Available online at www.isas.org.in/jisas
JOURNAL OF THE INDIAN SOCIETY OF
AGRICULTURAL STATISTICS 74(1) 2020 41–50

New Series of Optimal Covariate Designs in CRD and RBD set-ups

Hiranmoy Das[1], Anurup Majumder[2], Manoj Kumar[3] and Dikeshwar Nishad[4]

[1]*ICAR- Indian Institute of Soil Science, Bhopal*
[2]*Bidhan Chandra Krishi Viswavidyalaya, Nadia*
[3]*ICAR-Central Institute of Agricultural Engineering, Bhopal*
[4]*Pt. Shiv Kumar Shastri College of Agriculture and Research Station, Rajnandgaon*

Received 22 April 2019; Revised 30 August 2019; Accepted 04 January 2020

SUMMARY

The study of optimal design for covariate models in CRD set-up was initiated by Troya (1982a, 1982b). Das *et al.* (2003) followed up the study and extended for RBD set-up. Recently Das *et al.* (2015) published a book on 'Optimal Covariate Designs'. In the present study, one new series of global optimal covariate designs in CRD set-up and two new series in RBD set-up have been developed. The new OCDs in CRD or RBD designs require only two Hadamard matrices of order 2 and 4. The developed global optimal covariate designs in CRD set-up have v (=0; mod 4 or =2; mod 4) number of treatments, and the developed first series of global optimal covariate designs in RBD set-up have the treatment number **v** (=0; mod 4) for any even number of replications or blocks, **b** and do not dependent on the existence of $\mathbf{H}_v$ and $\mathbf{H}_b$. The second series of global optimal covariate designs in RBD set-up require only the existence of $\mathbf{H}_v$. The paper is enriched with examples of optimal covariates. All the developed optimal covariate designs in the present article are not available in the existing literature.

Keywords: ANCOVA, Hadamard Matrix, Optimal Covariate Design (OCD), CRD, RBD.

1. INTRODUCTION

Analysis of Covariance or 'ANCOVA' is a known method by which the error affecting the treatment comparisons may be minimized. The experimental results can be improved by suitably classifying or reclassifying the existing experimental units through a study of the associated covariates or by first suitably choosing the covariate values from a larger lot and then identifying the associated experimental units from a larger pool. The choice or selection of experimental units with suitably defined values of the covariates for a particular design set-up so as to attain the minimum variance or maximum precision for estimating the regression parameters has fascinated the interest of statisticians for the last three decades or a little more. Several authors, like Harville (1974, 1975), Haggstrom (1975) and Wu (1981) had studied the ANCOVA models on the problems of inference on varietal contrasts corresponding to qualitative factors. But the problem of determining the optimum designs for the estimation of regression parameters corresponding to controllable covariates was not a topic of research for many years. Troya (1982a and 1982b) was the pioneer in history in the topic of optimal covariate designs (OCDs) but she restricted to only Completely Randomized Design (CRD) set-up. After a long gap, Das *et al.* (2003) extended the work on OCDs to the block design set-up, viz., Randomized Block Design (RBD) and some series of Balanced Incomplete Block Design (BIBD). They also constructed OCDs for the estimation of covariate parameters. Rao *et al.* (2003) also revisited the problem in CRD and RBD set-ups. They identified that the solutions of construction of OCDs by using Mixed Orthogonal Arrays (MOAs) and thereby giving further insights and some new solutions. Dutta (2004) developed OCDs for BIBDs obtained through Bose's Difference Technique. He utilized conveniently different combinatorial arrangements and

Corresponding author: Hiranmoy Das
E-mail address: hiranmoydas.stat@gmail.com

tools such as Hadamard matrices and different kinds of products of matrices viz., Kronecker product to construct OCDs with as many covariates as possible. Dey and Mukherjee (2006) studied the problem of finding D-optimal designs in the presence of a number of covariates in the one-way set-up. They actually given an upper bound to the determinant of information matrix obtained through diagonal **C**-matrices. Dutta *et al.* (2007) studied the optimum choice of covariates for a series of balanced incomplete block designs (BIBDs). In case of incomplete block designs, the choice of the values of the covariates depends heavily on the allocation of treatments to the plots of blocks; more specifically on the method of construction of the incomplete block design. Based on this, they considered the situation where the block design is a member of the complementary series of balanced incomplete block design (BIBD) with parameters $b = v = s^N+s^{N-1}+\ldots+s+1$, $r = k = s^N$, $\lambda=s^N-s^{N-1}$ of symmetric balanced incomplete block design (SBIBD) obtained through projective geometry. Sinha (2009) gave the solution to accommodate maximum number of covariates in an optimal manner through combinatorially for the standard design layouts such as CRD, RBD, LSD and BIBD. Dutta *et al.* (2010b) considered the problem that when $n\neq0$ (mod 4), it is impossible to find designs attaining minimum variance for estimated covariate parameters. In this situation, they considered instead of using the criterion of attaining the lower bound (viz., σ^2/n) to the variance of each of the estimated covariate parameters γ, they found optimum designs with respect to covariate effects using D-optimality criterion retaining orthogonality with respect to treatment and block effect contrasts, where n=2 (mod 4). Dutta *et al.* (2014) extended the work of Dey and Mukherjee (2006) in the sense that for fixed replication numbers of each treatment, an alternative upper bound to the determinant of information matrix has been found through completely symmetric **C**-matrices for the regression coefficients and this upper bound includes the upper bound given in Dey and Mukherjee (2006). Recently, Das *et al.* (2015) has published a book, viz., 'Optimal Covariate Designs' with detail discussion on the topic. Mostly the designs developed by above mentioned authors are global optimal but the development of designs are dependent on existence of Hadamard matrix of order either v or b or k (v be the treatment numbers, b be the number of replications/ blocks in CRD/ RBD and k be the size of blocks in a variance balanced incomplete block design).

