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Efficient Row-Column Designs with Multiple units Per Cell Balanced for Spatial Effects

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SUMMARY

Generalized Row-Column (GRC) designs are defined as designs with v treatments in p rows and q columns such that the intersection of each row and column (cell) consists of k experimental units. In GRC designs, since there are more than one number of units in a cell, it is likely that the treatment applied to one experimental unit may affect the response of the neighbouring unit in the same cell if the units are placed linearly adjacent giving rise to spatial effects. The study in presence of spatial effects from neighbouring units requires construction of an arrangement in which the neighbouring units have to appear in a predetermined pattern. Here, series of GRC designs balanced for these spatial effects have been developed. The information matrices for estimating the contrasts pertaining to direct effect and spatial effect have been derived. The designs developed ensure that within a cell every treatment has every other treatment appearing as neighbour a constant number of times. A list of efficient designs has been prepared. Further, in order to give a readymade solution to the experimenters, a SAS macro has been developed that generates the layout of the designs with parameter v (prime), $p = v$, $q = v$-1, k = s ($3 \leq s \leq v$-1).

Keywords: Row-column design, Neighbour balanced, Variance balanced, Partially variance balanced, Canonical efficiency.

1. INTRODUCTION

When the heterogeneity present in the experimental material is from two sources, then two-dimensional blocking or double blocking of the experimental units is recommended for control or reduction of experimental error. The two blocking systems are referred to generally as row blocking and column blocking and the resulting designs are termed as Row-Column (RC) designs. When the number of treatments is large with limited experimental resources then RC designs with multiple units per cell can be used. These designs are called as Generalized Row-Column (GRC) designs and are defined as designs with v treatments in p rows and q columns such that the intersection of each row and column (cell) consists of k experimental units. Some of the experimental situations where these designs are useful are described below along with designs appropriate for such situations.

Example 1.1: To compare a number of dietary treatments on mice, the different breeds and different age groups constitute the two sources of variability. The cages available with the experimenter have two partitions accommodating two mice of same parity, one in each partition. Hence, corresponding to each breed-age combination there are two mice, each can receive a distinct treatment.

Example 1.2: A food sensory experiment where 6 food items are to be compared (Bailey, 1992). The

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experiment is conducted in 3 sessions. There are 6 panellists and each of them will taste 2 food items at each session. In this case, a GRC design with 3 rows (sessions), 6 columns (panellists) with each row-column intersection having cell of size 2 can be used. Following is the arrangement of such a design:

Sessions	Panellists					
	I	II	III	IV	V	VI
I	1 4	2 6	2 5	3 5	6 3	4 1
II	2 3	1 5	4 6	6 1	4 5	3 2
III	6 5	4 3	3 1	2 4	1 2	6 5

Example 1.3: An experiment was conducted where ten treatments are to be applied to sugar beet, which is grown in a 5 × 10 rectangular array of plots (Bailey, 1992). Each plot is a single long North-South row of sugar beet, so the 10 plots in a single row of the rectangle are close to each other and these rows are regarded as a nuisance factor. The beet is sown from five seed-drills on an arm, which protrudes from the right of the tractor. The tractor drives Northwards up the left-hand side of the array, sowing seed in the first five columns, then turns round and drives Southwards down the right-hand side of the array, sowing seed in the last five columns. Thus, the first and last columns are sown by the same drill and drills form a second nuisance factor. The following row-column design for ten treatments in five rows, five columns and the intersection of each row and each column contains a cell of two units ($k = 2$) can be used:

Rows (Plots of Sugar Beet)	Columns (Drill)									
	I	II	III	IV	V	V	IV	III	II	I
I	1	5	3	4	2	8	6	9	7	10
II	2	1	5	3	4	10	8	6	9	7
III	6	4	8	10	9	7	5	1	2	3
IV	9	8	10	7	5	3	1	2	6	4
V	8	3	7	2	1	6	9	4	10	5

For details on these designs, one may refer to Harshberger and Davis (1952), Darby and Gilbert (1958), Preece and Freeman (1983), Williams (1986), Bailey (1988, 1992), Edmondson (1998), Bedford and Whitaker (2001), Bailey and Monod (2001) and Parsad (2006). Subsequently Jaggi *et al.* (2010) and Datta *et al.* (2014, 2015, 2016, 2017) constructed more classes of GRC designs and studied various characterization properties of these designs.

There may arise experimental situations wherein the response from a unit may be affected by other units spatially belonging to the same cluster or group or cell. Like in agricultural experiments under block design setting, where the blocks are made up of plots which cannot be sufficiently isolated from each other, there could be spatial effects coming from the treatments applied to the neighbouring plots. Following are some of the situations of spatial effects:

- If the branches of a tree form plots while the tree serves as a block, spatial effects may arise from the treatments applied to the neighbouring branches.
- In fertilizer trials, plants in an unfertilized plot may rob a share of the plants in a nearby heavily fertilized plot, thereby resulting in spatial effects.
- In varietal trials, the yield of a variety may be depressed by more aggressive neighbouring varieties resulting in spatial effects.
- In fungicide experiments, an unsprayed plot provides a source of spores which can infect neighbouring treated plots resulting in spatial effects.
- In market studies, the sale of different brands on a store shelf may be affected by the brands in the neighbouring shelves.
- In the interpollination by natural hybridization of a group of genotypes, each clone has an equal chance of pollinating, or being pollinated by, any of the others.

