delta_2 . x_8 = [6t_1+4t_2+2t_4+4t_5+4t_6+6t_8+4t_9+2t_{11}+2t_{12}+6t_{15}+ 2t_{18}+2t_{19}+2t_{20}+6t_{21}+2t_{23}+4t_{24}+6t_{25}+4t_{26}+ 2t_{28}+2t_{29}+2t_{33}+t_{37}+t_{39}+t_{43}+2t_{44}+4t_{49}+ 2t_{50}+2t_{51}+2t_{62}+t_{70}+2t_{75}+2t_{93}+2t_{94}+8t_{97}+ 3t_{101}+2t_{104}+4t_{110}+2t_{112}]. \qquad (6.8)$$

We shall rename these terms, in order, as $y_{90}, y_{91}, \ldots y_{127}$.

Summary of Results for Intra-Class Reduction

It has been shown that by virtue of Theorem III, a large number of equalities exist, through factors of the transformation:

(123).(4567)

in the set of 663 terms obtained by applying δ_2 to the eight x's. It follows from equations (6.1), ..., (6.8) that these 663 terms have been, now, reduced to 127 terms as follows:

TABLE II *a*

Reduction of the number of terms generated by δ_2

i	1	2	3	4	5	6	7	8	Total
No. of reduced terms in $\delta_2 x_i$	8	8	22	5	9	18	19	38	127

We have renamed these terms, in order, as $y_1, y_2, y_3, \ldots y_{127}$.

It will be seen that to calculate A_7 and C_7 we need the values of each of the $\xi.x_i$ ($i = 1, 2, \ldots 8$) individually. The intra-class reduction has transformed the quantities $\delta_2.x_i$, and hence also the $\xi.x_i$, to weighted sums, and it is now necessary to give a comprehensive and explicit expression for them. These remarks are well illustrated by the following equation:

Notice that, since

$$\xi \equiv \eta.\delta_2,$$

we have,

$$\begin{aligned}\xi.x_i &= \eta.[\delta_2.x_i]\\ &= \eta\ [\Sigma w_i y_i]\\ &= \Sigma w_i.(\eta.y_i),\end{aligned} \tag{6.9}$$

where, using equations (6.1), ...(6.8), the appropriate weights w_i and the corresponding terms y_i are as given in the following table:

TABLE III

Reduction of the $\delta_2.x_i$ to the form $\Sigma w_i y_i$

Sl. No. i	w_i	y_i
		$\delta_2.x_1$
1	24	(1237) (1256) (1257) (1346) (2345) (3467)
2	12	(1237) (1257) (1267) (1346) (2345) (3456)
3	12	(1234) (1257) (1267) (1346) (2345) (3567)
4	24	(1234) (1236) (1267) (1567) (2345) (3457)
5	24	(1234) (1236) (1257) (1567) (2345) (3467)
6	12	$(1234)^2$.(1267) (1567) (2345) (3567)
7	6	$(1234)^2$.(1235) (1567) (2567) (3467)
8	2	$(1234)^3$.(1567) (2567) (3567)
		$\delta_2.x_2$
9	4	(1237).$(1267)^2$.(1345) (2345) (3456)
10	8	(1237) (1256) (1267) (1345) (2345) (3467)
11	16	(1235) (1247) (1267) (1367) (2345) (3456)
12	16	(1235) (1247) (1256) (1367) (2345) (3467)
13	16	(1235) (1237) (1267) (1467) (2345) (3456)
14	16	(1235) (1237) (1256) (1467) (2345) (3467)
15	32	(1234) (1237) (1256) (1467) (2345) (3567)
16	8	(1234) (1235) (1237) (1567) (2467) (3456)

TABLE III—(*Contd.*)

