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Optimal Nested Block Designs

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Summary

This article studies nested block designs with unequal block sizes. The numbers of subblocks and the subblock sizes within each block are unequal. The recovery of treatment information from the subblocks within the blocks, or from the blocks themselves is not considered in this article. The optimality of nested block designs with unequal block sizes is also studied. It is seen that a nested, binary subblocks balanced design is universally optimal over a class of designs with fixed numbers of treatments, blocks and subblocks within a block.

Key Words : Universal optimality, Balanced design.

Introduction

The use of incomplete block designs is widely recognized in many fields of experimentation. With some types of experimental material, however, there may be more sources of variation than can be eliminated by ordinary block designs. For such situations, Preece [13] introduced nested balanced incomplete block designs, where sub-blocks are nested within each block of a balanced incomplete block design, the sub-blocks constituting another balanced incomplete block design. Homel and Robinson [7] extended the results of Preece to nested partially balanced incomplete block designs. Singh and Dey [15] defined and discussed balanced incomplete block designs with nested rows and columns for eliminating heterogeneity in two directions within each block. Preece and Cameron [14], Singh and Dey [15], Agrawal and Prasad [1][2], Cheng [4], Dey, *et al* [5] and Sreenath [16] gave some methods of construction of balanced incomplete block designs with nested rows and columns.

The results obtained by earlier workers are restricted to a situation when the block design is proper and the common value of the block sizes is a composite number. In animal experiments where litters form natural blocks, the assumption of a common block size is unrealistic. Moreover, the common value of the block size may be a prime number. The purpose of this article is to study nested block designs with unequal block sizes. The present study is, however, restricted to a situation when there is only one more additional source of variability within blocks. In other words, we are considering a setting where within the blocks of a block design is nested another incomplete block design. The universal optimality of block designs with nested blocks is also investigated. Some universally optimal designs are reported. For nested block designs with different block sizes and different sub-block sizes, however, there is no valid randomisation procedure and Nelder's [11][12] randomisation theory does not apply.

2. *Analysis of the Designs*

Suppose that v treatments are arranged in a nested block design involving b blocks, there being q_j mutually exclusive and exhaustive sub-blocks within the j^{th} block, j=1, . . ., b, so that $b_1 = \Sigma\ q_j$ is the overall total number of sub-blocks. Let $\mathbf{N}=(n_{ij})$ be the vxb treatments-blocks incidence matrix, where n_{ij} denotes the number of replications of the i^{th} treatment in the j^{th} block, i=1, . . ., v. The row sums of $\mathbf{N}$ are denoted by $r = (r_1, \ldots, r_v)'$ and the column sum by $\mathbf{k} = (k_1, \ldots, k_b)'$, where r_i and k_j denote respectively the replication number of the i^{th} treatment and the size of the j^{th} block. Also $r_1 + \ldots + r_v = k_1 + \ldots + k_b = n$, the total number of experimental units. Let $\mathbf{M} = (m_{ij'(j)})$ denote the vxb_1 treatments sub-blocks matrix, where $m_{ij'(j)}$ denotes the replication number of the i^{th} treatment in the j'^{th} sub-block nested within the j^{th} block, $j' = 1, \ldots, q_j$. The row sums of $\mathbf{M}$ are the elements of r while its column sums are the elements of the b_1x1 vector $\mathbf{h} = (\mathbf{h}'_{(1)}, \ldots \mathbf{h}'_{(b)})'$, where $\mathbf{h}'_{(j)} = (h_{1(j)}, \ldots, h_{q_j(j)})$, $\mathbf{1}'\mathbf{h}_{(j)} = k_j$ and $\mathbf{1}_t$ is a tx1 vector of ones. Here $h_{j'(j)}$ denotes the size of the j'^{th} sub-block nested in the j^{th} block. Let $\mathbf{R}$, $\mathbf{H}_j$ and $\mathbf{H}$ denote respectively the diagonal matrices whose diagonal elements are the successive elements of $\mathbf{r}$, $\mathbf{h}_j$ and $\mathbf{h}$. Let $\mathbf{W}$ be the bxb_1 blocks-sub-blocks incidence matrix.

For analysis, consider the model

$$y_{ij'(j)u} = \mu + \tau_i + \beta_j + \eta_{j'(j)} + e_{ij'(j)u} \quad (2.1)$$

where $y_{ij'(j)u}$ is the u^{th} observation obtained from the i^{th} treatment in the j'^{th} sub-block of the j^{th} block, $u = 1, \ldots, m_{ij'(j)}$, μ is the general mean, τ_i is the i^{th} treatment effect, β_j is the j^{th} block effect, $\eta_{j'(j)}$ is the effect of the j'^{th} sub-block nested within the j^{th} block, and the quantities $e_{ij'(j)u}$ are uncorrelated errors with mean zero and common variance σ^2. In this article we do not consider recovery of treatment information from the sub-blocks within the blocks, or from the blocks themselves. The observations are assumed to be arranged in the order of (j, j'). Then $w = \Sigma^+ \mathbf{h}'_j$, where Σ^+ denotes the direct sum. Defining a $b_1 x b$ matrix $\mathbf{L} = \Sigma^+ \mathbf{1}_{q_j}$, the following relations are true:

$$\mathbf{N} = \mathbf{ML}, \quad \mathbf{K} = \mathbf{L}' \mathbf{HL}, \quad \mathbf{W} = \mathbf{L}'\mathbf{H}, \quad \mathbf{n} = \mathbf{1}'_{b_1} \mathbf{H1}_{b_1}, \quad \mathbf{k} = \mathbf{L}' \mathbf{H1}_{b_1}$$