In case of a GRC design, there are more than one units in a cell and the treatment applied to one experimental unit in a cell may affect the response on neighbouring units in the same cell. Treatments such as fertilizer, irrigation, or pesticide may spread to adjacent units causing neighbour effects. Such experiments exhibit spatial effects, because the effect of having no treatment as a neighbour is different from the neighbour effects of any treatment. Thus, spatial effects resulting in competition between neighbouring units may contribute to variability in experimental results and lead to substantial losses in efficiency. In order to compare the effects of treatments in this situation, designs balanced for spatial effects are considered where effects from the treatments applied in adjacent experimental units are known to exist. Thus, neighbour-balanced designs wherein the allocation of treatments is such that every treatment occurs equally often with

every other treatment as neighbour(s), are used for these situations. These designs permit the estimation of direct and neighbour effect(s) of treatments.

It is seen in the literature that most of the work on designs with neighbour effects is concentrated under block design set up. There is a few work done related to study of neighbour balanced RC designs. Freeman (1979) has given some row-column designs balanced for neighbours with and without border plots. Federer and Basford (1991) have given three methods of constructing balanced nearest neighbour row-column or competition effect designs. Chan and Eccleston (2003) have given an algorithm which generates neighbour balanced row–column designs. Varghese *et al.* (2014) obtained row-column designs incorporating directional neighbour effects.

In this study, it is assumed that the effect of a treatment applied to a given unit in a cell is the sum of the direct effect due to the treatment applied to the unit, spatial effect from the treatment applied to the immediate left-neighbouring unit and spatial effect from the treatment applied to the immediate right-neighbouring unit within the cell. It is further assumed that the spatial effects from both the adjacent units are same. The purpose of this paper is to give methods of constructing series of GRC designs balanced for spatial effects. The general expression for the joint information matrix for estimating contrasts pertaining to direct effect and spatial effect has been derived. The efficiency factor of the designs has also been worked out.

2. MODEL AND EXPERIMENTAL SETUP

We consider a GRC design with v treatments arranged in p rows, q columns and in each row-column intersection (i.e. cells) there are k units resulting in total $n= pqk$ experimental units or observations. In order to capture the spatial effect of treatments from neighbouring units, the following fixed effect model is considered:

$$Y_{m(ij)} = \mu + \tau_{m[i,j]} + \delta_{(m-1)[i,j]} + \delta_{(m+1)[i,j]} + \alpha_i + \beta_j + e_{m(ij)}\ , \quad (2.1)$$

$$\text{i} = 1,2,\ldots,p;\ \text{j} = 1,2,\ldots,q;\ \text{m} = 2,\ldots,k$$

where $Y_{m(ij)}$ is the response from the m^{th} unit corresponding to the intersection of i^{th} row and j^{th} column. μ is the general mean, $\tau_{m[i,j]}$ is the effect of the treatment appearing in the m^{th} unit corresponding to the intersection of i^{th} row and j^{th} column, $\delta_{(m-1)[i,j]}$ is the neighbour effect due to the treatment applied in the adjacent left unit, $\delta_{(m+1)[i,j]}$ is the neighbour effect due to the treatment applied in the adjacent right unit, α_i is the i^{th} row effect and βj is the j^{th} column effect. $e_{m(ij)}$ is the error term identically and independently distributed and following normal distribution with mean zero and constant variance.

The above model can be written in matrix notation as follows:

$$\mathbf{Y} = \mu\mathbf{1} + \Delta'\boldsymbol{\tau} + \Delta_1'\,\boldsymbol{\delta} + \mathbf{D}_1'\boldsymbol{\alpha} + \mathbf{D}_2'\boldsymbol{\beta} + \mathbf{e} \quad (2.2)$$

where $\mathbf{Y}$ is a n × 1 vector of observations, μ is the grand mean, $\mathbf{1}$ is the $n \times 1$ vector of ones, Δ' is $n \times v$ incidence matrix of observations versus treatments, τ is a $v \times 1$ vector of direct treatment effects, Δ_1' is $n \times v$ incidence matrix of observations versus neighbouring treatments $\mathbf{D}_1'$ is $n \times p$ incidence matrix of observations versus rows, α is $p \times 1$ vector of row effects, $\mathbf{D}_2'$ is $n \times q$ incidence matrix of observations versus columns, β is $q \times 1$ vector of column effects and $\mathbf{e}$ is $n \times 1$ vector of random errors with $E(\mathbf{e})=\mathbf{0}$ and $D(\mathbf{e})=\sigma^2\mathbf{I}_n$. Further, $\Delta'\mathbf{1}_v = \Delta_1'\mathbf{1}_v = \mathbf{D}_1'\mathbf{1}_p = \mathbf{D}_2'\mathbf{1}_q = \mathbf{1}_n$.