In the present piece of investigation, an effort has been made to construct global optimal covariate designs in CRD and RBD set-ups when Hadamard matrices of order $\mathbf{H}_v$ and $\mathbf{H}_b$ do not exist. The study contains five sections including the present introductory section. In section 2, the definition and properties of Special Array are presented. Section 3 and 4 describe the basic models, situations and conditions of the optimal covariate designs (OCDs) for CRD and RBD set-ups, respectively. Construction of a new series of global optimal covariate designs in CRD set-up has also been presented in section 3. Similarly, construction of two new series of global optimal covariate designs in RBD set-up has been given in section 4. Conclusion of the study has been given in section 5.

2. SPECIAL ARRAY; DEFINITION, PROPERTIES

2.1 Definition

A square matrix with elements 1, -1 and 0 of order h having r (≥ 1) number of rows (and columns) with all elements 0 and all the distinct row or column vectors except r rows (or columns) of the matrix are mutually orthogonal will be referred to as **Special Array** (SA) of order h. In SA, each row or column sum is zero except the first row or column. The simplest examples, one for order 3 and two for order 5 are given below:

$$\underset{r=1}{\begin{pmatrix}1&0&1\\0&0&0\\1&0&-1\end{pmatrix}},\ \underset{r=3}{\begin{pmatrix}1&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&-1\end{pmatrix}}\ \text{and}\ \underset{r=1}{\begin{pmatrix}1&1&0&1&1\\1&-1&0&-1&1\\0&0&0&0&0\\1&1&0&-1&-1\\1&-1&0&1&-1\end{pmatrix}}$$

2.2 Properties

Let the Special Array (SA) of order h be denoted as H_h^*, then

1) $\left|\det H_h^*\right| = 0$; when $r \geq 1$; with r = 0, H_h^* becomes a Hadamard Matrix.
2) $H_h^* H_h^{*T} = H_h^{*T} H_h^*$
3) Let H_1^* and H_2^* be two SA of order h_1 and h_2, respectively. Then the Kronecker product of H_1^* and H_2^* is also a SA of order h_1h_2.

3. OCDS IN CRD SET-UP

Let there be v treatments and c covariates in a design with total n experimental units. In matrix notation the model can be represented as

$$(\mathbf{Y}, \mathbf{X}\boldsymbol{\tau} + \mathbf{Z}\boldsymbol{\gamma}, \sigma^2\mathbf{I}_n) \tag{3.1}$$

where, for $1 \le i \le v$, $1 \le j \le n_i$ (n_i is the number of times the i^{th} treatment is replicated; clearly $\sum_{i=1}^{v} n_i = n$) and $1 \le t \le c$, **Y** is an observation vector and **X** is the design matrix corresponding to vector of treatment effects $\boldsymbol{\tau}^{vx1}$ and $\mathbf{Z}^{nxc} = ((z_{ij}^{(t)}))$ is the design matrix corresponding to vector of covariate effects $\boldsymbol{\gamma}^{cx1} = (\gamma_1, \gamma_2, \ldots, \gamma_c)'$. This is referred to as one–way model with covariates without general mean. In the above, **Z** is called covariate matrix of c covariates $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_c$. Here **z**'s are assumed to be controllable non-stochastic covariates. The n values $z_{i1}, z_{i2}, \ldots, z_{in}$ are assumed by the ith covariate $\mathbf{z}_i$ are such that they belong to a finite interval $[a_i, b_i]$ for each i and j, i.e.

$$a_i \le z_{ij} \le b_i \tag{3.2}$$

i.e. $$z_{ij} = \frac{a_i + b_i}{2} + \frac{b_i - a_i}{2} z_{ij}^* \tag{3.3}$$

so that z_{ij}^* lies in [-1, 1] for each i, j. Then replacing z_{ij} by z_{ij}^*'s, we get the same covariate model in a reparametrized scenario. So, without loss of generality, the covariate values z_{ij}'s to vary within [-1, 1].The information matrix with respect to model (3.1) is given by,

$$\sigma^{-2}\mathrm{I}(\eta) = \begin{pmatrix} X'X & X'Z \\ Z'X & Z'Z \end{pmatrix} \text{ where, } \eta' = (\tau', \gamma') \tag{3.4}$$

The problem is to suggest an optimal allocation scheme (for given design parameters n, v, c) for efficient estimation of the treatment effects and the covariate effects by ascertaining the values of the covariates for each one of them, assuming that each one is controllable and quantitative within a stipulated finite closed interval. The information matrix of γ is given by,

$$\sigma^{-2}\mathrm{I}(\gamma) = Z'Z - Z'X(X'X)^{-}X'Z \tag{3.5}$$

where, $(\mathbf{X}'\mathbf{X})^-$ is a generalized inverse of $\mathbf{X}'\mathbf{X}$. According to Rao (1973), $\mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'\mathbf{Z}$ is a positive semi-definite matrix. So from (3.5), it follows that

$$\sigma^{-2}\mathrm{I}(\gamma) \le Z'Z \tag{3.6}$$

Equality in (3.6) is attained whenever $\mathbf{X}'\mathbf{Z} = \mathbf{0}$ (3.7)

