UNBIASED RATIO ESTIMATORS

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CONSIDER a simple random sample of k units which is selected either with replacement or if without replacement from a population relatively large enough to let the sample be treated as with replacement and no loss of generality. Each sample unit will be observed for two characteristics (x, y) where y is the main characteristic of interest and x is a correlated supplementary characteristic. Let

$$\tilde{r} = \frac{\bar{y}}{\bar{x}} \text{ and } \bar{r} = \frac{1}{k}\sum_{i=1}^{k}\frac{y_i}{x_i} \tag{1}$$

be the two conventional biased estimates of the population ratio

$$Q = \frac{\bar{Y}}{\bar{X}} \tag{2}$$

Hartley,[1] Pascul,[2] Murty and Nanjamma[3] considered an "almost" unbiased estimate

$$r^* = \frac{k}{k-1}\tilde{r} - \frac{1}{k-1}\bar{r} \tag{3}$$

and investigated its efficiency in relation to $\tilde{r}$. In this paper, the same efficiency is investigated more explicitly and to different degrees of approximation considering that the unbiasedness of r^* is true to a second degree of approximation in terms of what are known as moment coefficients. At the same time an alternate estimator will be introduced in order to provide a completely unbiased estimate under certain special cases of frequent occurrence.

The necessary and sufficient condition for r^* to be more efficient than $\tilde{r}$ is given by

$$\text{Var}(r^*) - \text{Var}(\tilde{r}) < 0.$$

Writing

$$\text{Var}(r^*) = \frac{1}{(k-1)^2}[k^2 \text{Var}(\tilde{r}) + \text{Var}(\bar{r}) - 2k \text{Cov}(\bar{r}-\tilde{r})] \tag{4}$$

this condition can be simplified as

$$(2k - 1)\,\sigma_{\tilde{r}}^{2} + \sigma_{\bar{r}}^{2} - 2k\,\text{Cov}\,(\tilde{r}, \bar{r}) < 0 \tag{5}$$

The author has elsewhere[4] expressed $\sigma_{\tilde{r}}^{2}$ and $\sigma_{\bar{r}}^{2}$ in the following manner:

$$\sigma_{\tilde{r}}^{2} = E(\bar{y} - Q\bar{x})^2\, E\left(\frac{1}{\bar{x}^2}\right) + \text{Cov}\left(\frac{1}{\bar{x}^2}, \overline{\bar{y} - Q\bar{x}^2}\right)$$

$$- \text{Cov}^2\left(\frac{1}{\bar{x}}, \bar{y} - Q\bar{x}\right)$$

and

$$\sigma_{\bar{r}}^{2} = \frac{1}{k}\left[E(y - Qx)^2\, E\left(\frac{1}{x^2}\right) + \text{Cov}\left(\frac{1}{x^2}, \overline{y - Qx^2}\right)\right.$$

$$\left. - \text{Cov}^2\left(\frac{1}{x}, y - Qx\right)\right] \tag{6}$$

In terms of "moment coefficients" C_{rs} defined as

$$C_{rs} = E\,\frac{(x - \bar{X})^r\,(y - \bar{Y})^s}{\bar{X}^r\,\bar{Y}^s}$$

with order $(r + s)$, r^* is an unbiased estimate when expectations are taken up to and including second-order moment coefficients. For this reason it may be useful to investigate what the condition (5) will mean when the variance and covariance terms are expressed upto and including fourth-order moment coefficients. As far the variances at (6) are concerned, the following approximations can be derived from[4] under the assumption

$$\left|\frac{x - \bar{X}}{\bar{X}}\right| < 1 \tag{7}$$

$$\sigma_{\tilde{r}}^{2} \doteqdot \frac{1}{k}\,Q^2\,(1 + \beta - 2\rho\sqrt{\beta})\,C_{20} - \frac{Q^2}{k^2}\,C_{20}^{2}\,(1 - \rho\sqrt{\beta})^2$$

$$- Q^2\left[\frac{2}{k^2}\,(C_{12} - 2C_{21} + C_{30}) - \frac{3}{k^3}\{(C_{22} - 2C_{31} + C_{40})\right.$$