The design matrix $\mathbf{X}_{n\times(2v+p+q+1)}$ consisting of treatment effects, neighbour effects, row effects, column effects and mean can be partitioned into parameters of interest $\mathbf{X}_1$ and nuisance parameters $\mathbf{X}_2$.

$$\mathbf{X}_1 = \begin{pmatrix}\Delta' & \Delta_1'\end{pmatrix},\ \mathbf{X}_2 = \begin{pmatrix}\mathbf{1} & \mathbf{D}_1' & \mathbf{D}'\end{pmatrix}$$

with

$$\mathbf{X}_1'\mathbf{X}_1 = \begin{pmatrix}\Delta\Delta' & \Delta\Delta_1' \\ \Delta_1\Delta' & \Delta_1\Delta_1'\end{pmatrix} = \begin{pmatrix}\mathbf{R} & \mathbf{M} \\ \mathbf{M}' & \mathbf{G}\end{pmatrix}$$

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{pmatrix}\Delta\mathbf{1} & \Delta\mathbf{D}_1' & \Delta\mathbf{D}_2' \\ \Delta_1\mathbf{1} & \Delta_1\mathbf{D}_1' & \Delta_1\mathbf{D}'\end{pmatrix} = \begin{pmatrix}\mathbf{r} & \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{r}_1 & \mathbf{N}_3 & \mathbf{N}_4\end{pmatrix}$$

and

$$\mathbf{X}_2'\mathbf{X}_2 = \begin{pmatrix}\mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}_1' & \mathbf{1}'\mathbf{D}_2' \\ \mathbf{D}_1\mathbf{1} & \mathbf{D}_1\mathbf{D}_1' & \mathbf{D}_1\mathbf{D}_2' \\ \mathbf{D}_2\mathbf{1} & \mathbf{D}_2\mathbf{D}_1' & \mathbf{D}_2\mathbf{D}_2'\end{pmatrix} = \begin{pmatrix}n & \mathbf{p}' & \mathbf{q}' \\ \mathbf{p} & \mathbf{K} & \mathbf{W} \\ \mathbf{q} & \mathbf{W}' & \mathbf{H}\end{pmatrix}$$

Here, $\mathbf{N}_1$ is an incidence matrix of order $v \times p$ of direct treatments versus. rows; $\mathbf{N}_2$ is an incidence matrix of order $v \times q$ of treatments versus. columns; $\mathbf{N}_3$ is an incidence matrix of order $v \times p$ of neighbour treatments versus. rows; $\mathbf{N}_4$ is an incidence matrix of order $v \times q$ of neighbour treatments versus columns; $\mathbf{M}$ is an incidence matrix of order $v \times v$ of direct treatments

versus neighbour treatments; **W** is an incidence matrix of order $p \times q$ of rows versus columns; $\mathbf{r} = (r_1, r_{2,\ldots,} r_v)'$ is the $v \times 1$ replication vector of direct treatments $\mathbf{r}_1 = (r_{11,} r_{12,\ldots,} r_{1v})$ is the $v \times 1$ replication vector of the treatments as neighbour with r_{1n} (n = 1,2,…,v) being the number of times the n^{th} treatment appears as neighbour in the design; $\mathbf{R} = \text{diag}(r_1, r_2,\ldots, r_v)$ is the diagonal matrix of replication of treatments; $\mathbf{G} = \text{diag}(r_{11}, r_{12},\ldots, r_{1v})$ is the diagonal matrix of replication of treatments as neighbour; $\mathbf{p} = (p_1, p_2,\ldots, p_p)$ is the $p \times 1$ vector of row sizes; $\mathbf{q} = (q_1, q_2,\ldots, q_q)$ is the $q \times 1$ vector of column sizes; $\mathbf{K} = diag(k_1, k_2,\ldots,k_p)$ is the diagonal matrix of row sizes; $\mathbf{H} = diag(h_1, h_2,\ldots,h_q)$ is the diagonal matrix of column sizes.

The joint information matrix for estimating all the effects (direct and neighbors) can be obtained as:

$$\mathbf{C} = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{X}_1$$

where $(\mathbf{X}_2'\mathbf{X}_2)^{-}$ is the generalized inverse of $(\mathbf{X}_2'\mathbf{X}_2)$ and is obtained using the following result:

$$\mathbf{X} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{D} \end{pmatrix} \text{ then } \mathbf{X}^{-} = \begin{pmatrix} \mathbf{A}^{-} + \mathbf{F}\mathbf{E}^{-}\mathbf{F}' & -\mathbf{F}\mathbf{E}^{-} \\ -\mathbf{E}^{-}\mathbf{F}' & \mathbf{E}^{-} \end{pmatrix}$$

where $\mathbf{F} = \mathbf{A}^{-}\mathbf{B}$ and $\mathbf{E} = \mathbf{D} - \mathbf{B}'\mathbf{A}^{-}\mathbf{B}$.

