Jour. Ind. Soc. Ag. Statistics
Vol. LX, No. 1 (1988), pp. 1-8

SOME RESULTS ON T_1-CLASS OF LINEAR ESTIMATORS

PULAKESH MAITI
Indian Statistical Institute, Calcutta
(Received : May, 1984)

SUMMARY

T_1 class of linear estimators is examined to obtain a biased subclass of estimators, better than the sample mean $\bar{y}$.

Keywords: SRSWOR, Searls' estimator, UMMSE-estimator, Sampling strategy.

Introduction

Let $U = \{1, 2, \ldots, N\}$ be a finite population of N (given) units labelled 1 to N and y be a variable (real) which takes value y_i on the ith unit, $(i = 1, 2, \ldots, N)$.

Let

$$\bar{Y} = \sum_{i=1}^{N} y_i/N, \quad \sigma_y^2 = \sum_{i=1}^{N} (y_i - \bar{Y})^2/N \quad \text{and} \quad C_y = \sigma_y/\bar{Y}$$

be the population mean, variance and coefficient of variation of y respectively. It is desired to estimate $\bar{Y}$ on the basis of a sample of n units drawn by simple random sampling without replacement (SRSWOR).

The T_1-class of linear estimators for $\bar{Y}$ based on a sample of size n, may be defined by

$$\hat{T}_1 = \sum_{r=1}^{N} a_r y_r \tag{1.1}$$

where $a_r (r = 1, 2, \ldots, n)$ is the weight associated with the y-value of the unit appearing at the rth draw (Horvitz and Thompson [2], Koop [3] [4]).

When $a_r = \lambda/n$, for all $r = 1, 2, \ldots, n$, $\hat{T}_1$ reduces to

$$\hat{T}_1^* = \lambda\, \bar{y} \tag{1.2}$$

where $\bar{y}$ is the sample mean and the optimum value of λ which minimises the mean square error (MSE), $M(\hat{T}_1^*)$ of $\hat{T}_1^*$ is

$$\lambda_0 = 1/[1 + K\, C_y^2]$$

in the case of SRSWOR, where $K = (N - n)/n(N - 1)$. The resulting estimator discussed by Searls [5] is defined by

$$\hat{T}S = \bar{y}/[1 + K\, C_y^2]$$

with bias and MSE given by

$$B\,(\hat{T}s) = -\,K\, C_y^2\, \bar{Y}/[1 + K\, C_y^2]$$

and

$$M(\hat{T}s) = K\, \bar{Y}^2\, C_y^2/[1 + K\, C_y^2].$$

Obviously, $\hat{T}s$, a member of T_1-class is better than the sample mean $\bar{y}$ (in the sense of having a smaller MSE) and the relative efficiency of Searls' estimator $\hat{T}s$ over $\bar{y}$ is found to be

$$R(\hat{T}s/\bar{y}) = [1 + K\, C_y^2].$$

It is well known that in the case of general sampling designs, there does not exist a best linear unbiased estimator in the unbiased subclass of the class of linear estimators (Koop [3], [4]; Ajgaonkar [1]). However, in the case of SRSWOR, $\bar{y}$ is found to be the best in the unbiased subclass of the T_1-class. The question arises : does there exist the best linear (uniformly minimum mean square error UMMSE) estimator in the entire linear class T_1? Further, are there some biased estimators in T_1-class better than $\bar{y}$?

In this paper, these questions are answered confining to SRSWOR.

2. Existence of the UMMSE-estimator in T_1

THEOREM 2.1 : *If C_y is known exactly, then the sampling strategy (SRSWOR, $\hat{T}_S$) is the best in the class of strategies (SRSWOR, $\hat{T}_1$) for $\bar{Y}$.*

Proof: MSE of the estimator $\hat{T}_1$ is found to be

$$M(\hat{T}_1) = N\sigma_y^2 \sum_{r=1}^{n} a_r^2/(N-1) - \sigma_y^2\left(\sum_{r=1}^{n} a_r\right)^2 + \bar{Y}^2\left(\sum_{r=1}^{n} a_r - 1\right)^2 \tag{2.1}$$

It may be shown that $M(\hat{T}_1)$ would be a minimum for

$$a_r = 1/n\,(1 + K\,C_y^2), \qquad (2.2)$$

and in this case, $\hat{T}_1$ reduces to $\hat{T}_S$. Hence the result.