Sl. No. i	w_i	y_i
		$\delta_2 . x_3$
17	4	$(1234)^2.(1235)$ (1567) (2467) (3567)
18	8	(1234) (1235) (1267) (1467) (2345) (3567)
19	4	$(1234)^2.(1247)$ (1567) (2356) (3567)
20	8	(1234) (1236) (1247) (1567) (2345) (3567)
21	4	(1234) (1247) (1267) (1356) (2345) (3567)
22	4	(1234) (1235) (1247) (1567) (2356) (3467)
23	8	$(1234)^2.(1257)$ (1467) (2356) (3567)
24	8	(1234) (1236) (1257) (1467) (2345) (3567)
25	4	(1234) (1257) (1267) (1356) (2345) (3467)
26	4	(1234) (1235) (1236) (1567) (2467) (3457)
27	4	(1234) (1236) (1267) (1567) (2345) (3457)
28	16	(1234) (1235) (1267) (1467) (2356) (3457)
29	4	$(1234).(1267)^2.(1356)$ (2345) (3457)
30	4	(1234) (1246) (1257) (1356) (2347) (3567)
31	4	(1236) (1246) (1257) (1347) (2345) (3567)
32	4	(1235) (1246) (1267) (1356) (2347) (3457)
33	4	(1237) (1246) (1267) (1356) (2345) (3457)
34	4	(1234) (1247) (1256) (1356) (2347) (3567)
35	4	(1236) (1247) (1256) (1347) (2345) (3567)
36	4	(1234) (1237) (1256) (1467) (2356) (3457)
37	4	(1234) (1256) (1267) (1356) (2347) (3457)
38	4	(1236) (1256) (1267) (1347) (2345) (3457)
		$\delta_2 . x_4$
39	16	(1236) (1237) (1345) (1567) (2345) (2467)
40	32	(1235) (1237) (1367) (1467) (2345) (2456)
41	32	(1235) (1237) (1367) (1456) (2345) (2467)
42	32	(1235) (1236) (1237) (1467) (2345) (4567)
43	4	(1234) (1235) (1236) $(1237).(4567)^2$
		$\delta_2 . x_5$
44	16	(1236) (1237) (1347) (1567) (2345) (2456)
45	16	$(1236)^2.(1347)$ (1567) (2345) (2457)
46	16	(1235) (1236) (1347) (1567) (2345) (2467)
47	16	(1236) (1237) (1367) (1457) (2345) (2456)
48	8	(1235) (1237) (1367) (1467) (2345) (2456)
49	8	(1235) (1237) (1367) (1456) (2345) (2467)
50	16	$(1236).(1237)^2.(1456)$ (2345) (4567)
51	16	(1235) (1236) (1237) (1467) (2345) (4567)
52	2	(1234) (1235) (1236) $(1237).(4567)^2$

TABLE III—(*Contd.*)

Sl. No. i	w_i	y_i
		$\delta_2 . x_6$
53	6	(1237) (1267) (1346) (1457) (2345) (2356)
54	12	(1237) (1267) (1346) (1456) (2345) (2357)
55	6	(1237) (1257) (1346) (1467) (2345) (2356)
56	6	(1237) (1256) (1346) (1467) (2345) (2357)
57	12	(1237) (1257) (1346) (1456) (2345) (2367)
58	6	(1237) (1256) (1346) (1457) (2345) (2367)
59	6	(1234) (1236) (1467) (1567) (2345) (2357)
60	6	(1234) (1236) (1457) (1567) (2345) (2367)
61	3	$(1234)^2.(1567)^2.(2345)$ (2367)
62	6	(1234) (1236) (1267) (1457) (2345) (3567)
63	6	(1234) (1236) (1257) (1467) (2345) (3567)
64	6	$(1234)^2.(1267)$ (1567) (2345) (3567)
65	6	(1234) (1236) (1345) (1467) (2357) (2567)
66	6	(1234) (1236) (1345) (1457) (2367) (2567)
67	3	$(1234)^2.(1235)$ (1467) (2567) (3567)
68	1	$(1234)^3.(1567)$ (2567) (3567)
69	12	(1234) (1236) (1267) (1345) (2357) (4567)
70	6	$(1234)^2.(1235)$ (1567) (2367) (4567)
		$\delta_2 . x_7$
71	8	(1237) (1267) (1345) (1567) (2345) (2346)
72	4	(1237) (1267) (1345) (1456) (2345) (2367)
73	8	(1237) (1256) (1345) (1467) (2345) (2367)
74	4	(1235) (1267) (1367) (1457) (2345) (2346)
75	8	(1235) (1257) (1367) (1467) (2345) (2346)
76	4	(1234) (1257) (1367) (1567) (2345) (2346)
77	8	(1235) (1237) (1467) (1567) (2345) (2346)
78	4	(1234) $(1237).(1567)^2.(2345)$ (2346)
79	4	(1234) (1237) (1456) (1567) (2345) (2367)
80	4	(1235) (1237) (1267) (1456) (2345) (3467)
81	8	(1235) (1237) (1256) (1467) (2345) (3467)
82	8	(1234) (1237) (1256) (1567) (2345) (3467)
83	4	(1235) (1257) (1345) (1467) (2346) (2367)
84	4	(1234) (1257) (1345) $(1456).(2367)^2$
85	8	(1235) (1237) (1345) (1467) (2346) (2567)
86	4	(1234) (1237) (1345) (1456) (2367) (2567)
87	4	(1234) (1235) (1237) (1456) (2567) (3467)
88	12	(1235) (1237) (1267) (1345) (2346) (4567)
89	8	(1234) (1235) (1237) (1567) (2346) (4567)