$$\left. + (k - 1)\,C_{20}^{2}\,(\beta + 2\rho^2\beta - 6\rho\sqrt{\beta} + 3)\}\right] \tag{8}$$

and

$$\sigma_{\bar{r}}^2 \doteqdot \frac{1}{k} Q^2 (1 + \beta - 2\rho\sqrt{\beta})\, C_{20} - \frac{Q^2}{k} C_{20}^2 (1 - \rho\sqrt{\beta})^2$$

$$- \frac{Q^2}{k} [2 (C_{12} - 2C_{21} + C_{30}) - 3 (C_{22} - 2C_{31} + C_{40})] \tag{9}$$

where $\beta = C_{02}/C_{20}$ and ρ is the product moment coefficient correlation between x and y given by $\rho^2 = C_{11}^2/C_{20}C_{02}$.

The covariance term in (5) will be now evaluated to the same degree of approximation by using assumption (7).

$$\text{Cov}\,(\tilde{r}, \bar{r}) = \frac{1}{k} \sum \text{Cov}\,(\tilde{r}, r_i) = \text{Cov}\,(\tilde{r}, r),$$

$$\text{Cov}\,(\tilde{r}, r) = E\left(\frac{\bar{y}}{\bar{x}} \cdot \frac{y}{x}\right) - E\left(\frac{\bar{y}}{\bar{x}}\right) E\left(\frac{y}{x}\right)$$

Writing

$$\delta x = \frac{x - \bar{X}}{\bar{X}} \quad \text{and} \quad \delta y = \frac{y - \bar{Y}}{\bar{Y}},$$

$$E\left(\frac{\bar{y}}{\bar{x}}\right) E\left(\frac{y}{x}\right)$$

$$= Q^2 \{1 + (C_{20} - C_{11}) - (C_{30} - C_{21}) + \ldots\}$$
$$\times \{1 + (C'_{20} - C'_{11}) - (C'_{30} - C'_{21}) + \ldots\}.$$

Under assumption (4) where C_{rs}' is a moment coefficient with order $(r + s)$ defined like C_{rs} with $(\bar{x}, \bar{y})$ (*i.e.*),

$$C_{rs}' = \frac{E(\bar{x} - \bar{X})^r (\bar{y} - \bar{Y})^s}{\bar{X}^r \bar{Y}^s}.$$

Retaining moment coefficients up to and including those of the fourth-order in the expansion and expressing C_{rs}' in terms of $C_{rs} - s$, we have

$$E\left(\frac{\bar{y}}{\bar{x}}\right) E\left(\frac{y}{x}\right)$$

$$\doteqdot Q^2 \left\{1 + (C_{20} - C_{11}) \left(1 + \frac{1}{k}\right) - (C_{30} - C_{21}) \left(1 + \frac{1}{k^2}\right)\right.$$

$$+\frac{C_{40}+3(k-1)(C_{20}^2-C_{11}C_{20})-C_{31}}{k^3}$$

$$+\frac{(C_{20}-C_{11})^2}{k}+(C_{40}-C_{31})\Big\} \tag{10}$$

Similarly under assumption (7),

$$E\left(\frac{\bar{y}}{\bar{x}}\cdot\frac{y}{x}\right)$$

$$=Q^2E\,[(1+\delta\bar{y}+\delta y+\delta y\delta\bar{y})\{1-(\delta\bar{x}+\delta x)$$

$$+(\delta\bar{x}^2+\delta x\delta\bar{x}+\delta x^2)-(\delta\bar{x}^3+\delta x\delta\bar{x}^2+\delta x^2\delta\bar{x}+\delta x^3)$$

$$+(\delta\bar{x}^4+\delta\bar{x}^3\delta x+\delta\bar{x}^2\delta x^2+\delta\bar{x}\delta x^3+\delta x^4)\ldots\}] \tag{11}$$

