A NOTE ON THE CONSTRUCTION OF NEIGHBOUR DESIGNS

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1. Introduction

Rees [3] introduced the concept of neighbour designs, which are of use mainly in Serology.

A neighbour design may be defined as an arrangement of v antigens (called symbols) in b circular plates (called blocks) such that:

(i) each block has k symbols, not necessarily distinct,

(ii) each symbol appears r times in the design, and

(iii) each symbol is a 'neighbour' of every other symbol precisely λ times.

A neighbour design with at least one block having less than v distinct symbols may be called an incomplete block neighbour design. Methods of construction of incomplete block neighbour designs of constant block size have been discussed by Rees [3], Hwang [2] and Dey and Chakravarty [1].

2. Method of Construction

The following is a method of construction of neighbour designs suggested in [1]. This method may be called the method of symmetrically repeated differences.

Consider a module of m elements. To each element of the module let there correspond n elements (or symbols) and let the symbols corresponding to the u-th element of the module be denoted by u_i, $i = 1, 2, \ldots, n$. Thus we have exactly mn symbols. Symbols with the same suffix j will be said to belong to the j-th class.

Let a block s contain p_i symbols from the j-th class, denoted by $a_j(1)$, $a_j(2)$, ... $a_j(p_i)$, $j = 1,2, \ldots, n$. Then a difference of the form $a_j(u) - a_j(v)$, $u \neq v$, is said to be a 'pure forward'difference of the type(j,j) arising from S if $a_j(u)$ comes immediately after $a_j(v)$ in S. $a_j(v) - a_j(u)$ in this case is said to be a 'pure backward' difference of type (j, j). Similarly we define $a_i(u) - a_j(v)$, $i \neq j$ as a 'mixed forward (or backward)' difference of the type (i,j), depending upon whether $a_i(u)$ comes just after (or before) $a_j(v)$ in S.

Theorem 2.1. Consider a set of t basic blocks, each containing k distinct symbols, satisfying the following conditions:

(*i*) among the totality of pure forward and pure backward differences of the type (i, i) arising from the t basic blocks, each non-zero element of the module occurs equally frequently, say λ times each, for all i;

(*ii*) among the totality of mixed forward and mixed backward differences of the type (i, j) arising from the t basic blocks, every element of the module appears λ times each, for all i and j;

(*iii*) among the kt symbols occurring in the t basic blocks, exactly r symbols belong to each of the n classes.

Then by developing these basic blocks we get a neighbour design with parameters $v = mn$, $b = mt$, $r = k^t/n$, k, λ.

We now prove the following theorem.

Theorem 2.2. Let T be an indexing set having n elements, denoted by 0,1,2, . . . , (n-1). Then we have $n(n\text{-}1)$ ordered pairs of elements of T of the form (i, j). Let A be the set of non-trivial divisors of $n(n\text{-}1)$ and let B be the set $B = \{2n+1 : n > 0$ an integer$\}$. Then we can always construct a neighbour design with block size k and $\lambda = 2$, where $(k+1) \in (1+A) \cap B$, $(1+A)$ denoting the direct sum of the element 1 and the set A.

Proof. From the ordered pairs (i,j) of elements of T we can form triplets of the from (iji), $j > i$ and for any $k \in A \cap B$ we can form a chain $(i\, j\, i\, j\, i \ldots)$ of size $(k+1)$ corresponding to each such triplet. Consider also the 'dual-chains' $(j\, i\, j\, i\, j \ldots)$ of size $(k+1)$ corresponding to each such chain, which is obtained by interchanging i and j in the original chain. Let us call the set of chains and 'dual chains' as the array C.

Now when $(k+1) \in (1+A) \cap B$, we can form a set of chains of size $(k+1)$ such that in the set of chains each ordered pair of elements of T occur exactly once as neighbours and such that the initial and final symbols of each chain are identical. When $n = (k+1)$ we can construct this set by developing the chain (0 1 3 6...0) in mod (n) and when $n \neq k+1$, we can always construct this set by trial and error method. Let us call the set of chains as the array D.

Now form another array E of chains of size $(k+1)$ where the $(i+1)$th chain of the array is $(i\ i \dots i)$, $i=0, 1, 2, \dots, (n-1)$.

Let $d_1, d_2, \dots, d_k$ be the non-zero elements of mod $(k+1)$, in some order. Form a block

$$(0, d_1, d_1+d_2, \dots, d_1+d_2+\dots+d_k)$$

where the elements are reduced mod $(k+1)$. Since $(k+1)$ is odd, $\sum_{i=1}^{k} d_i=0$ always so that the first and last elements of the above block are 0.