TABLE III—(*Contd.*)

Sl. No. i	w_i	y_i
		$\delta_2 . x_8$
90	6	(1236) (1257) (1347) (1567) (2345) (2346)
91	4	(1236) (1256) (1347) (1567) (2345) (2347)
92	2	(1236) (1257) (1347) (1456) (2345) (2367)
93	4	(1236) (1256) (1347) (1457) (2345) (2367)
94	4	(1235) (1267) (1347) (1456) (2345) (2367)
95	6	(1237) (1267) (1356) (1457) (2345) (2346)
96	4	(1237) (1267) (1356) (1456) (2345) (2347)
97	2	(1237) (1256) (1356) (1467) (2345) (2347)
98	2	(1234) (1267) (1356) (1567) (2345) (2347)
99	6	(1236) (1237) (1457) (1567) (2345) (2346)
100	2	(1234) (1236). $(1567)^2$. (2345) (2347)
101	2	(1234) (1236) (1457) (1567) (2345) (2367)
102	2	(1234) (1235) (1467) (1567) (2345) (2367)
103	6	(1236) (1237) (1257) (1456) (2345) (3467)
104	2	(1235) (1237) (1267) (1456) (2345) (3467)
105	4	(1235) (1237) (1256) (1467) (2345) (3467)
106	6	(1234) (1236) (1257) (1567) (2345) (3467)
107	4	(1234) (1235) (1267) (1567) (2345) (3467)
108	2	(1236) (1256) (1345) (1567). $(2347)^2$
109	2	(1235) (1267) (1345) (1567) (2346) (2347)
110	2	(1235) (1256) (1345) (1467) (2347) (2367)
111	1	(1235) (1256) (1356) (1467). $(2347)^2$
112	1	(1234) (1256) (1356) (1567). $(2347)^2$
113	1	(1235) (1236) (1456) (1567). $(2347)^2$
114	2	$(1235)^2$. (1467) (1567) (2346) (2347)
115	4	$(1235)^2$. (1267) (1456) (2347) (3467)
116	2	$(1235)^2$. (1256) (1467) (2347) (3467)
117	2	(1234) (1235) (1256) (1567) (2347) (3467)
118	2	(1235) (1256) (1347) (1467) (2347) (2356)
119	1	$(1234)^2$. $(1567)^2$. (2347) (2356)
120	2	(1234) (1237) (1256) (1457) (2356) (3467)
121	2	(1234) (1235) (1236) (1457) (2567) (3467)
122	2	(1234). $(1235)^2$. (1467) (2567) (3467)
123	8	(1236) (1237) (1257) (1345) (2346) (4567)
124	3	(1234) (1235) (1267) (1345) (2367) (4567)
125	2	$(1235)^2$. (1267) (1347) (2346) (4567)
126	4	(1234) (1235) (1236) (1567) (2347) (4567)
127	2	$(1234)^2$. (1235) (1567) (2367) (4567).

§6 *b*. The Inter-Class Reduction of the $\delta_2 . x_i$

It has been seen—*cf.* equation (6.9)—that the problem of evaluating the $\xi . x_i$ ($i = 1, 2, \ldots 8$) reduces to that of finding the values of the $\eta . y_i$ ($i = 1, 2, \ldots 127$). Still further reduction is possible by means of the following inter-class equalities among the y's constituting the different $\delta_2 . x_i$, through the same transformation as used in the intra-class reduction above, *i.e.*, factors of the permutation:

$$(123).(4567).$$

We have—the figures in brackets denoting connecting permutations—

$\vec{y}_1$ $\vec{y}_{33}$ y_{56} y_{90}
(12) (456) (123) (56) (23) (45) (67)

y_2 $\vec{y}_{38}$ $\vec{y}_{76}$ y_{98}
(67) (13) (476) (576)