Here, $\mathbf{F} = \mathbf{K}^{-}\mathbf{W}$ and $\mathbf{E} = \mathbf{H} - \mathbf{W}'\mathbf{K}^{-}\mathbf{W}$, thus

$$(\mathbf{X}_2'\mathbf{X}_2)^{-} = \begin{pmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{K}^{-} + \mathbf{F}\mathbf{E}^{-}\mathbf{F}' & -\mathbf{F}\mathbf{E}^{-} \\ \mathbf{0} & -\mathbf{E}^{-}\mathbf{F}' & \mathbf{E}^{-} \end{pmatrix}$$

The joint information matrix for treatment and neighbour effects is

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{pmatrix} \qquad (2.3)$$

where,

$$\mathbf{C}_{11} = \mathbf{R} - (\mathbf{N}_1\mathbf{K}^{-}\mathbf{N}_1' + \mathbf{N}_1\mathbf{F}\mathbf{E}^{-}\mathbf{F}'\mathbf{N}_1' - \mathbf{N}_2\mathbf{E}^{-}\mathbf{F}'\mathbf{N}_1' - \mathbf{N}_1\mathbf{F}\mathbf{E}^{-}\mathbf{N}_2' + \mathbf{N}_2\mathbf{E}^{-}\mathbf{N}_2')$$

$$\mathbf{C}_{12} = \mathbf{M} - (\mathbf{N}_1\mathbf{K}^{-}\mathbf{N}_3' + \mathbf{N}_1\mathbf{F}\mathbf{E}^{-}\mathbf{F}'\mathbf{N}_3' - \mathbf{N}_2\mathbf{E}^{-}\mathbf{F}'\mathbf{N}_3' - \mathbf{N}_1\mathbf{F}E^{-}\mathbf{N}_4' + \mathbf{N}_2\mathbf{E}^{-}\mathbf{N}_4')$$

$$\mathbf{C}_{21} = \mathbf{M} - (\mathbf{N}_3\mathbf{K}^{-}\mathbf{N}_1' + \mathbf{N}_3\mathbf{F}\mathbf{E}^{-}\mathbf{F}'\mathbf{N}_1' - \mathbf{N}_3\mathbf{E}^{-}\mathbf{F}'\mathbf{N}_2' - \mathbf{N}_4\mathbf{F}E^{-}\mathbf{N}_1' + \mathbf{N}_4\mathbf{E}^{-}\mathbf{N}_2')$$

$$\mathbf{C}_{22} = \mathbf{G} - (\mathbf{N}_3\mathbf{K}^{-}\mathbf{N}_3' + \mathbf{N}_3\mathbf{F}\mathbf{E}^{-}\mathbf{F}'\mathbf{N}_3' - \mathbf{N}_4\mathbf{E}^{-}\mathbf{F}'\mathbf{N}_3' - \mathbf{N}_3\mathbf{F}\mathbf{E}^{-}\mathbf{N}_4' + \mathbf{N}_4\mathbf{E}^{-}\mathbf{N}_4')$$

The $2v \times 2v$ matrix **C** is symmetric, non negative definite with zero row and column sums. From the above, the information matrices for estimating the direct effects and neighbour effects are obtained respectively as

$$\mathbf{C}_{\hat{\mathrm{o}}} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21}$$

and $\mathbf{C}_{\ddot{\mathrm{a}}} = \mathbf{C}_{22} - \mathbf{C}_{12}\mathbf{C}_{11}^{-}\mathbf{C}_{21}$

Definition 2.1: A GRC design with v treatments in p rows and q columns is said to be balanced for spatial effects from neighbouring units if within a cell every treatment has every other treatment appearing as neighbour a constant number of times (say λ times). These designs are called here as Neighbour Balanced GRC (NBGRC) designs. Further, a NBGRC design, permitting the estimation of direct and neighbour effects, is called variance balanced if the variance of any estimated elementary contrast among the direct effects is constant.

3. NBGRC DESIGN CONSTRUCTION

Method 3.1: Consider v (prime) treatments. Develop the contents of i^{th} $(i=1,2,\ldots,v)$ row (mod v) with cell size $k = s$ $(3 \le s \le v\text{-}1)$ as follows:

i i+1 ... i+(s-1), i i+2 ... i+2(s-1), ..., i i+(v-1) ... i+(v-1)(s-1)

The design so obtained is a NBGRC design balanced for spatial effects with parameter v (prime), $p = v$, $q = v$-1, k = s $(3 \le s \le v\text{-}1)$, $r = s$ (v-1) and λ = 2(s-1).