In order to get an approximation in terms of moment coefficients up to the fourth order we only need to consider the expectations of products in δ—s up to the fourth order. While a product involving $(\bar{x}, \bar{y})$ or (x, y) alone directly gives expectations in terms of C_{rs}' or C_{rs} (where C_{rs}' can be as usual expressed in terms of C_{rs} — s), we may proceed in the following manner to obtain expectation of product involving both $(\bar{x}, \bar{y})$ and (x, y):

$$E(\delta y\delta\bar{y})$$

$$=\frac{1}{k}E(\delta y_i^2)+\frac{1}{k}\sum_{j\neq i}(\delta y_j\delta y_i)$$

$$=\frac{C_{02}}{k}.$$

$$E(\delta\bar{x}^2\delta x)$$

$$=\frac{1}{k^2}E\Big(\delta x_i^3+\delta x_i\sum_{j\neq i}\delta x_j^2+\delta x_i^2\sum_{1\neq i}\delta x_1$$

$$+\delta x_i^2\sum_{j\neq i}\delta x_j+\delta x_i\sum_{j\neq 1,i}\delta x_j\delta x_1\Big)$$

$$=\frac{C_{30}}{k^2}.$$

$$E(\delta y_i\delta\bar{x}\delta\bar{y})$$

$$=\frac{1}{k^2}E\Big\{\delta x_i\,\delta y_i^2+\delta y_i\sum_{j\neq 1}\delta x_j\delta y_j+(\delta y_i\delta x_i)\sum_{1\neq i}\delta y_1$$

$$+ \delta y_i^2 \sum_{j \neq} \delta x_j + \delta y_i \sum_{j \neq 1, i} \delta x_j \delta y_1 \Big\}$$

$$\doteq \frac{C_{12}}{k^2}.$$

$E(\delta x^2 \delta \bar{x}^2)$

$$= \frac{1}{k^2} E\Big(\delta x_i^4 + \delta x_i^2 \sum_{j \neq i} \delta x_j^2 + \delta x_i^3 \sum_{1 \neq i} \delta x_1$$

$$+ \delta x_i^3 \sum_{j \neq i} \delta x_j + \delta x_i^2 \sum_{j \neq 1, i} \delta x_j \delta x_1 \Big)$$

$$= \frac{1}{k^2} \{C_{40} + (k-1) C_{20}^2\}.$$

In so far as fourth-order terms are concerned it can be shown similarly,

$$E(\delta x \delta y \delta \bar{x} \delta \bar{y}) = \frac{1}{k^2} \{C_{22} + (k-1) C_{11}^2\}; \; E(\delta y \delta x^2 \delta \bar{y}) = \frac{C_{22}}{k}$$

$E(\delta y \delta \bar{y} \delta \bar{x}^2)$

$$= \frac{1}{k^3} \{C_{22} + (k-1) C_{02} C_{20} + 2(k-1) C_{11}^2\}; \; E(\delta x^3 \delta \bar{x})$$

$$= \frac{C_{40}}{k}.$$

$E(\delta \bar{x}^3 \delta x)$

$$= \frac{1}{k^3} \{C_{40} + 3(k-1) C_{20}^2\}; \; E(\delta x \delta \bar{y} \delta \bar{x}^2) = E(\delta y \delta \bar{x}^3)$$

$$= \frac{1}{k^3} \{C_{31} + 3(k-1) C_{20} C_{11}\}$$

$$E(\delta x^2 \delta \bar{x} \delta \bar{y}) = \frac{1}{k^2} \{C_{31} + (k-1) C_{20} C_{11}\} = E(\delta \bar{x}^2 \delta x \delta y)\}$$

$$E(\delta x^2 \delta y \delta \bar{x}) = E(\delta \bar{y} \delta x^3) = \frac{C_{31}}{k}.$$

Inserting these expressions into the fourth-order expansion of (11) and subtracting therefrom the approximated product of expectations as given at (10), Cov $(\tilde{r}, \bar{r})$ can be written to the fourth order of approximation in moment coefficients as

$$\begin{aligned}
&\operatorname{Cov}(\tilde{r}, \bar{r}) \\
&\quad \doteq Q^2 \Bigg[\frac{C_{20}}{k} (1 + \beta - 2\rho \sqrt{\beta}) - \left(\frac{1}{k} + \frac{1}{k^2} \right) \\
&\qquad \times (C_{12} - 2C_{21} + C_{30}) + \left(\frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} \right) \\
&\qquad \times (C_{40} - 2C_{31} + C_{22}) + C_{20}^2 \left\{ \left(\frac{2}{k^2} - \frac{3}{k^3} \right) (1 - \rho \sqrt{\beta})^2 \right. \\
&\qquad \left. + \left(\frac{1}{k^2} - \frac{1}{k^3} \right) \beta (1 - \rho^2) \right\} \Bigg] \qquad (12)
\end{aligned}$$