$\vec{y}_3$ y_{83} y_{110}
(123) (45) (67) (67)

$y_{27} = y_4$ y_{13} $y_{48} = y_{40}$ y_{66} $\vec{y}_{86}$ y_{117}
(45) (67) (23) (47) (56) (67) (23) (4567)

$y_{106} = \vec{y}_5$ $\vec{y}_{28}$ $\vec{y}_{44}$ $\vec{y}_{65}$ y_{82}
(456) (123) (46) (57) (457) (132) (67)

$y_{64} = y_6$ y_{19} y_{116}
(13) (57) (45) (67)

y_7 y_{17} y_{67} y_{122}, $y_8 = y_{68}$, $\vec{y}_9$ y_{112}
(23) (12) (12) (45) (13) (467)

$\vec{y}_{10}$ $\vec{y}_{74}$ y_{97}
(13) (4756) (576)

$\vec{y}_{11}$ $\vec{y}_{37}$ $\vec{y}_{71}$ y_{91}, $\vec{y}_{12}$ $\vec{y}_{30}$ y_{57} y_{93}
(4765) (13) (476) (576) (475) (13) (475) (67)

$y_{105} = y_{81} = y_{14}$ y_{20}
(45) (67)

y_{15} $y_{63} = y_{24}$ y_{46} $\vec{y}_{85}$ y_{103}
(67) (23) (45) (12) (67) (23) (4576)

y_{16} $\vec{y}_{26}$ y_{87} y_{121}, $\vec{y}_{18}$ $\vec{y}_{47}$ y_{107}
(67) (123) (67) (67) (123) (46) (57) (132) (4657)

$\vec{y}_{21}$ $\vec{y}_{54}$ $\vec{y}_{75}$ y_{96}
(13) (4756) (4675) (576)

$\vec{y}_{22}$ $y_{49} = y_{41}$ y_{62} $y_{104} = y_{80}$
(23) (476) (23) (45) (67) (45) (67)

$\vec{y}_{23}$ $\vec{y}_{45}$ y_{115}, $\vec{y}_{25}$ y_{53} y_{65}
(23) (465) (23) (4765) (13) (4756) (45)

$\vec{y}_{29}$ $\vec{y}_{84}$ y_{108}
(123) (67) (23) (467)

$\vec{y}_{31}$ y_{58} y_{92}, $\vec{y}_{32}$ $\vec{y}_{55}$ y_{109}
(13) (467) (67) (123) (576) (23) (4675)

$\vec{y}_{34}$ y_{72} y_{118}
(13) (4765) (57)

$\vec{y}_{35}$ y_{73} y_{94}, $\vec{y}_{36}$ $\vec{y}_{39}$ y_{120}
(13) (4567) (23) (46) (57) (23) (4765) (23) (467)

$y_{51} = \vec{y}_{42}$ $\vec{y}_{89}$ y_{126}, $y_{43} = y_{52}$, y_{50} $y_{127} = y_{70}$
(465) (567) (47) (56)

y_{59} y_{77} y_{99}, $y_{101} = y_{60}$ y_{79}, y_{61} y_{119}
(45) (67) (56) (67) (57)

y_{69} y_{88} y_{123}, y_{78} y_{100},
(45) (67) (12) (56) (67)

and,

$$y_{102},\ y_{111},\ y_{113},\ y_{114},\ y_{124},\ y_{125}.$$

Summary of Results for Inter-Class Reduction

This reduction shows that it is sufficient for us to calculate the $\eta . y_j$ only for the following set of 40 values of j:—

TABLE IV

The representative set of y_j's

Set	Values of j										
$\{y_j\}$	$j =$	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,
		11,	12,	14,	15,	16,	18,	21,	22,	23,	25,
		29,	31,	32,	34,	35,	36,	42,	43,	50,	59,
		60,	61,	69,	78,	102,	111,	113,	114,	124,	125.

Notice that, by virtue of Theorem III, the 663 terms generated by the operation $\delta_2 . x_i$ ($i = 1, 2, \ldots 8$), have been reduced by means of intra-class and inter-class reductions, to the 40 terms y_j as given above in Table IV.

§7. THE EVALUATION OF THE FORTY REPRESENTATIVE $\eta . y_j$

We have now to apply the four differential operations:

$$\eta \equiv \delta_3 . \delta_4 . \delta_5 . \delta_6$$

to the 40 y_j's of Table IV.