The structure of the various incidence matrices as per model (2.2) for this class of the designs obtained is as follows:

$$\Delta\Delta_1' = \mathbf{M} = 2(\mathrm{s}-1)[\mathbf{J}-\mathbf{I}]$$
$$\Delta\mathbf{D}_1' = \mathbf{N}_1 = (\mathrm{v}-\mathrm{s})\mathbf{I} + (\mathrm{s}-1)\mathbf{J}$$
$$\Delta\mathbf{D}_2' = \mathbf{N}_2 = \mathrm{s}\,\mathbf{J}$$
$$\Delta_1\mathbf{D}_1' = \mathbf{N}_3 = (\mathrm{v}-2\mathrm{s}+2)\mathbf{I} + (2\mathrm{s}-3)\mathbf{J}$$
$$\Delta_1\mathbf{D}_2' = \mathbf{N}_4 = 2(\mathrm{s}-1)\,\mathbf{J}$$
$$\mathbf{D}_1\mathbf{D}_2' = \mathbf{W} = \mathrm{s}\,\mathbf{J}$$

$$\Delta\Delta' = \mathbf{R}_{\hat{\mathrm{o}}} = \mathrm{s}(\mathrm{v}-1)\,\mathbf{I}$$
$$\Delta_1\Delta_1' = \mathbf{G} = [2(\mathrm{v}-1)(\mathrm{s}-1) - 2(\mathrm{s}-2)]\mathbf{I} + 2(\mathrm{s}-2)\mathbf{J}$$
$$\mathbf{D}_1\mathbf{D}_1' = \mathbf{K} = \mathrm{s}(\mathrm{v}-1)\mathbf{I}$$
$$\mathbf{D}_2\mathbf{D}_2' = \mathbf{H} = \mathrm{sv}\,\mathbf{I}$$

The components of $2v \times 2v$ joint information matrix for estimating the contrast pertaining to direct and neighbour effects as in (2.3) is obtained as below:

$$\mathbf{C}_{11} = \left[s(v\text{-}1) - \frac{(v\text{-}s)^2}{s(v\text{-}1)} \right] \mathbf{I} - \frac{2(v\text{-}s)(s\text{-}1)+v(s\text{-}1)^2}{s(v\text{-}1)} \mathbf{J}$$

$$\mathbf{C}_{12} = \mathbf{C}_{21} = -\left[2(s\text{-}1) + \frac{(v\text{-}s)(v\text{-}2s+2)}{s(v\text{-}1)} \right] \mathbf{I} + \left[\frac{(v\text{-}s)(2s\text{-}3)+(s\text{-}1)(v\text{-}2s+2)+v(s\text{-}1)(2s\text{-}3)}{s(v\text{-}1)} - 2(s\text{-}1) \right] \mathbf{J}$$

$$\mathbf{C}_{22} = \left[2(v\text{-}1)(s\text{-}1) - 2(s\text{-}2) - \frac{(v\text{-}2s+2)^2}{s(v\text{-}1)} \right] \mathbf{I} - \left[\frac{2(v\text{-}2s+2)(2s\text{-}3)+v(2s\text{-}3)^2}{s(v\text{-}1)} - 2(s\text{-}2) \right] \mathbf{J}$$

The information matrix for estimating the contrast for direct treatment effects is obtained as below:

$$\mathbf{C}_{\delta} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-}\mathbf{C}_{21} = \mathrm{A}\mathbf{I} - \mathrm{B}\mathbf{J}$$

where,

$$\mathrm{A} = \left((sa - \frac{f^2}{sa}) - \frac{(2abs + df)^2}{sa(2a^2bs - 2acs - d^2)} \right)$$

$$\mathrm{B} = \left(\frac{2fb + vb^2}{sa} - \mathrm{D} \right)$$

$$\mathrm{D} = \frac{1}{2a^2bs - 2acs - d^2}\left((ef + bd + vbe - 2sab)\right) - \frac{(2de + ve^2 - 2sac)[(2sab + df) + v(ef + bd + vbe - 2abs)]}{e(3v^2 - 4vs + 2s) - d^2 - 2vd} \times \left(\frac{ef + bd + vbe - 2sab}{sa} \right)$$

$a = (v\text{-}1)$, $b = (s\text{-}1)$, $c = (s\text{-}2)$, $d = (v\text{-}2s+2)$, $e = (2s\text{-}3)$ *and* $f = (v\text{-}s)$.

Example 3.1.1: For $v = 5$ and $s = 3$, following is a NBGRC design with parameters $v = 5$, $p = 5$, $q = 4$, $k = 3$, $r = 12$ and $\lambda = 6$:

	Columns			
Rows	1 2 3	1 3 5	1 4 2	1 5 4
	2 3 4	2 4 1	2 5 3	2 1 5
	3 4 5	3 5 2	3 1 4	3 2 1
	4 5 1	4 1 3	4 2 5	4 3 2
	5 1 2	5 2 4	5 3 1	5 4 3

For this design,

$$\mathbf{C}_{11} = 11.66\mathbf{I} - 2.33\mathbf{J}$$

$$\mathbf{C}_{12} = \mathbf{C}_{21} = -4.16\mathbf{I} + 0.83\mathbf{J}$$

$$\mathbf{C}_{22} = 13.92\mathbf{I} - 2.25\mathbf{J}$$

The information matrix for estimating direct treatment contrast is

$\mathbf{C}_k$ = 10.42 **I** - 2.08 **J**

Similarly, the information matrix for estimating neighbour treatment contrast is

$\mathbf{C}_{\delta}$ = 12.43 **I** – 1.95**J**.