If **Z** satisfies (3.7), then treatment effects and covariate effects are orthogonally estimated. In addition, the information matrix **I**(γ) reduces to **I**(γ) = **Z′Z**. The z-values are so chosen that **Z′Z** is positive definite, so that from (3.6)

$$Var(\hat{\gamma}_t) \ge \frac{\sigma^2}{\sum_{i=1}^{v}\sum_{j=1}^{n_i} z_{ij}^{(t)2}} \ge \frac{\sigma^2}{n} \tag{3.8}$$

as

$$z_{ij}^{(t)} \in [-1, 1]; \forall i, j, t$$

Now equality in (3.8) holds for all i if and only if the **Z**-matrix is such that

$$\mathbf{z}^{(s)\prime}\mathbf{z}^{(t)} = 0 \text{ for all } s \ne t \tag{3.9}$$

$$\text{and } z_{ij}^{(t)} = \pm 1 \tag{3.10}$$

Condition (3.7) implies that the estimators of ANOVA effects parameters or parametric contrasts do not interfere with those of the covariate effects and conditions (3.9) and (3.10) imply that the estimators of each of the covariate effects are such that these are pair wise uncorrelated, attaining the minimum possible variance. Thus, the covariate effects are estimated with the maximum efficiency if and only if

$$\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_c \tag{3.11}$$

along with (3.7). The designs allowing the estimators with the minimum variance are called globally optimal designs (Shah and Sinha, 1989) or optimal covariate design, to be abbreviated as OCD.

Visualizing the **Z**-matrix in a particular design set up satisfying conditions (3.7) and (3.11) is somewhat difficult. In the set-up of the model (3.1), it transpires from Troya Lopes (1982a) that optimal estimation of the treatment effects and the covariates effects is possible when the treatment replications are all necessarily equal, assuming that n is a multiple of v, the number of treatments. Set n = bv, where b is the common replication of treatments. Das *et al.* (2003) had represented each column of the **Z**-matrix by a v×b matrix, viz., **W** with elements of ±1. Condition (3.7) implies that the sum of each row of **W** should be zero. Further, condition (3.11) implies that the sum of products of the corresponding elements i.e. the Hadamard product of $\mathbf{W}^{(s)}$ and $\mathbf{W}^{(t)}$, should also be zero, $1 \le s < t \le c$. For orthogonality of sth and tth columns of **Z**, it is required that

$$\sum_{i=1}^{v}\sum_{j=1}^{b} w_{ij}^{(s)} w_{ij}^{(t)} = 0 \qquad (3.12)$$

In this case the ANOVA parameters as well as the covariate effect-parameters can be estimated orthogonally and/or most efficiently. This holds simultaneously for c covariates and one can deduce maximum possible value of c for this to happen. As already mentioned, the most efficient estimation of γ-components is possible when conditions (3.7) and (3.11) are simultaneously satisfied and these conditions reduce, in terms of **W**-matrices defined in above, to $\mathbf{C}_1$ and $\mathbf{C}_2$, where

$\mathbf{C}_1$: Each of the c **W**-matrices has all row-sums equal to zero;

$\mathbf{C}_2$**:** The grand total of all the entries in the Hadamard product of any two distinct **W**-matrices reduces to zero.

3.1 Construction of optimum W-matrices for covariate model in CRD set-up:

Definition 3.1: With respect to model (3.1), the c number of **W**-matrices corresponding to the c covariates are said to be optimum if they satisfy the conditions $\mathbf{C}_1$ and $\mathbf{C}_2$ simultaneously.

Under the realization of $\mathbf{C}_1$ and $\mathbf{C}_2$ in terms of optimum **W** matrices, we can develop the following theorem.

Theorem 3.1: If both v and b be two even numbers, then there exists c (=2) optimum covariates in a CRD set-up with v (=0; mod 4 or =2; mod 4) treatments with b replications for each treatment even if $\mathbf{H}_v$ and $\mathbf{H}_b$ do not exist.

Proof (by construction)**:** For construction of optimum **W** matrices of order vxb, we follow the steps given below.

Step 1. Let us consider two Hadamard matrices $\mathbf{H}_4$ and $\mathbf{H}_2$.

$$H_4 = (1, h_1, h_2, h_3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad H_2 = (1, h_1^*) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Step 2. Using $\mathbf{H}_4$ and $\mathbf{H}_2$, we construct the following three **W*** matrices ($\mathbf{W}_1$*, $\mathbf{W}_2$* and $\mathbf{W}_3$*) of order 4x2 by Kronecker product of the columns (with zero sums) of these two matrices.

$$W_1^* = h_1 \otimes h_1^{*'} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad W_2^* = h_2 \otimes h_1^{*'} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$W_3^* = h_3 \otimes h_1^{*'} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}$$

Step 3. Firstly, repeat each of the $\mathbf{W}_i$* (i = 1, 2, 3) vertically side by side q-1 (≥1) times such that b = 2q. Let the newly matrix be denoted as $\mathbf{W}_i$** (i = 1, 2, 3) of order 4xb.

$$W_1^{**} = \left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{matrix} \ldots \\ \ldots \\ \ldots \\ \ldots \end{matrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}}^{q-1} \right)$$

Similarly, construct the W_2** and W_3**.