Substituting from (8), (9) and (12) in (5), the condition can now be written in terms of moment coefficient as

$$\begin{aligned}
&-\frac{(k-1)^2}{k^3} \{2k (C_{12} - 2C_{21} + C_{30}) + (2k + 3) \\
&\qquad \times (C_{40} - 2C_{31} + C_{22})\} \\
&-\frac{(k-1)}{k^3} C_{20}^2 \{-(4k - 3)\beta - (7k - 6)\rho^2\beta \\
&\qquad + (11k - 9)(2\rho\sqrt{\beta} - 1)\} < 0. \qquad (13)
\end{aligned}$$

Defining

$$\mu_1 = \frac{1}{\bar{X}\bar{Y}^2} \operatorname{Cov} \{x, (y - Qx)^2\}$$

and

$$\mu_2 = \frac{1}{\bar{X}^2\bar{Y}^2} \operatorname{Cov} \{x^2, (y - Qx)^2\}$$

it can be shown that

$$C_{12} - 2C_{21} + C_{30} = \mu_1$$

and

$$C_{40} - 2C_{31} + C_{22} = \mu_2 - 2\mu_1 + C_{20}^2 (1 + \beta - 2\rho\sqrt{\beta}).$$

Using these relations (13) can be simplified as

$$\begin{aligned}
&\frac{(k-1)^2}{k^3} \{2(k + 3)\mu_1 - (2k + 3)\mu_2\} \\
&\qquad + \frac{k-1}{k^3} C_{20}^2 (ak^2 + bk + c) < 0 \qquad (14)
\end{aligned}$$

where

$$\left.\begin{aligned} a &= -2\,\{(1-\rho\sqrt{\beta})^2 + \beta\,(1-\rho^2)\} < 0 \\ b &= 10\,\{(1-\rho\sqrt{\beta})^2 + \beta\,(1-\rho^2)\} + 7\,(\rho^2 - \beta) \\ \text{and} \quad c &= -6\,(1-\rho\sqrt{\beta})^2. \end{aligned}\right\} \tag{15}$$

μ_1 and μ_2 are likely to be small positive constants of a negligible order in most of the cases where a ratio estimate will be usually considered. Between the two μ_2 is likely to be of a higher order and if we assume μ_2 as at least twice μ_1 in any case, the first term in (14) can be seen as negative so that the condition can be generally reduced as

$$ak^2 + bk + c < 0. \tag{16}$$

Since the coefficient a is always negative, this will mean that the sample size k should be outside (k_0, k_1) which are the roots of the quadratic if the unbiased ratio estimate r^* is to be more efficient than the conventional ratio estimate $\bar{r}$. For the usual values of ρ and $\sqrt{\beta}$ that we normally come across the lower root seems to be very near the zero value or negative so that the higher root should be considered as the lower limit for k. From the values of that higher root tabulated below for a range of $(\rho, \sqrt{\beta})$ combinations that we normally come across, it is seen that the condition is almost invariably satisfied since the samples are always at least that large. It may be thus concluded that to the degree of approximation considered and under situations when a ratio estimate is normally considered, the unbiased estimate generally turns out to be more efficient than the conventional biased ratio estimate.

Integral Part in the Higher Root k_1, of the Quadratic

$$ak^2 + bk + c = 0$$

$\sqrt{\beta}$	ρ = 0·5	0·7	0·9
0·7	3	4	9
1·0	2	1	1
1·5	1	0	0

Substituting in (4) the expressions for Var $(\tilde{r})$ Var $(\bar{r})$ and Cov $(\tilde{r}, \bar{r})$ and after simplification to the fourth degree of approximation it can be shown that

$$\text{Var } (r^*)$$

$$= \frac{Q^2}{k} C_{20} (1 + \beta - 2\rho \sqrt{\beta})$$

$$+ \frac{Q^2}{k^2} \left[- 2\mu_2 + \frac{C_{20}^2}{(k-1)} \right.$$

$$\left. \times \{2(3k-2)(1 - \rho\sqrt{\beta})^2 + k\beta(1 - \rho^2)\} \right] \tag{17}$$

