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# Balanced Bipartite Generalized Row-Column Designs

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## SUMMARY

This article deals with generalized row-column designs when there are two sets of treatments, one set consisting of test treatments and the other of control treatments called Bipartite Generalized Row-Column Designs. The two sets are disjoint in the sense that there are no treatments in common between the two. The interest here is to estimate the contrasts pertaining to test treatments vs. control treatments with as high precision as possible. Series of Bipartite Generalized Row-Column designs for comparing a set of test treatments to a set of control treatments have been obtained. These designs ensure that all the contrasts pertaining to test vs. control are estimated with less variance in comparison to those pertaining to test vs. test treatments.

*Keywords:* Row-column design, Disjoint set, Test treatments, Control treatments, Bipartite.

## 1. INTRODUCTION

Generalized Row-Column (GRC) design is an arrangement of v treatments in p rows and q columns such that the intersection of each row and column consists of $k$ (>1) units. For instance, an experiment was conducted on tobacco plants at Rothamsted Experimental Station to check whether a mechanism to inhibit tobacco mosaic virus had been carried over to following generations (Bailey and Monod, 2001). Each treatment was a solution made from an extract of one of the offspring plants. The solution was rubbed onto several half-leaves of normal tobacco plants. The number of lesions per half leaf was measured. There were eight plants and pair of half leaves at four heights. This can be considered as generalized row- column design where leaf heights represent rows, plants columns and there are two plots in the intersection of each row and column.

For details on these designs, one may refer to Harshberger and Davis (1952), Darby and Gilbert (1958), Preece and Freeman (1983), Williams (1986), Bailey (1988, 1992), Edmonson (1998), Bailey and Monod (2001), Bedford and Whitaker (2001), Jaggi *et al.* (2010) and Datta *et al.* (2014, 2015, 2016).

In the conventional GRC designs, the interest is to make all possible pair-wise comparisons among the treatment effects. However, there may arise experimental situations where it is desired to compare treatments belonging to two disjoint sets i.e. there are no common treatments between the two. The interest here is to estimate the contrasts of the type ($\tau_i$ - $\tau_j$) with as high precision as possible, $\tau_i$ and $\tau_j$ belongs to 1st and 2nd set of treatments respectively. For example, in agricultural experiments the aim is to test a set of new varieties of a crop with a set of already existing varieties and to determine which of the varieties performs better in comparison to the existing ones. The designs that are efficient for all pair-wise comparisons may not be efficient for this subset of comparisons.

The earliest work on comparing treatments from one set (test treatments) with one or more treatment in second set (control) was carried out by Dunnett (1955). A lot of work has been done under different experimental settings for comparing treatments from one set with a single treatment from other set (Hedayat *et al.*1988; Majumdar and Tamhane 1996; Jaggi *et al.* 1996; Parsad *et al.* 1996; Jaggi and Gupta 1997; Gupta *et al.* 1998; Gupta and Parsad 2001; Parsad and Gupta

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2001; Jacroux 2003; Abeynayake and Jaggi 2009; and Sarkar *et al.* 2013).

This article deals with constructing GRC designs for comparing a set of test treatments to a set of control treatments. The interest here is to estimate the contrasts pertaining to test treatments vs. control treatments with as high precision as possible.

## 2. EXPERIMENTAL SETUP AND MODEL

We consider a GRC design with $v=v_1+v_2$ ($v_1$ test treatments and $v_2$ control treatments) treatments arranged in $p$ rows, $q$ columns and in each row-column intersection (i.e. cells) there are $k$ units or plots resulting in total $n=pqk$ experimental units or observations. The following three-way classified model with treatments, rows and columns, is considered:

$$Y_{l(ij)} = \mu + \alpha_i + \beta_j + \tau_{l(ij)} + e_{l(ij)}; \tag{2.1a}$$

$$i = 1,2,\ldots,p;\ j = 1,2,\ldots,q;\ l = 1,2,\ldots,k$$

where $Y_{l(ij)}$ is the response from the $l^{th}$ unit corresponding to the intersection of $i^{th}$ row and $j^{th}$ column. $\mu$ is the general mean, $\alpha_i$ is the $i^{th}$ row effect, $\beta_j$ is the $j^{th}$ column effect and $\tau_{l(ij)}$ is the effect of the treatment appearing in the $l^{th}$ unit corresponding to the intersection of $i^{th}$ row and $j^{th}$ column. $e_{l(ij)}$ is the error term identically and independently distributed and following normal distribution with mean zero and constant variance.