As an illustration, we give below in outline the calculations necessary to evaluate $\eta . y_1$. The other $\eta . y_j$'s may be similarly evaluated. The forty values of the $\eta . y_j$ thus obtained are given in Table V, at the end of the following example:

Example

To evaluate $\eta . y_1$, *i.e.*,

$$\delta_3 . \delta_4 . \delta_5 . \delta_6 \, [(1237)\,(1256)\,(1257)\,(1346)\,(2345)\,(3467)].$$

We shall first apply the operator $\delta_3 \equiv D_{(123567)}$ to the term y_1. We divide the work into four stages, according as 1 is deleted from the first, second, third or fourth factor of y_1. Next, using the language at the beginning of §5, in each stage, we shall delete, in order, the numbers 1, 2, 3 and next 5, 6, 7 in all possible ways.

The terms generated in this manner are given below after rearrangement of the factors within themselves in ascending order. It is well to remember that for some of the other y_j's, those containing a squared factor give the common multiplier 2 !, and those containing a cubed factor give the common multiplier 3 !.

Hence the third operator $\delta_3 \equiv D_{(123567)}$ acting on y_1, generates the *sum* of the following 36 terms:

Application of δ_3 to y_1

Serial No.	Term
1	(125) (146) (156) (234) (237) (347)
2	(127) (134) (156) (237) (245) (346)
3	(125) (134) (156) (234) (237) (467)
4	(125) (146) (157) (234) (237) (346)
5	(126) (134) (157) (237) (245) (346)
6	(125) (126) (146) (237) (345) (347)
7	(125) (127) (146) (237) (345) (346)
8	(125) (126) (134) (237) (345) (467)
9	(125) (137) (146) (234) (256) (347)
10	(127) (134) (137) (245) (256) (346)
11	(125) (134) (137) (234) (256) (467)
12	(127) (134) (157) (234) (256) (346)
13	(123) (146) (157) (234) (256) (347)
14	(123) (134) (157) (234) (256) (467)
15	$(127)^2$.(134) (256) (345) (346)
16	(123) (127) (146) (256) (345) (347)
17	(123) (127) (134) (256) (345) (467)
18	(125) (137) (146) (234) (257) (346)
19	(126) (134) (137) (245) (257) (346)
20	(127) (134) (156) (234) (257) (346)
21	(123) (146) (156) (234) (257) (347)
22	(123) (134) (156) (234) (257) (467)
23	(126) (127) (134) (257) (345) (346)
24	(123) (126) (146) (257) (345) (347)
25	(123) (126) (134) (257) (345) (467)
26	(125) (126) (137) (245) (346) (347)
27	(125) (127) (137) (245).$(346)^2$
28	$(125)^2$.(137) (234) (346) (467)
29	(125) (127) (156) (234) (346) (347)
30	(123) (127) (156) (245) (346) (347)
31	(125) (127) (157) (234).$(346)^2$
32	(123) (126) (157) (245) (346) (347)
33	(123) (125) (157) (234) (346) (467)
34	(125) (126) (127) (345) (346) (347)
35	(125).$(127)^2$.(345).$(346)^2$
36	(123) (125) (127) (345) (346) (467)

We shall designate these terms, for convenience, as $z_1, z_2, \ldots z_{36}$, so that

$$\delta_3 . y_1 = z_1 + z_2 + \ldots + z_{36}. \tag{7.1}$$

We have now to apply the three operations:

$$\zeta \equiv \delta_4 . \delta_5 . \delta_6$$

to these 36 terms.

Equalities among such terms, obtained by applying δ_3 to the y_j's, occur by Theorem III through permutations which keep the operator ζ invariant, *i.e.*, factors of the transformation:

$$(1234).(567).$$

Thus, for example, we have:

$$z_{14} \underset{(67)}{\quad} z_{22}.$$

Extensive use has been made of these equivalences in the evaluation of the $\eta . y_j$.'s.

Application of the operator $\zeta \equiv \delta_4 . \delta_5 . \delta_6$ has been illustrated in the table below for a few of the 36 terms occurring in (7.1).

Column 1 of the table gives the name of the term. Column 2 gives the result of applying the operator $\delta_4 \equiv D_{(123467)}$ to the term in column 1. In applying this operator δ_4, we divide the work into three stages, according as 1 is deleted from the first, second or third factor of the term z_i. Next, using the language of §5, in each stage we shall delete, in order, the numbers 1, 2, 3 and next 4, 6, 7 in all possible ways.