Alternate unbiased ratio estimate.—We shall now consider the case when the regression of y on x is linear and y has a constant variance for any given x; the regression however does not necessarily pass through origin (*i.e.*) the regression coefficient B is not necessarily equal to Q. In this case the expected values of the two conventional ratio estimates $\tilde{r}$ and $\bar{r}$ including expressions for bias can be written as (4),

$$E(\tilde{r}) = Q + (Q - B)\left(\frac{\bar{X}}{\bar{X}_h'} - 1\right) \tag{18}$$

and

$$E(\bar{r}) = Q + (Q - B)\left(\frac{\bar{X}}{\bar{X}_h} - 1\right) \tag{19}$$

when $\bar{X}_h'$ and $\bar{X}_h$ are the harmonic means of $\bar{x}$ and x.

Multiplying both sides of (18) by $(\bar{X}/\bar{X}_h - 1)$, both sides of (19) by $(\bar{X}/\bar{X}_h' - 1)$ and subtracting the latter from the former, we shall have

$$E\left\{\frac{\left(\frac{1}{\bar{X}_h} - \frac{1}{\bar{X}}\right)\tilde{r} - \left(\frac{1}{\bar{X}_h'} - \frac{1}{\bar{X}}\right)\bar{r}}{\left(\frac{1}{\bar{X}_h} - \frac{1}{\bar{X}_h'}\right)}\right\} = Q \tag{20}$$

Assuming knowledge of $\bar{X}$ (which may not be always available) we shall define an estimate $\hat{r}$ taking the clue from (20) as

$$\hat{r} = \frac{\left(\frac{1}{\bar{x}_h} - \frac{1}{\bar{X}}\right)\tilde{r} - \left(\frac{1}{\bar{x}} - \frac{1}{\bar{X}}\right)\bar{r}}{\left(\frac{1}{\bar{x}_h} - \frac{1}{\bar{x}}\right)} \tag{21}$$

where $\bar{x}$ and $\bar{x}_h$ are the usual arithmetic and harmonic means from a simple random sample of k observations. We shall show that this is a completely unbiased estimate of Q. According to our assumption of linear regression of y on x

$$E(y|\,x) = (\bar{Y} - B\bar{X}) + Bx$$

so that

$$E\left(\frac{y}{x}\,\Big|\,x\right) = \frac{\bar{Y} - B\bar{X}}{x} + B \tag{22}$$

and

$$E(\bar{r}\mid x) = \frac{\bar{Y} - B\bar{X}}{\bar{x}_h} + B.$$

Similarly

$$E(\bar{y}\mid x) = (\bar{Y} - B\bar{X}) + B\bar{x}$$

so that

$$E(\tilde{r}\mid x) = \frac{\bar{Y} - B\bar{X}}{\bar{x}} + B \tag{23}$$

$$\begin{aligned} E(\hat{r}) &= EE(\hat{r}\mid x) \\ &= E\left(\frac{1}{\bar{x}_h} - \frac{1}{\bar{x}}\right)\left\{\left(\frac{1}{\bar{x}_h} - \frac{1}{\bar{X}}\right)\left(\frac{\bar{Y} - B\bar{X}}{\bar{x}} + B\right)\right. \\ &\qquad \left. - \left(\frac{1}{\bar{x}} - \frac{1}{\bar{X}}\right)\left(\frac{\bar{Y} - B\bar{X}}{\bar{x}_h} + B\right)\right\} = Q. \end{aligned}$$

We shall obtain the variance of $\hat{r}$ by using the approach

$$\text{Var}(\hat{r}) = E[\text{var}(\hat{r})\mid x] + V[E(\hat{r}\mid x)] \tag{24}$$

Since $E(\hat{r}\mid x) = Q$, the second expression in the above formula will be zero. Writing

$$\xi = \left(\frac{1}{\bar{x}_h} - \frac{1}{\bar{X}}\right) \quad \text{and} \quad \eta = \left(\frac{1}{\bar{x}} - \frac{1}{\bar{X}}\right)$$

$$V(\hat{r}\mid x) = \frac{\xi^2\, V(\tilde{r}\mid x) + \eta^2\, V(\bar{r}\mid x) - 2\xi\eta\, \text{Cov}(\tilde{r}, \bar{r}\mid x)}{(\xi - \eta)^2} \tag{25}$$