Corresponding to each chain, say, $(f_1 f_2 \dots f_{k+1})$ of C and E form blocks

$$[0_{f_1}, (d_1)_{f_2}, (d_1+d_2)_{f_3}, \dots, (d_1+\dots+d_{k-1})_{f_k}\ 0_{f_{k+1}}]$$

and corresponding to each chain $(g_1\, g_2 \dots g_{k+1})$ of D form blocks $(0g_1, 0g_2, \dots, 0g_{k+1})$.

Then these blocks corresponding to the chains of C, D and E form a family of basic blocks for construction of a neighbour design.

For, it is to be noted that since $(k+1)$ is odd, every chain of C have the same initial and final symbols. It is also to be noted that in C any ordered pair (i, j) occurs in all possible neighbouring positions of the chains. Thus the basic blocks formed from C and D are such that among the mixed difference arrising from these all the elements of mod $(k+1)$ occur each exactly twice and among the pure differences arising from tne basic blocks formed from E all the non-zero elements of mod $(k+1)$ occur each exactly twice. The nature of the arrays C, D and E are such that the elements of T occur equally frequently in the first k positions of each of these arrays, when their chains are arranged in rows. Thus by ommitting the last symbols of each of the basic blocks and then developing these in mod $(k+1)$ we get the design. Hence the proof.

Example : Let us illustrate the method of construction by an example to clarify the procedure.

Let the indexing set $T=(0, 1, 2, 3)$. Hence $n=4$. Therefore, $A=(2, 3, 4, 6)$, $B=(3, 5, 7, 9, 11, \dots)$, so that $(1+A)=(3, 4, 5, 7)$ and $(3, 5, 7)=(1+A) \cap B$. Hence, in this case we can construct neighbour designs with block sizes $k=2$, 4, and 6. Let us construct a neighbour design with $k=4$. The corresponding arrays C, D and E of elements of T are :

$$C=\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 2 \\ 0 & 3 & 0 & 3 & 0 \\ 3 & 0 & 3 & 0 & 3 \\ 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 1 & 3 & 1 & 3 & 1 \\ 3 & 1 & 3 & 1 & 3 \\ 2 & 3 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 3 \end{bmatrix}, \quad D=\begin{matrix} 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 3 & 1 & 3 & 0 \end{matrix}, \quad E=\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{matrix}.$$

The non-zero elements $d_1, d_2, \ldots, d_k$ of mod $(k+1)$ are 1, 2, 4, 3 from which we get a block (0 1 3 2 0) corresponding to $(0, d_1, d_1+d_2+\ldots+d_k)$. Thus basic blocks generating a neighbour design are :

$$\begin{array}{lll} (0_0\ 1_1\ 3_0\ 2_1), & (0_2\ 1_1\ 3_2\ 2_1), & (0_0\ 0_3\ 0_1\ 0_3), \\ (0_1\ 1_0\ 3_1\ 2_0), & (0_1\ 1_3\ 3_1\ 2_3), & (0_0\ 1_0\ 3_0\ 2_0), \\ (0_0\ 1_2\ 3_0\ 2_2), & (0_3\ 1_1\ 3_3\ 2_1), & (0_1\ 1_1\ 3_1\ 2_1), \\ (0_2\ 1_0\ 3_2\ 2_0), & (0_2\ 1_3\ 3_2\ 2_3), & (0_2\ 1_2\ 3_2\ 2_2), \\ (0_0\ 1_3\ 3_0\ 2_3), & (0_3\ 1_2\ 3_3\ 2_2), & (0_3\ 1_3\ 3_3\ 2_3), \\ (0_3\ 1_0\ 3_3\ 2_0), & (0_0\ 0_1\ 0_2\ 0_1), & \\ (0_1\ 1_2\ 3_1\ 2_2), & (0_0\ 0_2\ 0_3\ 0_2), & \end{array}$$

Developing each of these blocks in mod (5) we get a neighbour design with parameters $v=20$, $b=19\times 5$, $r=19$, $k=4$, $\lambda=2$.

CONCLUSION

One Method of obtaining a class of basic blocks satisfying properties (*i*), (*ii*) and (*iii*) of Theorem 2.1 is given.

REFERENCES

[1] Dey, A and Chakravarty Rupak (1977) : "On the construction of some classes of neighbour designs", *Jour. Ind. Soc. Agrl. Stat.*, Vol. 29, 97-104.

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