Lastly, column 3 gives the result of applying the operators $\delta_5 . \delta_6$ to the terms in column 2. Notice that the last two operations $\delta_5 . \delta_6$ may be carried out orally. The operation $\delta_5 \equiv D_{(123457)}$ is easily performed if we start by deleting 7, then the number paired with 7, then the number paired with this number, and so on in the same manner. If this process stops in the middle, we can follow a similar procedure with the numbers still remaining to be deleted. The last operation $\delta_6 \equiv D_{(123456)}$ need not be performed, as there is only one way of writing down the last row of a Latin square, when all the previous rows have been written down.

The total of column 3 clearly represents the result of applying the operator $\zeta \equiv \delta_4 . \delta_5 . \delta_6$ to the term in column 1. This total appears in the last row, and is written as $\zeta (z_i)$. Notice that,

$$\zeta (z_1) + \zeta (z_2) + \ldots + \zeta (z_{36}) = \zeta . \delta_3 y_1 = \eta . y_1. \tag{7.2}$$

Application of the operator ' ζ ' to the terms generated by $\delta_3 y_1$

Term	δ_4	$\delta_5 . \delta_6$
z_1	(25) (34).(27) (16) (15) (34) +	2
	,, ,, .(47) (16) (15) (23) +	1
	(25) (37).(24) (16) (15) (34) +	1
	(46) (15).(24) (15) (23) (37) +	2
	,, ,, .(27) (15) (23) (34) +	2
	,, ,, .(47) (15) (23) (23) +	4
	(56) (15).(24) (14) (23) (37) +	1
	,, ,, .(27) (14) (23) (34) +	1
	,, ,, .(47) (14) (23) (23).	2
	$\zeta(z_1) =$	16
z_2	(27) (45).(14) (15) (23) (36) +	2
	,, ,, .(46) (13) (15) (23) +	1
	(34) (17).(46) (15) (23) (25) +	1
	(34) (37).(46) (12) (15) (25) +	2
	(34) (45).(27) (12) (15) (36) +	1
	(56) (17).(14) (23) (25) (34) +	1
	(56) (37).(14) (12) (25) (34) +	1
	(56) (45).(27) (12) (13) (34).	1
	$\zeta(z_2) =$	10
z_{15}	2 × [(27) (56).(14) (12) (35) (34) +	2 [1
	,, ,, .(45) (12) (13) (34) +	1
	(34) (17).(45) (12) (25) (36) +	1
	,, ,, .(46) (12) (25) (35).]	1]
	$\zeta(z_{15}) =$	8

Proceeding in the above manner, we obtain the following set of values for the $\zeta(z_i)$, $i = 1, 2, \ldots 36$.

$\zeta(z_1) = 16$	$\zeta(z_{13}) = 14$	$\zeta(z_{25}) = 14$
$\zeta(z_2) = 10$	$\zeta(z_{14}) = 16$	$\zeta(z_{26}) = 14$
$\zeta(z_3) = 14$	$\zeta(z_{15}) = 8$	$\zeta(z_{27}) = 12$
$\zeta(z_4) = 14$	$\zeta(z_{16}) = 10$	$\zeta(z_{28}) = 12$
$\zeta(z_5) = 12$	$\zeta(z_{17}) = 14$	$\zeta(z_{29}) = 12$
$\zeta(z_6) = 14$	$\zeta(z_{18}) = 14$	$\zeta(z_{30}) = 10$
$\zeta(z_7) = 14$	$\zeta(z_{19}) = 14$	$\zeta(z_{31}) = 16$
$\zeta(z_8) = 10$	$\zeta(z_{20}) = 12$	$\zeta(z_{32}) = 10$
$\zeta(z_9) = 14$	$\zeta(z_{21}) = 14$	$\zeta(z_{33}) = 22$
$\zeta(z_{10}) = 14$	$\zeta(z_{22}) = 16$	$\zeta(z_{34}) = 8$
$\zeta(z_{11}) = 14$	$\zeta(z_{23}) = 12$	$\zeta(z_{35}) = 16$
$\zeta(z_{12}) = 12$	$\zeta(z_{24}) = 12$	$\zeta(z_{36}) = 12$