Example 3.1.2: For $v = 5$ and $s = 4$, following is a NBGRC design with parameters $v = 5$, $p = 5$, $q = 4$, $k = 4$, $r = 16$ and $\lambda = 4$:

	Columns			
Rows	1 2 3 4	1 3 5 2	1 4 2 5	1 5 4 3
	2 3 4 5	2 4 1 3	2 5 3 1	2 1 5 4
	3 4 5 1	3 5 2 4	3 1 4 2	3 2 1 5
	4 5 1 2	4 1 3 5	4 2 5 3	4 3 2 1
	5 1 2 3	5 2 4 1	5 3 1 4	5 4 3 2

Here,

$$\mathbf{C}_{11} = 15.93\mathbf{I} - 3.18\mathbf{J}$$

$$\mathbf{C}_{12} = \mathbf{C}_{21} = -5.94\mathbf{I} + 1.19\mathbf{J}$$

$$\mathbf{C}_{22} = 19.94\mathbf{I} - 3.19\mathbf{J}$$

The information matrix for estimating direct treatment contrast is

$\mathbf{C}_k$ = 14.17 **I** - 2.38 **J**.

Similarly, the information matrix for estimating neighbour treatment contrast is

$\mathbf{C}_{\delta}$ = 17.73 **I** – 2.75**J**.

Thus, we see that the developed series of NBGRC design is variance balanced for estimating the contrast pertaining to direct treatments and also pertaining to neighbour effects.

Method 3.2: Consider a Balanced Incomplete Block (BIB) design with parameters (v^*, b^*, r^*, k^*, *and* λ^*). Let $v^* = 4t+3 = x^n$, where x is a prime and n ($\geq$1) is a positive integer. Consider the odd powers of the primitive number of GF(x^n) as set 1 and the even powers of the primitive number of GF(x^n) as set 2. The block contents of set 1 comprises the 1st column of resulting GRC design and set 2 comprises the 2nd column of resulting GRC design. The parameters of the developed design are $v = v^*$, $p = v^*$, $q = 2$, $k = k^*$, r = r* and λ_i (i = 1,2,…, $\frac{v-1}{2}$). Thus, a GRC design

with neighbour effects obtained through initial blocks of a BIB design is always a partially balanced design for estimating elementary direct treatment contrasts following a varying circular association scheme.

Example 3.2.1: Consider a BIB design with parameters (7,7,3,3,2). Following is a GRC design with neighbour effects with parameters $v = 7$, $p = 7$, $q = 2$, r = 6, $k = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 1$:

	Columns	
Rows	1 2 4	3 6 5
	2 3 5	4 7 6
	3 4 6	5 1 7
	4 5 7	6 2 1
	5 6 1	7 3 2
	6 7 2	1 4 3
	7 1 3	2 5 4

The information matrix for estimating direct treatment contrasts is given by

$$\mathbf{C}_{\hat{\delta}} = \begin{bmatrix} 4.54 & -0.52 & -0.89 & -0.85 & -0.85 & -0.89 & -0.52 \\ -0.52 & 4.54 & -0.52 & -0.89 & -0.85 & -0.85 & -0.89 \\ -0.89 & -0.52 & 4.54 & -0.52 & -0.89 & -0.85 & -0.85 \\ -0.85 & -0.89 & -0.52 & 4.54 & -0.52 & -0.89 & -0.85 \\ -0.85 & -0.85 & -0.89 & -0.52 & 4.54 & -0.52 & -0.89 \\ -0.89 & -0.85 & -0.85 & -0.89 & -0.52 & 4.54 & -0.52 \\ -0.52 & -0.89 & -0.85 & -0.85 & -0.89 & -0.52 & 4.54 \end{bmatrix}$$

The information matrix for estimating neighbour treatment contrasts is given by

$$\mathbf{C}_{\delta} = \begin{bmatrix} 5.39 & -0.89 & -0.47 & -0.67 & -0.67 & -0.47 & -0.89 \\ -0.89 & 5.39 & -0.89 & -0.47 & -0.67 & -0.67 & -0.47 \\ -0.47 & -0.89 & 5.39 & -0.89 & -0.47 & -0.67 & -0.67 \\ -0.67 & -0.47 & -0.89 & 5.39 & -0.89 & -0.47 & -0.67 \\ -0.67 & -0.67 & -0.47 & -0.89 & 5.39 & -0.89 & -0.47 \\ -0.47 & -0.67 & -0.67 & -0.47 & -0.89 & 5.39 & -0.89 \\ -0.89 & -0.47 & -0.67 & -0.67 & -0.47 & -0.89 & 5.39 \end{bmatrix}$$

It can be seen that treatment number 1 has treatment 2 and 7 as first associates (these treatments appear as neighbour twice), treatment 3 and 6 as second associates (these treatments appear as neighbour once) and remaining 4 and 5 as third associates (these treatments appear as neighbour once).

A SAS code (given in Annexure I) has been written in PROC IML to calculate the information matrix (**C**-matrix) of treatment effects and neighbour effects and to study the properties of the designs under the three-way model with spatial effects.