Step 4. Next, repeat the first pair of rows of each of the $\mathbf{W}_i$** matrix horizontally p-2 (≥1) times such that v = 2p. Let the constructed matrix be denoted as $\mathbf{W}_i$ (i = 1, 2, 3) of order vxb.

$$W_1 = \left(\begin{array}{l} \begin{pmatrix} 1\text{-}1\ 1\text{-}1\ 1\text{-}1 \ldots\ 1\text{-}1 \\ \text{-}1\ 1\text{-}1\ 1\text{-}1\ 1 \ldots \text{-}1\ 1 \end{pmatrix} \\ \;\; 1\text{-}1\ 1\text{-}1\ 1\text{-}1 \ldots\ 1\text{-}1 \\ \;\; \text{-}1\ 1\text{-}1\ 1\text{-}1\ 1 \ldots \text{-}1\ 1 \\ \left.\begin{array}{l} \begin{pmatrix} 1\text{-}1\ 1\text{-}1\ 1\text{-}1 \ldots\ 1\text{-}1 \\ \text{-}1\ 1\text{-}1\ 1\text{-}1\ 1 \ldots \text{-}1\ 1 \end{pmatrix} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \begin{pmatrix} 1\text{-}1\ 1\text{-}1\ 1\text{-}1 \ldots\ 1\text{-}1 \\ \text{-}1\ 1\text{-}1\ 1\text{-}1\ 1 \ldots \text{-}1\ 1 \end{pmatrix} \end{array}\right\} p\text{-}2 \end{array} \right)$$

Similarly, construct the W_2 and W_3 matrices.

Step 5. Among the three **W** matrices, either the pair ($\mathbf{W}_1$, $\mathbf{W}_3$) or **(**$\mathbf{W}_2$, $\mathbf{W}_3$) are satisfying the conditions $\mathbf{C}_1$ and $\mathbf{C}_2$ simultaneously for global optimality.

For easy understanding of the above steps, the following example will be useful.

Example 3.1: Let us consider a CRD with v = 6 and b = 6. The constructional procedure of two optimum **W** matrices is given below:

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

By Kronecker product of the columns (with zero sums) of these two matrices, we get

$$W_1^* = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad W_2^* = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad W_3^* = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}$$

In each $\mathbf{W}_i^*$, (i=1,2,3), the pair of columns replicated vertically twice and we get

$$W_1^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}, \quad W_2^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}, \quad W_3^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

Again in each $\mathbf{W}_i^{**}$, (i=1,2,3), first pair of rows further replicated one time and we get

$$W_1 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

$\{\mathbf{W}_1, \mathbf{W}_3\}$ and $\{\mathbf{W}_2, \mathbf{W}_3\}$ are the two sets, each having two optimum **W** matrices satisfying the conditions $\mathbf{C}_1$ and $\mathbf{C}_2$ simultaneously.

4. OCDS IN RBD SET-UP

For two-way layout, the set-up can be written as

$$(\mathbf{Y}, \mu\mathbf{1}+\mathbf{X}_1\boldsymbol{\tau}+\mathbf{X}_2\boldsymbol{\beta}+\mathbf{Z}\boldsymbol{\gamma}, \sigma^2\mathbf{I}) \qquad (4.1)$$

where μ, as usual, stands for the general effect, $\boldsymbol{\tau}^{vx1}$, $\boldsymbol{\beta}^{bx1}$ represent vectors of treatment and block effects, respectively, $\mathbf{X}_1^{nxv}$ and $\mathbf{X}_2^{nxb}$ are the corresponding incidence matrices, respectively. **Y** and **Z** as usual, represents an observation vector of order nx1 and the design matrix of order nxc corresponding to vector of covariate effects $\boldsymbol{\gamma}^{cx1}$, respectively.

The information matrix for the whole set of parameters $\eta = (\mu, \tau', \beta', \gamma')'$ underlying a design d with $\mathbf{X}_{1d}$, $\mathbf{X}_{2d}$ and $\mathbf{Z}_d$ as the versions of $\mathbf{X}_1$, $\mathbf{X}_2$ and **Z** in (4.1):

$$I_d(\eta) = \begin{pmatrix} n & 1'X_{1d} & 1'X_{2d} & 1'Z_d \\ & X_{1d}'X_{1d} & X_{1d}'X_{2d} & X_{1d}'Z_d \\ & & X_{2d}'X_{2d} & X_{2d}'Z_d \\ & & & Z_d'Z_d \end{pmatrix} \qquad (4.2)$$