The above model can be written in matrix notation as follows:

$$\boldsymbol{Y} = \mu\mathbf{1} + \Delta'\boldsymbol{\tau} + \boldsymbol{D}_1'\boldsymbol{\alpha} + \boldsymbol{D}_2'\boldsymbol{\beta} + \boldsymbol{e} \tag{2.1b}$$

where $\boldsymbol{Y}$ is a $n \times 1$ vector of observations, $\mu$ is the grand mean, $\mathbf{1}$ is the $n \times 1$ vector of ones, $\Delta'$ is $n \times v$ incidence matrix of observations versus treatments, $\boldsymbol{\tau}$ is a $v \times 1$ vector of treatment effects, $\boldsymbol{D}_1'$ is $n \times p$ incidence matrix of observations versus rows, $\boldsymbol{\alpha}$ is $p \times 1$ vector of row effects, $\boldsymbol{D}_2'$ is $n \times q$ incidence matrix of observations versus columns, $\boldsymbol{\beta}$ is $q \times 1$ vector of column effects and $\boldsymbol{e}$ is $n ' 1$ vector of random errors with E($\mathbf{e}$) = 0 and D($\mathbf{e}$) = $\sigma^2\boldsymbol{I}_n$. Further,

$$\Delta' 1_{\boldsymbol{v}} = \boldsymbol{D}_1' 1_{\boldsymbol{p}} = \boldsymbol{D}_2' 1_{\boldsymbol{q}} = 1_{\boldsymbol{n}}$$

$$\Delta\boldsymbol{D}_1' = \boldsymbol{N}_1 = \begin{bmatrix} \boldsymbol{N}_{11} \\ \boldsymbol{N}_{12} \end{bmatrix},$$ $(v_1 + v_2)'p$ matrix with $\boldsymbol{N}_{11}$ as the incidence of treatments of first set versus row and $\boldsymbol{N}_{12}$ as the incidence of treatments of second set versus row,

$$\Delta\boldsymbol{D}_2' = \boldsymbol{N}_2 = \begin{bmatrix} \boldsymbol{N}_{21} \\ \boldsymbol{N}_{22} \end{bmatrix},$$ $(v_1 + v_2)'q$ matrix with $\mathrm{N}_{21}$ as the incidence of first set of treatments versus column and $\boldsymbol{N}_{22}$ as the incidence of second set of treatments versus column and $\mathbf{W}$ is the incidence matrix of rows versus columns.

$\mathbf{r} = [\mathbf{r}_{\tau 1}' \ \ \mathbf{r}_{\tau 2}']'$ is the $(v_1 + v_2) \times 1$ replication vector of treatments with $\boldsymbol{r}_{\tau 1}$ as the replication vector of first set treatments and $\boldsymbol{r}_{\tau 2}$ as the replication of second set treatments and

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_2 \end{bmatrix}$$

with $\boldsymbol{R}_1$ $(\boldsymbol{R}_2)$ as the diagonal matrix of replication of first (second) set of treatments.

$\boldsymbol{k}_\alpha = (k_{\alpha 1}, k_{\alpha 2}, \ldots, k_{\alpha p})'$ is the $p \times 1$ vector of row sizes with $\boldsymbol{K}_\alpha = diag\ (k_{\alpha 1}, k_{\alpha 2}, \ldots, k_{\alpha p})$, the diagonal matrix of row-sizes.

$\boldsymbol{k}_\beta = (k_{\beta 1}, k_{\beta 2}, \ldots, k_{\beta q})'$ is the $q \times 1$ vector of column sizes with $\boldsymbol{K}_\beta = diag\ (k_{\beta 1}, k_{\beta 2}, \ldots, k_{\beta q})$ as the diagonal matrix of column-sizes.

The information matrix for a GRC design for two sets of treatments is thus obtained as

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{R}_1 - \boldsymbol{K}_{11} & -\boldsymbol{K}_{12} \\ -\boldsymbol{K}_{21} & \boldsymbol{R}_2 - \boldsymbol{K}_{22} \end{pmatrix} \tag{2.2}$$

where,

$$\boldsymbol{K}_{11} = \boldsymbol{N}_{11}\boldsymbol{K}_\alpha^- \boldsymbol{N}_{11}' + \boldsymbol{N}_{11}\boldsymbol{F}\boldsymbol{Z}^-\boldsymbol{F}'\boldsymbol{N}_{11}' - \boldsymbol{N}_{21}\boldsymbol{Z}^-\boldsymbol{F}'\boldsymbol{N}_{11}' + \boldsymbol{N}_{11}\boldsymbol{F}\boldsymbol{Z}^-\boldsymbol{N}_{11}' + \boldsymbol{N}_{21}\boldsymbol{Z}^-\boldsymbol{N}_{21}'$$
$$\boldsymbol{K}_{12} = \boldsymbol{N}_{11}\boldsymbol{K}_\alpha^- \boldsymbol{N}_{12}' + \boldsymbol{N}_{11}\boldsymbol{F}\boldsymbol{Z}^-\boldsymbol{F}'\boldsymbol{N}_{12}' - \boldsymbol{N}_{21}\boldsymbol{Z}^-\boldsymbol{F}'\boldsymbol{N}_{12}' - \boldsymbol{N}_{11}\boldsymbol{F}\boldsymbol{Z}^-\boldsymbol{N}_{22}' + \boldsymbol{N}_{21}\boldsymbol{Z}^-\boldsymbol{N}_{22}'$$
$$\boldsymbol{K}_{21} = \boldsymbol{N}_{12}\boldsymbol{K}_\alpha^- \boldsymbol{N}_{11}' + \boldsymbol{N}_{12}\boldsymbol{F}\boldsymbol{Z}^-\boldsymbol{F}'\boldsymbol{N}_{11}' - \boldsymbol{N}_{22}\boldsymbol{Z}^-\boldsymbol{F}'\boldsymbol{N}_{11}' - \boldsymbol{N}_{12}\boldsymbol{F}\boldsymbol{Z}^-\boldsymbol{N}_{21}' + \boldsymbol{N}_{22}\boldsymbol{Z}^-\boldsymbol{N}_{21}'$$
$$\boldsymbol{K}_{22} = \boldsymbol{N}_{12}\boldsymbol{K}_\alpha^- \boldsymbol{N}_{12}' + \boldsymbol{N}_{12}\boldsymbol{F}\boldsymbol{Z}^-\boldsymbol{F}'\boldsymbol{N}_{12}' - \boldsymbol{N}_{22}\boldsymbol{Z}^-\boldsymbol{F}'\boldsymbol{N}_{12}' - \boldsymbol{N}_{12}\boldsymbol{F}\boldsymbol{Z}^-\boldsymbol{N}_{22}' + \boldsymbol{N}_{22}\boldsymbol{Z}^-\boldsymbol{N}_{22}'$$