Adding up the values of the $\zeta(z_i)$, $i = 1, 2, \ldots 36$, we obtain from (7.2),

$$\begin{aligned}\eta . y_1 &= \zeta(z_1) + \zeta(z_2) + \zeta(z_3) + \ldots + \zeta(z_{36}) \\ &= 16 + 10 + 14 + \ldots + 12 \\ &= 472.\end{aligned}$$

The above example illustrates the evaluation of $\eta . y_1$. Proceeding n a similar manner we can evaluate all the $\eta . y_j$'s for values of j mentioned in Table IV, and thus obtain the following table:

TABLE V

Values of the 40 *representative* $\eta . y_j$'s

y_j	$\eta . y_j$
$y_4, y_5, y_{11}, y_{21}, y_{22}$	456
$y_3, y_{10}, y_{18}, y_{23}, y_{29}, y_{32}, y_{34}, y_{36}, y_{59}, y_{60}, y_{69}$	464
$y_1, y_{12}, y_{15}, y_{25}, y_{35}$	472
$y_{102}, y_{111}, y_{113}, y_{114}, y_{124}, y_{125}$	480
y_{31}	496
$y_2, y_6, y_7, y_9, y_{14}, y_{16}, y_{42}, y_{50}, y_{61}, y_{78}$	512
y_8, y_{43}	576

§8. THE VALUES OF THE $\xi . x_i$ ($i = 1, 2, \ldots 8$)

The use of Table V along with the inter-class equalities of §6 *b*, clearly gives us the values of the $\eta . y_i$ for all the 127 y'sof Table III.

We have seen before that—*cf.* equation (6.9)—

$$\xi . x_i = \Sigma \, w_i \, (\eta . y_i) \tag{8.1}$$

Using the weights w_i given in Table III, and the values of the representative $\eta . y_i$ just obtained in Table V, it is now easy to construct Table VI (see next page).

From Table VI and equation (8.1), we readily obtain:

$$\begin{aligned} \xi . x_1 &= (456 \times 48) + (464 \times 12) + (472 \times 24) + (512 \times 30) + (576 \times 2) \\ &= 21{,}888 + 5{,}568 + 11{,}328 + 15{,}360 + 1{,}152 \\ &= 55{,}296. \end{aligned}$$

$$\begin{aligned} \xi . x_2 &= (456 \times 32) + (464 \times 8) + (472 \times 48) + (512 \times 28) \\ &= 14{,}592 + 3{,}712 + 22{,}656 + 14{,}336 \\ &= 55{,}296. \end{aligned}$$

$$\begin{aligned} \xi . x_3 &= (456 \times 32) + (464 \times 32) + (472 \times 24) + (496 \times 4) + (512 \times 24) \\ &= 14{,}592 + 14{,}848 + 11{,}328 + 1{,}984 + 12{,}288 \\ &= 55{,}040. \end{aligned}$$

$$\begin{aligned} \xi . x_4 &= (456 \times 64) + (464 \times 16) + (512 \times 32) + (576 \times 4) \\ &= 29{,}184 + 7{,}424 + 16{,}384 + 2{,}304 \\ &= 55{,}296. \end{aligned}$$

$$\begin{aligned} \xi . x_5 &= (456 \times 32) + (464 \times 32) + (472 \times 16) + (512 \times 32) + (576 \times 2) \\ &= 14{,}592 + 14{,}848 + 7{,}552 + 16{,}384 + 1{,}152 \\ &= 54{,}528. \end{aligned}$$

$$\begin{aligned} \xi . x_6 &= (456 \times 30) + (464 \times 30) + (472 \times 30) + (496 \times 6) + (512 \times 18) + (576 \times 1) \\ &= 13{,}680 + 13{,}920 + 14{,}160 + 2{,}976 + 9{,}216 + 576 \\ &= 54{,}528. \end{aligned}$$

$$\begin{aligned} \xi . x_7 &= (456 \times 32) + (464 \times 40) + (472 \times 16) + (512 \times 28) \\ &= 14{,}592 + 18{,}560 + 7{,}552 + 14{,}336 \\ &= 55{,}040. \end{aligned}$$

$$\begin{aligned} \xi . x_8 &= (456 \times 18) + (464 \times 36) + (472 \times 26) + (480 \times 11) + (496 \times 2) + (512 \times 22) \\ &= 8{,}208 + 16{,}704 + 12{,}272 + 5{,}280 + 992 + 11{,}264 \\ &= 54{,}720. \end{aligned}$$

TABLE VI

Evaluation of the $\xi . x_i$ $(i = 1, 2, \ldots 8)$.