From (23), we have

$$\tilde{r} - E(\tilde{r} \mid x) = \frac{1}{\bar{x}}\{(\bar{y} - \bar{Y}) - B(\bar{x} - \bar{X})\}.$$

Again according to the basic assumption

$$E[\{(y - \bar{Y}) - B(x - \bar{X})\}^2 \mid x) = Sy^2$$

so that

$$V(\tilde{r} \mid x) = E[\{\tilde{r} - E(\tilde{r} \mid x)\}^2 \mid x]$$

$$= \frac{1}{\bar{x}^2} E[\{\bar{y} - \bar{Y}) - B(\bar{x} - \bar{X})\}^2 \mid x]$$

$$= \frac{1}{\bar{x}^2} \frac{Sy^2}{k} \tag{26}$$

On the other hand from (22) we have

$$\frac{y}{x} - E\left(\frac{y}{x} \middle| x\right) = \frac{1}{x}\{(y - \bar{Y}) - B(x - \bar{X})\}$$

so that

$$V\left(\frac{y}{x} \middle| x\right) = \frac{1}{x^2} Sy^2.$$

and

$$V(\bar{r} \mid x) = \frac{Sy^2}{k} \frac{1}{w_h} \tag{27}$$

where w_h is the harmonic average of x^2 in the sample.

$$\text{Cov } \{(\tilde{r}, \bar{r}) \mid x\}$$

$$= \frac{1}{k} \sum E\left[\frac{1}{x_i}\{(y_i - \bar{Y}) - B(x_i - \bar{X})\}\right.$$

$$\left.\times \frac{1}{\bar{x}} \frac{1}{k} \sum_i \{(y_i - \bar{Y}) - B(x_i - \bar{X})\}\right]$$

$$= \frac{1}{k} \sum \frac{Sy^2}{k\bar{x}x_i} = \frac{Sy^2}{k\bar{x}\bar{x}_h} \tag{28}$$

assuming sampling with replacement or individual sample values as independent.

Substituting in (25) from (26), (27) and (28) we have

$$V(\hat{r} \mid x) = \frac{Sy^2}{k}\left\{\frac{1}{\bar{X}^2} + \frac{\left(\frac{1}{\bar{x}} - \frac{1}{\bar{X}}\right)^2}{\left(\frac{1}{\bar{x}} - \frac{1}{\bar{x}_h}\right)^2}\left(\frac{1}{w_h} - \frac{1}{\bar{x}_h{}^2}\right)\right\} \tag{29}$$

we may now notice that

$$\frac{1}{w_h} - \frac{1}{\bar{x}_h{}^2} = \frac{1}{k}\left(\sum \frac{1}{x_i{}^2} - \frac{k}{\bar{x}_h{}^2}\right)$$

$$= \frac{k-1}{k} \text{ (sample mean square of } 1/x)$$

$$= \frac{k-1}{k} S_{1/x}{}^2 = \frac{1}{\bar{x}_h{}^2} C_{1/x}{}^2 \tag{30}$$

where $C_{1/x}{}^2$ is the sample relative variance of $1/x$.

Using (30), (29) can be rewritten as

$$V(r \mid x) = \frac{Sy^2}{k\bar{X}^2}\left\{1 + \frac{(\bar{x} - \bar{X})^2}{(\bar{x} - \bar{X}_h)^2} C_{1/x}{}^2\right\} \tag{31}$$

so that from (24) we can get

$$V(r) = \frac{Sy^2}{k\bar{X}^2}\left[1 + E\left\{\frac{(\bar{x} - \bar{X})^2}{(\bar{x} - \bar{X}_h)^2} C^2{}_{1/x}\right\}\right] \tag{32}$$

whereas a sample estimate of (32) can be computed from (31) with Sy^2 replaced by its sample estimate

$$sy^2 = \frac{1}{(k-2)} \sum_i \{(y_i - \bar{y}) - b_i (x - \bar{x})\}^2.$$

References

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