For the covariates, without loss of generality, the (location scale)-transformed version, $\left|z_{ij}^{(t)}\right| \le 1$; *i*, *j*, *t* . From (4.2), it is evident that orthogonal estimation of treatment and block effect contrasts on one hand and covariate effects on the other is possible when the conditions

$$\mathbf{X}_{1d}'\mathbf{Z}_d = \mathbf{0}, \text{ and } \mathbf{X}_{2d}'\mathbf{Z}_d = \mathbf{0} \qquad (4.3)$$

are satisfied. It is to be noted that under (4.3), $\mathbf{1}'\mathbf{Z}_d = \mathbf{0}'$ also holds. Further, the most efficient estimation of γ-components is possible whenever, in addition to (4.3), we can also ascertain

$$\mathbf{Z}_d'\mathbf{Z}_d = n\mathbf{I}_c \qquad (4.4)$$

For an RBD set-up, following Das *et al.* (2003), we recast each column of the $\mathbf{Z}^{nxc}$ = (±1) matrix by a **W**-matrix of order vxb. Corresponding to the treatment x block classifications, conditions (4.3) and (4.4) reduce, in terms of **W**-matrices, to $\mathbf{C}_1^*$ - $\mathbf{C}_3^*$ where

$\mathbf{C}_1^*$: Each **W**-matrix has all column-sums equal to zero;

$\mathbf{C}_2^*$: Each **W**-matrix has all row-sums equal to zero;

$\mathbf{C}_3^*$: The grand total of all the entries in the Hadamard product of any two distinct **W**-matrices reduces to zero.

4.1 Construction of optimum W-matrices for covariate model in RBD set-up:

Definition 4.1: With respect to model (4.1), the c number of **W**-matrices corresponding to the c covariates are said to be optimum if they satisfy conditions $\mathbf{C}_1^*$, $\mathbf{C}_2^*$ and $\mathbf{C}_3^*$ simultaneously.

Now, under the realization of $\mathbf{C}_1^*$, $\mathbf{C}_2^*$ and $\mathbf{C}_3^*$ in terms of optimum **W** matrices, we can develop the following theorem.

Theorem 4.1: If both v and b be two even numbers, then there exists c (= 3) optimum covariates in a Randomized Complete Block Design (RCBD or RBD) with v (=0; mod 4) treatments and b number of blocks even if $\mathbf{H}_v$ and $\mathbf{H}_b$ do not exist.

Proof (by construction)**:** For construction of optimum W matrices of order vxb, we follow the steps given below.

Step 1. Let us consider two Hadamard matrices $\mathbf{H}_2$ and $\mathbf{H}_4$ as shown in step 1 of theorem 3.1.

Step 2. Using $\mathbf{H}_2$ and $\mathbf{H}_4$, we construct three **W*** matrices ($\mathbf{W}_1$*, $\mathbf{W}_2$* and $\mathbf{W}_3$*) of order 2x4 by Kronecker product of the columns (with zero sums) of these two matrices.

$$W_1^* = h_1 \otimes h_1^{*'} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix},\quad W_2^* = h_1 \otimes h_2^{*'} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix},$$

$$W_3^* = h_1 \otimes h_3^{*'} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

Step 3. Firstly, repeat each of the $\mathbf{W}_i$* (i = 1, 2, 3) vertically side by side q-1 (≥1) times such that v = 4q. Let the newly matrix be denoted as $\mathbf{W}_i$** (i = 1, 2, 3) of order 2xv.

$$W_1^{**} = \left(\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}}_{q-1} \right)$$

Similarly, construct the W_2** and W_3**.

Step 4. Repeat each of the $\mathbf{W}_i$** matrix horizontally p-1 (≥1) times such that b = 2p. Let the constructed matrix be denoted as $\mathbf{W}_i$ ***(i = 1, 2, 3) of order bxv.

$$W_1^{***} = \begin{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 \end{pmatrix} \\ \left.\begin{matrix} \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 \end{pmatrix} \\ \cdots \\ \cdots \\ \cdots \\ \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 \end{pmatrix} \end{matrix}\right\} p-1 \end{pmatrix}$$

Similarly, construct the W_2*** and W_3***.

Step 5. After taking the transpose of each $\mathbf{W}_i$*** matrices, we get the set of desired covariate matrices ($\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$) satisfying the conditions $\mathbf{C}_1^*$, $\mathbf{C}_2^*$ and $\mathbf{C}_3$* simultaneously for global optimality.

For easy understanding of the above steps, the following example will be useful.

Example 4.1: Let us consider a RBD with v = 20 and b = 6. The method of construction of three **W** matrices are given below:

By taking the Kronecker product of the columns (with zero sums) of $\mathbf{H}_2$ and $\mathbf{H}_4$ matrices, we get,

$$W_1^* = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix},\quad W_2^* = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix},\quad W_3^* = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

In each $\mathbf{W}_i$*, (i=1,2,3), whole set of columns replicated vertically four times and we get

$$W_1^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix},$$

$$W_2^{**} = \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

$$W_3^{**} = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

Again in each $\mathbf{W}_i$**, (i=1,2,3), the whole set of rows further replicated horizontally two times, then we get,

$$W_1^{***} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix},$$

$$W_2^{***} = \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

$$W_3^{***} = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

After transpose of each $\mathbf{W}_i$***, (i=1,2,3), we get the ultimate three optimum **W** matrices satisfying the conditions $\mathbf{C}_1^*$, $\mathbf{C}_2^*$ and $\mathbf{C}_3$* simultaneously, e.g., $W_1 = W_1^{***'}$, $W_2 = W_2^{***'}$ and $W_3 = W_3^{***'}$.

Corollary 4.1: The optimal covariate design in RBD developed by theorem 4.1 is true for CRD with similar v and b.

Proof: Straight forward from the definition of CRD.