Sl. No.	w_i	ηy_i
	$\xi . x_1$	
1	24	472
2	12	512
3	12	464
4	24	456
5	24	456
6	12	512
7	6	512
8	2	576
	$\xi . x_2$	
9	4	512
10	8	464
11	16	456
12	16	472
13	16	456
14	16	512
15	32	472
16	8	512
	$\xi . x_3$	
17	4	512
18	8	464
19	4	512
20	8	512
21	4	456
22	4	456
23	8	464
24	8	472
25	4	472
26	4	512
27	4	456
28	16	456
29	4	464
30	4	472
	$\xi . x_3$—*Contd.*	
31	4	496
32	4	464
33	4	472
34	4	464
35	4	472
36	4	464
37	4	456
38	4	512
	$\xi . x_4$	
39	16	464
40	32	456
41	32	456
42	32	512
43	4	576
	$\xi . x_5$	
44	16	456
45	16	464
46	16	472
47	16	464
48	8	456
49	8	456
50	16	512
51	16	512
52	2	576
	$\xi . x_6$	
53	6	472
54	12	456
55	6	464
	$\xi . x_6$—*Contd.*	
56	6	472
57	12	472
58	6	496
59	6	464
60	6	464
61	3	512
62	6	456
63	6	472
64	6	512
65	6	456
66	6	456
67	3	512
68	1	576
69	12	464
70	6	512
	$\xi . x_7$	
71	8	456
72	4	464
73	8	472
74	4	464
75	8	456
76	4	512
77	8	464
78	4	512
79	4	464
80	4	456
81	8	512
82	8	456
83	4	464
84	4	464
85	8	472
86	4	456
87	4	512
88	12	464
89	8	512

TABLE VI—*Contd.*

Sl. No.	w_i	ηy_i	Sl. No.	w_i	ηy_i	Sl. No.	w_i	ηy_i
$\xi . x_8$			$\xi . x_8$—*Contd.*			$\xi . x_8$—*Contd.*		
			102	2	480	115	4	464
90	6	472	103	6	472	116	2	512
91	4	456	104	2	456	117	2	456
92	2	496	105	4	512	118	2	464
93	4	472	106	6	456	119	1	512
94	4	472	107	4	464	120	2	464
95	6	472	108	2	464	121	2	512
96	4	456	109	2	464	122	2	512
97	2	464	110	2	464	123	8	464
98	2	512	111	1	480	124	3	480
99	6	464	112	1	512	125	2	480
100	2	512	113	1	480	126	4	512
101	2	464	114	2	480	127	2	512

Collecting together the values of the $\xi \cdot x_i$ just found, we obtain Table VII as follows:—

TABLE VII

Values of the $\xi . x_i$ $(i = 1, 2, \ldots 8)$

i	1	2	3	4	5	6	7	
$\xi . x_i$	55,296	55,296	55,040	55,296	54,528	54,528	55,040	54,720

We shall now use these values of the $\xi . x_i$ for the evaluation of A_7, C_7 and hence R_7 in the next section.

§9. THE VALUES OF A_7, C_7 AND R_7

From Table VII and equations (4·7) and (4.13), we immediately have:

$$A_7 = \xi . [8x_1 + 12x_2 + 24x_3]$$
$$= 2,426,880.$$

and

$$C_7 = \xi . [3x_4 + 6x_5 + 8x_6 + 12x_7 + 24x_8]$$
$$= 2,903,040.$$

The value of R_7

We have, using equation (4.1), and the values of A_7 and C_7 found above,

$$R_7 = A_7 + 5.C_7$$
$$= 16{,}942{,}080$$

which is the total number of different standard 7×7 Latin squares.

The number of Latin squares reached by Norton was 16,927,968 which falls short of the number reached by us by 14,112. Presumably this is due to one or more *species* having been missed by Norton in the process of enumeration.

§10. SUMMARY

In Part I, MacMahon's formula for enumerating Latin squares was simplified and applied to the enumeration of 6×6 squares. In this part the formula has been applied to the enumeration of Latin squares of order $n = 7$. It is found that there are exactly 16,942,080 different standard 7×7 Latin squares which number is greater by 14,112 than the number previously reached by Norton.

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