Using least squares, the co-efficient matrix of the reduced normal equations for obtaining best linear unbiased estimates of treatment contrasts is

$$\mathbf{F} = \mathbf{R} - \mathbf{M} \mathbf{H}^{-1} \mathbf{M}' \qquad (2.2)$$

The matrix $\mathbf{F}$ is the same as the usual C matrix that is obtained if blocks are ignored and the design is analysed treating sub-blocks as blocks. Therefore, in so far as estimation of treatment effects is considered, it is only the sub-blocks structure that matters.

The vxv matrix $\mathbf{F}$ is symmetric, non-negative definite with zero row sums. For a connected nested block design the rank of $\mathbf{F}$ is v-1. A connected nested block design is variance balanced if and only if

$$\mathbf{F} = \theta \left(\mathbf{I_v} - \frac{\mathbf{11}'}{v} \right) \qquad (2.3)$$

where θ, a scalar constant, is the unique positive eigenvalue of $\mathbf{F}$ with multiplicity v-1 and $\mathbf{I}_v$ is an identity matrix of order v.

Example 2.1 : The following is a nested variance balanced design with parameters v = 6, b =9, b_1 = 18, k = $(6\mathbf{1}'_3, 5\mathbf{1}'_6)'$, $\mathbf{h} = (3\mathbf{1}'_7$, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2)'.

{(1,2,3), (2,4,6)},	{(1,4,5), (3,5,6)},	{(1,4,5), (3,5,6)},
{(1,2,3), (1,6)},	{(1,4,5), (2,5)},	{(2,4,6), (3,4)},
{(1,2,3), (2,5)},	{(2,4,6), (1,6)},	{(3,5,6), (3,4)}.

3. *Optimality Tool*

Suppose D is a class of competing designs under the specified design parameters. Let Φ, a functional of F_d for d ε D, denote an optimality criterion. A design $d^* \in D$ is said to be Φ-optimal over D if $\Phi(F_{d^*}) \leq \Phi(F_d)$ for all $d \in D$. Kiefer [10] proposed a strong optimality criterion, termed universal optimality, which includes many optimality criteria including *D*-, *E*- and *A*- optimality and many others. Kiefer's definition of universal optimality and the sufficient condition for achieving universal optimality are stated below.

Definition 3.1 : Let $\mathcal{B}_{v,0}$ be the class of all the vxv symmetric nonnegative-definite matrices with zero row sums. Suppose a function Φ from $\mathcal{B}_{v,0}$ to $(-\infty, +\infty]$ is such that

(a) Φ is matrix convex.

(b) For any $F \in \mathcal{B}_{v,0}$, $\Phi(aF)$ is nonincreasing in the scalar $a \geq 0$.

(c) Φ is invariant under each simultaneous permutation of rows and columns of F in $\mathcal{B}_{v,0}$.

A design $d^* \in D$ is said to be universally optimal over D if d^* is Φ-optimal over D for all Φ satisfying (a), (b) and (c).

Theorem 3.1 (Kiefer, 1975) : Suppose a class $\mathcal{F} = \{F_d : d \in D\}$ of matrices in $\mathcal{B}_{v,0}$ contains an F_{d^*} for which

(i) F_{d^*} is completely symmetric, *i.e.*, F_{d^*} is of the form $a\mathbf{I}_v + b\mathbf{1}_v\mathbf{1}_v'$, where a and b are scalars.

(ii) $\text{trace}(\mathbf{F}_{d^*}) = \max_{d \in D} \text{trace}(\mathbf{F}_d)$.

Then d^* is universally optimal over D. Definition 3.1 and Theorem 3.1 hold for all experimental settings where vxv information matrices are symmetric, nonnegative-definite with zero row sums and rank v−1. For example, this result is applicable to all block designs, nested block designs, designs for two-way elimination of heterogeneity, multi-way heterogeneity designs, etc.

4. *The Main Result*

We now prove the universal optimality of nested block designs under the model (2.1). Before giving the main result we have the following definition:

Definition 4.1 : A block design with nested blocks is binary if $m_{ij'(j)} = 0$ or 1 for all i, j' and j. It is not, however, required that n_{ij} should also take only the two values 0 or 1.

Let $D(v; q_1, \ldots, q_b; n)$ denote the class of all connected nested block designs with b blocks of sub-blocks $q_1, \ldots, q_b$, v treatments and total number of observations n. The replication numbers of treatments, the block sizes and the sub-block sizes are fixed for designs belonging to this class. Throughout, the suffix d will denote a design in D.