Theorem 4.2: The existence of a Hadamard matrix of order v, $\mathbf{H}_v$ and a Special Array of order b, $\mathbf{H}_b$* (b=0; mod 4) with r rows and columns with all zero elements in middle, implies the existence of either (i) $(r-1)^2$ or (r-1)(v-1) optimal covariates when $(r-1)^2$ or (r-1)(v-1) is less than (v-1)(b-r-1) or (ii) (v-1)(b-r-1) optimal covariates when $(r-1)^2$ or (r-1)(v-1)≥(v-1)(b-r-1) of a RBD with v treatments in b blocks provided $\mathbf{H}_r$ and $\mathbf{H}_{b-r}$ exist and r = v/m, where, m is any real valued positive integer number.

Proof (by construction): For construction of optimum W matrices of order vxb, we follow the steps given below.

Step 1. Let us consider a Hadamard matrix of order v, $\mathbf{H}_v=(\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{v-1})$.

Step 2. Let us construct a Special Array $\mathbf{H}_b$* of order b from $\mathbf{H}_{b-r}$ with r rows and columns with all zero elements in middle, i.e., $(\mathbf{1}^*, \mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_{(b-r)/2-1}^*, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{h}_{(b-r)/2}^*, \ldots, \mathbf{h}_{b-r-1}^*)$.

$$H_b^* = \begin{pmatrix} 1 & 1 & \overbrace{0 \ldots\ldots 0}^{r} & 1 & 1 \\ 1 & -1 & 0 \ldots\ldots 0 & -1 & 1 \\ 0 & 0 & 0 \ldots\ldots 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots\ldots 0 & 0 & 0 \\ 1 & 1 & 0 \ldots\ldots 0 & -1 & -1 \\ 1 & -1 & 0 \ldots\ldots 0 & 1 & -1 \end{pmatrix} \updownarrow r$$

Step 3. Using $\mathbf{H}_b$*and $\mathbf{H}_v$, we get (b-r-1) sets of (v-1) W_{ij}* matrices of order bxv (without considering the first column and r columns with all zeros) by taking the Kronecker product of the columns (with zero sums) of the above matrices, where $i=1,2,\ldots,(b\text{-}r\text{-}1)$ and $j=1,2,\ldots,(v\text{-}1)$. In each of the W_{ij}* matrix there are r rows with all elements zero in the middle.

$W_{ij}^* = h_i^* \otimes h'_j$, $\otimes$ denotes the Kronecker product

Step 4. As $\mathbf{H}_v$ and $\mathbf{H}_r$ both exist, following the Theorem 3.4.1 (Das *et al.*, 2015), we construct orthogonal W** matrices either $(r-1)^2$ numbers of order r or (r-1)(v-1) numbers of order rxv.

Step 5. In each **W*** matrix, insert the first W** matrix of order r in the first r columns of W_1* matrix and replicate the selected W** matrix (v-r)/r times or insert the first W** matrix of order rxv in the r rows with all elements zero in the middle of W_1* matrix, such that all the r rows with all elements zero has been replaced by ±1. Let the resulting matrix be W_1'. Repeat the procedure with other W** matrices in the remaining **W*** matrices till all W** matrices or all W* matrices have been covered totally. So, we get either (i) $(r-1)^2$or (r-1)(v-1) W′ matrices of order bxv when $(r-1)^2$ or (r-1)(v-1) < (b-r-1)(v-1) or (ii) (b-r-1)(v-1) W′ matrices of order bxv when $(r-1)^2$ or (r-1)(v-1) ≥ (b-r-1)(v-1), which are orthogonal to each other and all the **W′** matrix has all column-sums and row-sums equal to zero. Finally, the desired **W** matrices of order vxb satisfying the conditions $\mathbf{C}_1^*$, $\mathbf{C}_2^*$ and $\mathbf{C}_3$* simultaneously can be developed by taking the transpose of W'_{ij} matrices.

Remark 4.1: If v ≠ mr, then either (i) (r-1)(v-1) optimal covariates exists for (r-1)(v-1)<(v-1)(b-r-1) or (ii) (v-1)(b-r-1) optimal covariates exists for (r-1)(v-1)≥(v-1)(b-r-1) of RBD with v treatments in b blocks provided $\mathbf{H}_r$ and $\mathbf{H}_{b-r}$ exists.

Remark 4.2: When $\mathbf{H}_r$ do not exist, then (i) $(a-1)^2$ or (a-1)(v-1) optimal covariates exists when $(a-1)^2$ or (a-1)(v-1)<(v-1)(b-r-1) and (ii) (v-1)(b-r-1) optimal covariates exists when $(a-1)^2$ or (a-1)(v-1)≥(v-1)(b-r-1) of RBD with v treatments in b (0 or 2; mod 4) blocks where r can be partitioned in such a way that r = a+e+...+u, provided $\mathbf{H}_a$, $\mathbf{H}_e$,..., $\mathbf{H}_u$ exists and a = min (a, e,..., u).

For easy understanding of the above steps, the following example will be useful.