4. SAS MACRO

For readymade solution for the experimenters, a SAS macro for the generation of NBGRC designs with parameter v (prime), $p = v$, $q = v-1$, $k = s$ $(3 \le s \le v-1)$, $r = s(v-1)$ and $\lambda = 2(s-1)$ has been developed. The SAS macro is given in Annexure II. In order to generate the design, user has to enter the number of treatments i.e. 'v' and 'k'.

5. EFFICIENCY OF DESIGNS

The canonical efficiency of the NBGRC designs is obtained as follows:

$$E = \frac{H}{r}, \quad H = \left(\frac{1}{v-1} \sum_{i=1}^{v-1} \theta_i^{-1} \right)^{-1},$$

where θ_i are the eigen-values of **C**- matrix (obtained for direct treatment effects and neighbour treatment effects). Here, r is the number of replications of the treatments and is assumed to be same for the developed design and the orthogonal design to which it is compared.

The parameters of NBGRC designs obtained using Method 3.1.1 described above have been listed in Table 5.1. The list contains number of treatments ($v \le 11$), cell sizes (k), number of rows (p), number of columns (q) and replications (r). The canonical efficiency of the developed designs for direct treatment effects and neighbour treatment effects are also reported in the Table 1.

It is seen that the efficiency of direct treatment effects of NBGRC designs constructed is more as compared to neighbour treatment effects. The efficiency factor increases with increase in cell size for a given number of treatments.

6. CONCLUSIONS

Two series of GRC designs balanced for spatial effects have been developed. One series is variance balanced for estimating the contrasts pertaining to direct treatment effects and also for estimating the contrasts pertaining to neighbour treatment effects. The second series is partially balanced for estimating elementary treatment contrasts for direct and neighbour treatment effects following circular association scheme. Further, the efficiency of the NBGRC designs have been worked out and are found to be quite high for estimating the direct treatment effects. For future line of work an attempt will be made to obtain a series

Table 1. Parameters and efficiency factor of NBGRC designs

S. No.	v	k	p	q	λ_i	Efficiency Factor (direct treatment effects)	Efficiency Factor (neighbour treatment effects)	Series
1	5	3	5	4	4	0.86	0.45	I
2	5	4	5	4	6	0.88	0.45	I
3	7	3	7	6	4	0.89	0.53	I
4	7	4	7	6	6	0.82	0.54	I
5	7	5	7	6	8	0.94	0.54	I
6	7	6	7	6	10	0.95	0.55	I
7	11	3	11	10	4	0.89	0.62	I
8	11	4	11	10	6	0.94	0.63	I
9	11	5	11	10	8	0.94	0.63	I
10	11	3	11	10	4	0.89	0.62	I
11	11	4	11	10	6	0.94	0.63	I
12	11	5	11	10	8	0.94	0.63	I
13	11	6	11	10	10	0.94	0.64	I
14	11	7	11	10	12	0.95	0.65	I
15	11	8	11	10	14	0.96	0.65	I
16	11	9	11	10	16	0.96	0.66	I
17	11	10	11	10	18	0.97	0.66	I

of GRC designs balanced for spatial effects for new parametric combinations and to study the spatial effects between cell of GRC designs.