Although, the sampling strategy (SRSWOR, $\hat{T}_S$) is the best in the class of strategies (SRSWOR, $\hat{T}_1$), it can be shown through numerical illustration that the efficiency of $\hat{T}_S$ over $\bar{y}$ is almost negligible when $K < 0.01$ and $C_y \lessdot 1$. Thus the Searls' estimator should be used only in other situations provided the exact value of C_y is known.

It may be shown that $\hat{T}_1^*$ would be better than $\bar{y}$ under SRSWOR, iff

$$[1 - K\,C_y^2]/[1 + K\,C_y^2] < \lambda < 1 \qquad (2.3)$$

and hence a sufficient condition for $\hat{T}_1^*$ to be better than $\bar{y}$ would be

$$[1 - K\,C_{(1)}^2]/[1 + K\,C_{(1)}^2] \leqslant \lambda < 1 \qquad (2.4)$$

which may be modified to

$$1/[1 + K\,C_{(1)}^2] \leqslant \lambda < 1$$

where $C_{(1)}$ is any quantity such that $C_{(1)}^2 \leqslant C_y^2$

Let us call $\hat{T}_1^*$ with λ satisfying (2.4), a modified Searls' estimator $\hat{T}_S'$, i.e.,

$$\hat{T}_S' = \lambda\,\bar{y},\ \lambda \varepsilon\ [(1 - K\,C_{(1)}^2)/(1 + K\,C_{(1)}^2),\ 1] \text{ or } \lambda \varepsilon\ [1/(1 + KC_{(1)}^2),\ 1].$$

The following Table 2.1 shows the percent relative efficiency of the estimators $\hat{T}_S = \bar{y}\,/[1 + K\,C_y^2]$ and $\hat{T}_S' = \lambda\bar{y}$, $\lambda \varepsilon\ [(1 - K\,C_{(1)}^2)/(1 + K\,C_{(1)}^2),\ 1]$ over $\bar{y}$ to observe the sensitivity of the estimators $\hat{T}_S'$ to departures of optimum choice of λ in $\hat{T}_2^* = \lambda\bar{y}$.

For this, we have considered the populations of having $C_y > 0.5$. Let $N = 5$, $n = 5$ and $C_{(1)} = 0.5$.

From (2.4), it may be shown that $\hat{T}_S' = \lambda\,\bar{y}$ will be better than $\bar{y}$ for all λ satisfying

$$0.9200 \leqslant \lambda \lessdot 1.$$

3. Estimators in T_1 Better than the Sample Mean

In this section, we search for biased estimators in T_1 based on SRSWOR, but better than $\bar{y}$.

TABLE 2.1—PERCENT RELATIVE EFFICIENCY OF $\hat{T}_S$ AND $\hat{T}'_S$ OVER $\bar{y}$, FOR DIFFERENT VALUES OF C_y AND λ's

λ	C_y^2 0.25	1	2.25	4.00
(1)	(2)	(3)	(4)	(5)
λ_0	104.17 (0.96)*	116.23 (0.86)*	137.30 (0.72)*	166.39 (0.60)*
0.94	103.08	110.47	111.95	112.48
0.95	103.89	108.99	109.98	110.34
0.97	103.89	105.67	106.01	106.13
0.98	103.09	103.86	104.01	104.05

*Values in the bracket denote the optimum choice of λ.

Let

$$l = \sum_{r=1}^{n} a_r,\ l_0 = \sum_{r=1}^{n} a_r^2 \quad \text{and} \quad Q = l^2 + \left(\frac{N}{n} - 1\right) - Nl_0.$$

Next we have the following

THEOREM 3.1 : *Let $a_1, a_2, \ldots, a_n$ be chosen such that $Q > 0$. Then a necessary and sufficient condition for the sampling strategy (SRSWOR, $\hat{T}_1$) to be better than the strategy (SRSWOR, $\bar{y}$) is*

$$(N-1)(l-1)^2/Q \leqslant C_y^2 \tag{3.1}$$

Proof: From 2.1, the MSE of T_1 is found to be

$$M(\hat{T}_1) = \bar{Y}^2\left[(l-1)^2 + \frac{(Nl_0 - l^2)}{(N-1)}\, C_y^2\right] \tag{3.2}$$

and $V(\bar{y}) = K\,\bar{Y}^2 C_y^2$. (3.3)

Comparing (3.2) with (3.3), the result follows.