Example 4.2: Let us consider a RBD with v = 16 and b = 20. When r = 4, the nine **W** matrices are given below:

Step 1. Let us consider a Hadamard matrix of order 16, $\mathbf{H}_{16}$.

$$H_{16}=(1,h_1,h_2,\ldots,h_{15})=\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
1&1&-1&-1&1&1&-1&-1&1&1&-1&-1&1&1&-1&-1\\
1&-1&-1&1&1&-1&-1&1&1&-1&-1&1&1&-1&-1&1\\
1&1&1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1&-1\\
1&-1&1&-1&-1&1&-1&1&1&-1&1&-1&-1&1&-1&1\\
1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1&-1&1&1\\
1&-1&-1&1&-1&1&1&-1&1&-1&-1&1&-1&1&1&-1\\
1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1\\
1&-1&1&-1&1&-1&1&-1&-1&1&-1&1&-1&1&-1&1\\
1&1&-1&-1&1&1&-1&-1&-1&-1&1&1&-1&-1&1&1\\
1&-1&-1&1&1&-1&-1&1&-1&1&1&-1&-1&1&1&-1\\
1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1&1&1&1&1\\
1&-1&1&-1&-1&1&-1&1&-1&1&-1&1&1&-1&1&-1\\
1&1&-1&-1&-1&-1&1&1&-1&-1&1&1&1&1&-1&-1\\
1&-1&-1&1&-1&1&1&-1&-1&1&1&-1&1&-1&-1&1
\end{pmatrix}$$

Step 2. Let us construct a Special Array of order 20 from $\mathbf{H}_{16}$ with 4 rows and columns with all zero elements in the middle, $\mathbf{H}_{20}$* i.e., ($\mathbf{1}$*, $\mathbf{h}_1$*, $\mathbf{h}_2$*,…,$\mathbf{h}_7$*,0,0,0,0,$\mathbf{h}_8$*,…,$\mathbf{h}_{15}$*).

$$H^*_{20}=(1^*,h_1{}^*,h_2{}^*,\ldots,h_7{}^*,0,0,0,0,h_8{}^*,\ldots,h_{15}{}^*)$$

$$=\begin{pmatrix}
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1\\
1&-1&1&-1&1&-1&1&-1&0&0&0&0&1&-1&1&-1&1&-1&1&-1\\
1&1&-1&-1&1&1&-1&-1&0&0&0&0&1&1&-1&-1&1&1&-1&-1\\
1&-1&-1&1&1&-1&-1&1&0&0&0&0&1&-1&-1&1&1&-1&-1&1\\
1&1&1&1&-1&-1&-1&-1&0&0&0&0&1&1&1&1&-1&-1&-1&-1\\
1&-1&1&-1&-1&1&-1&1&0&0&0&0&1&-1&1&-1&-1&1&-1&1\\
1&1&-1&-1&-1&-1&1&1&0&0&0&0&1&1&-1&-1&-1&-1&1&1\\
1&-1&-1&1&-1&1&1&-1&0&0&0&0&1&-1&-1&1&-1&1&1&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&-1&-1&-1&-1&-1&-1&-1&-1\\
1&-1&1&-1&1&-1&1&-1&0&0&0&0&-1&1&-1&1&-1&1&-1&1\\
1&1&-1&-1&1&1&-1&-1&0&0&0&0&-1&-1&1&1&-1&-1&1&1\\
1&-1&-1&1&1&-1&-1&1&0&0&0&0&-1&1&1&-1&-1&1&1&-1\\
1&1&1&1&-1&-1&-1&-1&0&0&0&0&-1&-1&-1&-1&1&1&1&1\\
1&-1&1&-1&-1&1&-1&1&0&0&0&0&-1&1&-1&1&1&-1&1&-1\\
1&1&-1&-1&-1&-1&1&1&0&0&0&0&-1&-1&1&1&1&1&-1&-1\\
1&-1&-1&1&-1&1&1&-1&0&0&0&0&-1&1&1&-1&1&-1&-1&1
\end{pmatrix}$$

Step 3. Using $\mathbf{H}_{20}$*and $\mathbf{H}_{16}$, by Kronecker product of these two matrices, we get 15 set of 15 W_{ij}* matrices of order 20x16 where i=1,2,…,15 and j=1,2,…,15. In each of the W_{ij}* matrix there are 4 rows with all elements zero in the middle. First matrix of first set W_{11}* is the following.

$$W^*_{11}=h^*_1\otimes h'_1$$

$$=\begin{pmatrix}1\\-1\\1\\-1\\1\\-1\\1\\-1\\0\\0\\0\\0\\1\\-1\\1\\-1\\1\\-1\\1\\-1\end{pmatrix}\otimes(1\ -1\ 1\ -1\ 1\ -1\ 1\ -1\ 1\ -1\ 1\ -1\ 1\ -1\ 1\ -1)$$

$$=\begin{pmatrix}
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1
\end{pmatrix}$$

Similarly, we can easily construct the others.