Definition 4.2 : A design $d \in D\ (v;\ q_1, \ldots, q_b\ ;\ n)$ is called a nested binary sub-blocks balanced design if

(i) $$\left| m_{dij'(j)} - \frac{h_{dj'(j)}}{v} \right| < 1 \text{ for all } i=1, \ldots, v,\ j'=1, \ldots, q_j,\ j = 1 \ldots, b .$$

(ii) $$\frac{\sum_{j=1}^{b} \sum_{j'=1}^{q_j} m_{dij'(j)}\, m_{di'j'(j)}}{h_{dj'(j)}} = \lambda, \text{ a constant for all } i \neq i'.$$

Condition *(i)* above implies that the frequency of treatments appearing in any subblock should differ by at most one. This condition enables us to prove the universal optimality of the nested binary subblocks balanced designs. Condition *(ii)* ensures that all the off-diagonal elements of the F-matrix of a nested block design are equal and the design is variance balanced. We have the following theorem:

Theorem 4.1 : A nested binary sub-blocks balanced design d^*, whenever existent, is universally optimal over $D(v;\ q_1, \ldots, q_b;\ n)$.

Proof : To prove this theorem we appeal to Theorem 3.1. Complete symmetry of F_{d^*} is ensured by *(ii)* of Definition 4.2 using the fact that the row sums of $\mathbf{F_d}$ are zero. Further

$$\text{trace } \mathbf{F_d} = n - \sum_i \sum_j \sum_{j'} \frac{m^2_{dij'(j)}}{h_{dj'(j)}} \leq n - b_1 = \text{trace } \mathbf{F_d}^*,$$

using the fact that

$$\sum_{i=1}^{v} m^2_{dij'(j)} \geq \sum_{i=1}^{v} m_{dij'(j)} = h_{dj'(j)} .$$

The proof is thus complete.

Remark 4.1 Using theorem 4.1 it is a established that the nested balanced incomplete block designs of Preece [13] are universally optimal over $D(v;b_1;n)$.

The construction of nested binary sub-blocks balanced designs does not pose any problems. There are many binary, balanced block designs available in the literature (c.f. Kageyama, [8]; Khatri, [9]; Gupta and Jones, [6]; Calvin, [3]). Using these designs it is easy to construct nested binary sub-blocks balanced designs which are universally optimal over $D(v;\ q_1, \ldots, q_b;\ n)$.

Example 4.1 : The following design is nested binary sub-blocks balanced and is universally optimal over D(6; $21'_4$, 3; 26):

{(1,2,3,4), (5,6)}, {(1,2,3,4), (1,5)}, {(1,6), (2,5)},

{(2,6), (3,5)}, {(3,6), (4,5), (4,6)}.

Example 4.2 : The following is a nested binary sub-blocks balanced design and is universally optimal over D(8; $21'_3$, $31'_4$; 40):

{(1,3,5,7), (4,7)}, {(2,4,6,8), (3,8)}, {(4,5), (2,7)}, {(1,2),

(3,4), (5,6)}, {(7,8), (1,6), (2,5)} {(1,4), (3,2), (5,8)},

{(7,6), (8,1), (6,3)}.

Example 4.3 : The following is a nested binary sub-blocks balanced design and is universally optimal over D(9; $21'_5$, $31'_6$; 64):

{(1,2,3,4), (5,9)}, {(1,2,3,4), (7,9)}, {5,6,7,8), (1,9)},

{(5,6,7,8), (2,9)}, {(3,8), (4,7)}, {(1,5), (3,9), (2,6)}, {(4,9),

(1,6), (2,5)}, {(6,9), (1,7), (4,8)}, {(8,9), (2,7), (3,5)},

{(1,8), (3,6), (4,5)}, {(2,8), (3,7), (4,6)}.

5. *Optimality with Correlated Observations*

We have so far discussed the optimality of nested block designs under the assumption that the observations are independent with a common variance σ^2. However, because of nested sub-blocks it is logical to assume that the observations within sub-blocks are correlated, although observations from different sub-blocks within a block as well as observations from different blocks may be assumed to be uncorrelated. We therefore assume once again the model in (2.1) with

$$D(e) = \sigma^2 \sum_{j=1}^{b}{}^{+} A_j \tag{5.1}$$

and

$$A_j = \sum_{j=1}^{q_j}{}^{+} (1-\rho)\, I_{h_{j'(j)}} + \rho \mathbf{11}'$$

where D(e) is the covariance matrix of the residuals. It is also assumed that A_j is non-singular and $-1 \le \rho \le 1$. Under this set-up the co-efficient matrix of the reduced normal equations for obtaining the generalized least squares estimates of linear functions of treatment effects is $F_d^c = (1-\rho) F_d$, where F_d, as given in (2.2), is the co-efficient matrix of the reduced normal equations for obtaining the least squares estimates of linear functions of treatment effects under the usual additive, homoscedastic model. It therefore follows that a design which is universally optimal over D(v; $q_1, \ldots, q_b$; n) under the usual homoscedastic model is also universally optimal over D(v; $q_1, \ldots, q_b$; n) under the heteroscedastic setting (5.1).

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