Obviously, the inequality (3.1) can never be satisfied if $Q \leqslant 0$. In fact a_r's should be so chosen that $Q > 0$ is satisfied. The checking of the inequality (3.1) does not always require the exact knowledge of C_y^2. If $C_{(1)}^2$ be a quantity ($\leqslant C_y^2$), then a sufficient condition for $\hat{T}_1$ to be better than $\bar{y}$ would be given by (3.1) with C_y^2 replaced by $C_{(1)}^2$. Thus when C_y is not known exactly, Searls' estimator can not be used at all and in that case, using the knowledge of $C_{(1)}$ only, an estimator from T_1-class of linear estimators can be detected to behave better than $\bar{y}$, better in the sense of having smaller mean square error.

For an illustration, let $N = 25$ and $n = 5$. The weights a_r's. in $\hat{T}_1$ are taken arbitrarily with $l = \Sigma a_r = 0.8$ and such that $Q > 0$ and (3.1) with C_y^2 being replaced by $C_{(1)}^2 = 1.0$ is satisfied.

Table 3.1 shows that one may generate estimators from $\hat{T}_1$ with arbitrary weights better than $\bar{y}$ even when C_y is not known exactly, the case in which Searls' estimator $\hat{T}_S$ can not be used.

TABLE 3.1—RELATIVE EFFICIENCY OF $\hat{T}_1$ OVER $\bar{y}$ FOR ARBITRARY WEIGHTS $N = 25$, $n = 25$, $C_y > 1$, $l = 0.8$, $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.2$, $a_4 = 0.1$, $a_5 = 0.2$.

Relative Efficiency	C_y						
	1.0	1.5	2.0	2.5	3.0	3.5	4.0
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$[V(\bar{y})/M(\hat{T}_S)]$	116.66	137.48	166.64	204.12	249.94	304.08	366.56
$[V(\bar{y})/M(\hat{T}_1)]$	104.29	121.22	128.51	132.20	134.29	135.58	136.44

3.1 *Guidelines to the Practitioner for the Choice of the Coefficients a_r*

Now in what follows, a procedure is given for making choices of a_r's in $\hat{T}_1$ such that the results stated in Theorem 3.1 may be implemented in practice.

From Theorem 3.1, $\hat{T}_1$ defined in (1.1) would be better than $\bar{y}$, if,

$$(N-1)\,(l-1)^2/Q \leqslant C_{(1)}^2 \tag{3.4}$$

Let $a_r = r/\lambda$, where $r(1 \leqslant r \leqslant n)$ is a positive integer and λ is any real number satisfying $Q > 0$. Then from (3.4), we have the following inequality

$$q(\lambda) \leqslant 0 \tag{3.5}$$

where,

$$q(\lambda) = \alpha\,\lambda^2 + \beta\,\lambda + \gamma$$

$$\alpha = (N-1)\,(1 - K\,C_{(1)}^2)$$

$$\beta = -\,(N-1)\,n(n+1)$$

and

$$\gamma = \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}\,(N-1-C^2_{(1)}) + \frac{N(2n+1)\,C^2_{(1)}}{3}\right]$$

Let D be the discriminant of $q(\lambda) = 0$ and let $f = n/N$ be the sampling fraction. Then after routine calculation, D is found to be

$$(N-1)\,N^2\,C^2_{(1)}\,f(Nf+1)\left[(Nf+1)\left\{1 - \frac{C^2_{(1)}\,(1-f)}{f(N-1)}\right\} - \frac{2(2Nf+1)}{3}\left\{1 - \frac{C^2_{(1)}\,(1-f)}{f(N-1)}\right\}\right]$$

and hence, it may be shown that a sufficient condition for $q(\lambda) = 0$ to admit two real roots is given by

$$f < \min\left\{\frac{2}{3},\ \frac{C^2_{(1)}}{N-1+C^2_{(1)}}\right\}$$

Let λ_1 and λ_2 be two roots of $q(\lambda) = 0$. Then the inequality (3.5) will always be satisfied for those λ satisfying

$$\lambda < \lambda_1 \quad \text{or} \quad \lambda > \lambda_2, \quad \text{when } \alpha < 0$$

$$\text{or} \quad \lambda_1 < \lambda < \lambda_2, \quad \text{in case } \alpha > 0$$