Step 4. As $\mathbf{H}_{16}$ and $\mathbf{H}_4$ both are exists, following the Theorem 3.4.1 (Das *et al.*, 2015), we construct either 9W** matrices of order 4 or 45 W** matrices of order 4x16, which are orthogonal to each other. Here, in this case, 9W** matrices of order 4 has been considered.

$$W_1^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \quad W_2^{**} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad W_3^{**} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix},$$

$$W_4^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \quad W_5^{**} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \quad W_6^{**} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix},$$

$$W_7^{**} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad W_8^{**} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \text{ and } W_9^{**} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

Step 5. In each W* matrix, insert the first W** matrix of order 4 in the first 4 columns of W_1* matrix and replicate the selected W** matrix 3 times, such that all the 4 rows with all elements zero has been replaced by +1 or -1. Let the resulting matrix be W_{11}'. Repeat the procedure with other W** matrices in the remaining W* matrices till W** matrices or W* matrices has been utilized totally. So, we get 9 W′ matrices of order 20x16 as 9 < 225 which are orthogonal to each other and all the W′ matrix has all column-sums and row-sums equal to zero. Finally, the desired W matrices of order 16x20 satisfying the conditions $\mathbf{C}_1^*$, $\mathbf{C}_2^*$ and $\mathbf{C}_3$* simultaneously can be developed by taking the transpose of W'_{ij} matrices where i=1,2,…,15 and j=1,2,…,15. Here, W_1** matrix is inserted in W^*_{11} matrix and replicate W_1** matrix 3 times, we get the following matrix W'_{11}.

$$W_{11}' = \begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{pmatrix}$$

Finally, the desired $\mathbf{W}_1$ matrix of order 16x20 can be developed by taking the transpose $\mathbf{W}'_{11}$ matrix i.e., $\mathbf{W}_1 = (\mathbf{W}'_{11})'$. Similarly, we can find out the others. Here, we can construct 9**W** matrices. Alternately, we can construct 45**W**matrices by using 45**W**** matrices of order 4x16. For the RBD with v=16 and b=20, the other possible alternatives are shown in the following Table 4.1.

Table 4.1. The other possible alternatives for RBD with v=16 and b=20.

No. of rows (columns) with all elements zero in the SA of order 20 (r)	No. of optimum covariates (c)
8	49 or 105
12	105
16	45
18	15

Corollary 4.2: The optimal covariate design in RBD developed by theorem 4.2 is true for CRD with similar v and b.

Proof: Straight forward from the definition of CRD.

5. CONCLUSION

New global optimal covariate designs in CRD and RBD set-ups have been presented in section 3 and 4. In Theorem 3.1 and Theorem 4.1, the developed designs require only the Hadamard matrices $\mathbf{H}_2$ and $\mathbf{H}_4$. There is no need to existence of Hadamard matrices $\mathbf{H}_v$ and $\mathbf{H}_b$, where v is the number of treatments and b is the number of replications or blocks. The Theorem 4.2 yields several OCDs in RBD set-up by using $\mathbf{H}_v$ and special array of order b; $\mathbf{H}_b$ does not exist. The developed optimal covariate designs based on the above theorems are not available in the existing literature.

ACKNOWLEDGEMENT

The authors are grateful to the referee for giving valuable suggestions about the article which are included in the manuscript.

REFERENCES

Das, K., Mandal, N.K. and Sinha, B.K. (2003). Optimal experimental designs with covariates. *J. Stat. Plan. Inference*, **115**, 273-285.

Das, P., Dutta, G., Mandal, N.K. and Sinha, B.K. (2015). *Optimal covariate design-Theory and applications*. Springer, India.

Dey, A. and Mukherjee, R. (2006). D-optimal designs for covariate models. *Statistics*, **40**, 297-305.

Dutta, G. (2004). Optimum choice of covariates in BIBD set-up. *Cal. Stat. Assoc. Bull.*, **55**, 39-55.

Dutta, G., Das, P. and Mandal, N.K. (2007). Optimum choice of covariates for a series of SBIBDs obtained through projective geometry. *J. Modern Appld. Stat. Methods*, **6**, 649-656.

Dutta, G., Das, P. and Mandal, N.K. (2010b). D-optimal designs for covariate parameters in block design set-up. *Commun. Stat. Theory Methods*, **39**, 3434-3443.

Dutta, G., Das, P. and Mandal, N.K. (2014). D-Optimal designs for covariate models. *Commun. Stat. Theory Methods*, **43**, 165-174.

Haggstrom, G.W. (1975). *Pitfalls Manpow Exp*. RAND Corporation, Santa Monica.

Harville, D.A. (1974). Nearly optimal allocation of experimental units using observed covariate values. *Technometrics*, **16**, 589-599.

Harville, D.A. (1975). Computing optimum designs for covariate models. *IN: Srivastava JN (ed) A survey of statistical design and linear models*. 209-228, Amsterdam, North Holland.

Rao, C.R. (1973). *Linear statistical inference and its applications*. Wiley, New York.

Rao, PSSNVP., Rao, S.B., Saha, G.M. and Sinha, B.K. (2003). Optimal designs for covariates' models and mixed orthogonal arrays. Electronic Notes. *Discret. Math*., **15**, 157-160.

Shah, K.R. and Sinha, B.K. (1989). Theory of optimal designs. *Lecture notes in statistics*, **series 54**, Springer, New York.

Sinha, B.K. (2009). A reflection on the choice of covariates in the planning of experimental designs. *J. Indian Soc. Agric. Stat.*, **63(3)**, 219-225.

Troya, Lopes, J. (1982a). Optimal designs for covariate models. *J. Stat. Plan. Inference*, **6**, 373-419.

Troya, Lopes, J. (1982b). Cyclic designs for a covariate model. *J. Stat. Plan. Inference,***7**, 49-75.

Wu, CFJ. (1981). Iterative construction of nearly balanced assignments I: Categorical covariates. *Technometrics*, **23**, 37-44.