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ANNEXURE I

SAS CODE FOR OBTAINING THE C-MATRIX AND THE CANONICAL EFFICIENCY FACTOR FOR NBGRC DESIGNS

```
/* NBGRC Design */
proc iml;
/*design [put non-zero values]*/
a={
1 2 3 4 1 3 5 2 1 4 2 5 1 5 4 3,
 2 3 4 5 2 4 1 3 2 5 3 1 2 1 5 4,
 3 4 5 1 3 5 2 4 3 1 4 2 3 2 1 5,
 4 5 1 2 4 1 3 5 4 2 5 3 4 3 2 1,
 5 1 2 3 5 2 4 1 5 3 1 4 5 4 3 2
};
/*define cell sizes [b is defined here as cell size (k)]*/
b={ 4 4 4 4 ,
4 4 4 4 ,
```

```
4 4 4 4 ,
4 4 4 4 ,
4 4 4 4
};
cc=b[+, ];
dd=b[ ,+];
bb=j(nrow(b)*ncol(b),1,0);
k=1;
do i=1 to nrow(b);
do j=1 to ncol(b);
bb[k, ]= b[i,j];
k=k+1;
end;
end;
b1=bb[loc(bb>0),];
*print b1;
aa=j(nrow(a)*ncol(a),1,0);
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
aa[k, ]= a[i,j];
k=k+1;
end;
end;
*print aa;
m1=j(nrow(a)*ncol(a),1,1);/*mean vector*/
/*print m1;*/
dir=j(nrow(a)*ncol(a),max(a),0);/*design matrix -obs VS direct treatment*/
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
if a[i,j]>0 then
        do;
        dir[k,a[i,j]]=1;
        k=k+1;
        end;
end;
end;
*print dir;
r=j(nrow(a)*ncol(a),nrow(dd),0);/*design matrix -obs VS row*/
k=1;
do i=1 to nrow(a);
do j=1 to ncol(a);
r[k,i]=1;
k=k+1;
end;
end;
*print r;
c=j(nrow(a)*ncol(a),ncol(b),0);/*design matrix - obs VS column*/
k=1;
do i=1 to nrow(b);
do j=1 to ncol(b);
do l=1 to b[i,j];
c[k,j]=1;
k=k+1;
end;
end;
end;
*print c;
cell=j((nrow(a)*ncol(a)),nrow(b1),0);/*design matrix - obs VS cell*/
kk=1;
z=0;
do k=1 to nrow(b1);
do j=1 to b1[k];
if aa[z+j, ]>0 then
    do;
    cell[kk,k]=1;
    kk=kk+1;
    end;
end;
```

```
z=z+b1[k];
end;
*print cell;
l_neig = j(nrow(a)*ncol(a),max(a),0);
k=2;
z=0;
do i = 1 to nrow(b1);
do j = 1 to b1[i]-1;
    if aa[z+j, ]>0 then l_neig[k,aa[z+j, ]]=l_neig[k,aa[z+j, ]]+1;
    k=k+1;
end;
z=z+b1[i];
k=k+1;
end;
*print l_neig;
r_neig = j(nrow(a)*ncol(a),max(a),0);
k=1;
z=0;
do i = 1 to nrow(b1);
do j = 2 to b1[i];
    if aa[z+j, ]>0 then r_neig[k,aa[z+j, ]]=r_neig[k,aa[z+j, ]]+1;
    k=k+1;
end;
z=z+b1[i];
k=k+1;
end;
*print r_neig;
neigh=l_neig+r_neig;
x1=dir||neigh;
x2=m1||r||c;
c_mat=(x1`*x1)-(x1`*x2*(ginv(x2`*x2))*x2`*x1)/*C matrix*/;
print c_mat;
c11=j(max(a),max(a),0);
do i=1 to max(a);
do j=1 to max(a);
c11[i,j]=c_mat[i,j];
end;
end;
*print c11;
c12=j(max(a),ncol(c_mat)-max(a),0);
do i=1 to max(a);
k=1;
do j=max(a)+1 to ncol(c_mat);
c12[i,k]=c_mat[i,j];
k=k+1;
end;
end;
*print c12;
c22=j(nrow(c_mat)-max(a),nrow(c_mat)-max(a),0);
k=1;
do i=max(a)+1 to nrow(c_mat);
kk=1;
do j=max(a)+1 to nrow(c_mat);
c22[k,kk]=c_mat[i,j];
kk=kk+1;
end;
k=k+1;
end;
*print c22;
c_dir=c11- c12*ginv(c22)*c12`;
print c_dir;
c_neigh= c22- c12*ginv(c11)*c12`;
print c_neigh;
eig=eigval(c_dir);
*print eig;
eig1=eig[loc(eig>0.0000001),];/*positive eigen values*/
rep=dir`*dir;
eig2=eig1/(rep[1,1]);
eig3=1/eig2;
```

```
CanEffFactor=nrow(eig3)/sum(eig3);
print CanEffFactor;
eig=eigval(c_neigh);
*print eig;
eig4=eig[loc(eig>0.0000001),];/*positive eigen values*/
rep=neigh`*neigh;
eig5=eig4/(rep[1,1]);
eig6=1/eig5;
CanEffFactor=nrow(eig6)/sum(eig6);
print CanEffFactor;
quit;
quit;
```

ANNEXURE II

SAS macro for the generation of NBGRC designs with parameter *v* (prime), *p = v, q = v-1, k = s (3 ≤ s ≤ v-1), r = s(v-1)* and λ=*2(s-1)*

```
%let v=5;/* Enter the number of treatments (Treament number should be prime number)*/
%let s=3;/*Enter the cell sizes(it varies from 3 to (v-1)*/
ods rtf file= 'output.rtf' startpage=no;
proc iml;
TRT1=j(&v,&s*(&v-1),0);
k=1;
do i=1 to &s;
do j=1 to &v;
TRT1[j,i]=(j+(i-1));
if TRT1[j,i]>&v then TRT1[j,i]=TRT1[j,i]-&v;
end;
end;
kk=&s+1;
do k=1 to &v-1;
do i=1 to &s;
do j=1 to &v;
TRT1[j,kk]=TRT1[j,kk-(&s)]+(i-1);
if TRT1[j,kk]>&v then do;
TRT1[j,kk]=TRT1[j,kk]-&v;
end;
end;
kk=kk+1;
end;
end;
varNames2= "Column1":"Column"+strip(char(&v-1));
varNames3= "Row1":"Row"+strip(char(&v));
do i=1 to (&v-1);
do j=1 to &s;
columns=varNames2[ ,i];
columns1=columns1||columns;
end;
end;
GRC_Design=char(TRT1,5,0);
print 'Generalized Row Column (GRC) Design Balanced for Spatial Indirect Effects';
print GRC_Design[rowname=varNames3 colname=columns1];
print 'Number of Rows =' &v;
print 'Number of Columns ='(&v-1);
print 'Number of treatments in each Row-Column Intersection is =' &s;
ods rtf close;
quit;
```

The output of the SAS macro for $v = 5$ and $k = 3$ is shown in the following figure.