Let $R_{0\lambda}$, $R_{1\lambda}$, $R_{2\lambda}$ and $R_{3\lambda}$ denote the ranges for λ for which $Q > 0$, $\lambda_1 < \lambda$, $\lambda > \lambda_2$ and $\lambda_1 < \lambda < \lambda_2$ respectively. Then obviously from Theorem 3.1, the estimators

$$\hat{T}_1' = \frac{1}{\lambda}\,\Sigma\, r\, y_r$$

will be better than $\bar{y}$, if

$$\lambda \in R_{0\lambda} \cap R_{1\lambda} \quad \text{or } \lambda \in R_{0\lambda} \cap R_{2\lambda}$$

$$\text{and} \quad \lambda \in R_{0\lambda} \cap R_{3\lambda}.$$

As an illustration, let us consider a population with $N = 51$, $C_y > 4$. Let us take $C^2_{(1)} = 10$ and $n = 5$. This gives

$$Q = 9.2 - (2580/\lambda^2).$$

Obviously, for all $\lambda > 17$ or $\lambda \leqslant -17$, we shall have $Q > 0$. Now the roots of $q(\lambda) = 0$ are given by

$$\lambda_1 = -54.45 \quad \text{and } \lambda_2 = 16.25$$

Therefore for any

$$\lambda > \max(17, 16.95) \quad \text{or} \quad \lambda < \min(-17, -54.45)$$

the estimator in $\hat{T}_1'$ will be better than sample mean $\bar{y}$.

Remarks: (i) As a general procedure to generate the weights a_r's so that $\hat{T}_1$ is better than $\bar{y}$, we proceed as follows. For given N, n and $C_{(1)}$, we find a λ such that $q(\lambda) < 0$ is satisfied, then for $a_r = r/\lambda$, $(r = 1, 2, \ldots, n)$ in $\hat{T}_1$ the resulting estimator will be better than $\bar{y}$.

(ii) Though the expression for $q(\lambda)$ in (3.5) looks somewhat complicated, but once N, $C^2_{(1)}$ and n are known, the coefficients α, β and γ can easily be computed and hence the roots λ_1, λ_2 of λ such that $q(\lambda) = 0$ may be obtained without any difficulty.

4. Unequal Weights in $\hat{T}_1$ Versus Equal Weights

Theorem 3.1 assures the superiority of an estimator $\hat{T}_1 = \sum_{r=1}^{n} a_r y_r$ over $\bar{y}$, but it does not guarantee whether $\hat{T}_1$ will be better than $\hat{T}_S'$.

In this section, we observe that there always exists at least one set of choice $(a_1, a_2, \ldots, a_n)$ with all $a_r \neq \lambda \ (\neq \lambda_0)$ such that the strategy (SRSWOR, $\hat{T}_1$) is better than (SRSWOR, $\hat{T}_S'$) and hence the strategy (SRSWOR, $\bar{y}$).

Let l and l_0 be the same as in Theorem 3.1 and let

$$[2/(1 + K C^2_{(1)})] - \lambda < l < \lambda \tag{4.1}$$

then, we have the following

THEOREM 4.1 : *A sufficient condition that the strategy (SRSWOR, $\hat{T}_1$) is better than the strategy (SRSWOR, $\hat{T}_S'$) and hence the strategy (SRSWOR, $\bar{y}$) would be*

$$l^2/n < l_0 < \frac{1}{n}\left[\lambda^2 - \{2(\lambda - l)/(1 + K C^2_{(1)})\}\right]$$

Proof: From (2.1) and $M(\hat{T}_S')$, it may be shown that

$$M(\hat{T}_1) \leqslant M(\hat{T}_S')$$

$$\text{iff} \quad \frac{N}{N-1} C_y^2 \, l_0 + l^2 \left(1 - \frac{C_y^2}{N-1}\right) < \lambda^2 (1 - K C_y^2) - 2(\lambda - l), \tag{4.2}$$

Since $l_0 \geqslant l^2/n$, a sufficient condition for (4.2) is obtained by replacing l^2 by $n\, l_0$, where it is assumed that $C_y^2 < (N-1)$. Thus $M(\hat{T}_1) < M(\hat{T}'_s)$

if $l_0 < (1/n)\,[\lambda^2 - 2(\lambda - l)/(1 + K\, C^2_{(1)})]$

provided $\lambda > l$.

ACKNOWLEDGEMENT

The author is grateful to Dr. T. P. Tripathi while preparing this paper and also wishes to thank the referee for his valuable comments in improving the earlier version.

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