$$\boldsymbol{F} = \boldsymbol{K}_{\acute{a}}^-\boldsymbol{W}$$

$$\boldsymbol{Z} = \boldsymbol{K}_{\hat{a}} - \boldsymbol{W}'\boldsymbol{K}_{\acute{a}}^-\boldsymbol{W}$$

The $(v_1 + v_2) \times (v_1 + v_2)$ matrix $\boldsymbol{C}$ is symmetric, non-negative definite with zero row and column sums. Considering this information matrix, the GRC design for two disjoint sets of treatments is now defined

**Definition:** A GRC design with $p$ rows and $q$ columns with intersection of each row-column having

$k$ units in a cell is said to be a Balanced Bipartite Generalized Row-Column (BBP-GRC) design for comparing a set of $v_1$ treatments to a set of $v_2$ treatments if and only if its C matrix is of the form

$$\mathbf{C}=\begin{bmatrix} (\mathrm{f}_1\text{-}\mathrm{f}_2)\boldsymbol{I}_{v_1}+\mathrm{f}_2\mathbf{1}_{v_1}\mathbf{1}'_{v_1} & \mathrm{f}_3\boldsymbol{1}_{v_1}\boldsymbol{1}'_{v_2} \\ \mathrm{f}_3\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & (\mathrm{f}_4\text{-}\mathrm{f}_5)\mathbf{I}_{v_2}+\mathrm{f}_5\boldsymbol{1}_{v_2}\boldsymbol{1}'_{v_2} \end{bmatrix}$$

such that $f_1 + (v_1\text{-}1)f_2 + f_3v_2 = 0$ and $f_4 + (v_2\text{-}1)f_5 + f_3v_1 = 0$ where $f_1$, $f_2$, $f_3$, $f_4$ and $f_5$ are integers. The parameter of a BBP-GRC design can be represented as $v_1$, $v_2$, $p$, $q$, $k$, $r_1$ (replication of treatments of first set also called as test treatments) and $r_2$ (replication of treatments of second set also called as control treatments).

**Note**: If the first term is not of the form $(\mathrm{f}_1\text{-}\mathrm{f}_2)\mathbf{I}_{v_1}+\mathrm{f}_2\mathbf{1}_{v_1}\mathbf{1}'_{v_1}$, then it may result in a Partially Balanced Bipartite Generalized Row-Column Design.

## 3. METHODS OF CONSTRUCTING BBP-GRC DESIGNS

**Method 3.1**: Consider a Balanced Incomplete Block (BIB) design with parameters $v^*$, $b^*$, $r^*$, $k^*$, $\lambda^*$ and it's complementary with parameters $v^*$, $b^*$, $b^*\text{-}r^*$, $v^*\text{-}k^*$, $v^*\text{-}2r^*+\lambda^*$. Arrange the blocks of the BIB design in the first row giving rise to $q=b^*$ columns. The blocks obtained from the complements are arranged in the second row.

**Case I:** If $v^*>2k^*$, then augment $v_2= v^*\text{-}2k^*$ last treatments called as control treatments to all the cells of the first row. The resulting design will be a BBP-GRC design with parameters $v_1= 2k^*$, $v_2= v^*\text{-}2k^*$, $p= 2$, $q= b^*$, $r_1= b^*$, $r_2= 2b^*$ and $k= v^*\text{-}k^*$.

**Case II:** If $v^*< 2k^*$, then augment $v_2= 2k^*\text{-} v^*$ last treatments called as control treatments to all the cells of the second row. The resulting design will be a BBP-GRC design with parameters $v_1= 2(v^*\text{-}k^*)$, $v_2= 2k^*\text{-} v^*$, $p= 2$, $q= b^*$, $r_1= b^*$, $r_2= 2b^*$ and $k= k^*$.

**Particular Case IA**: Consider a BIB design of the form $v^*= s^2$, $b^*= s(s+1)$, $r^*= s+1$, $k^*= s$, $\lambda^*=1$. A BBP-GRC design with $v=v_1+v_2$, where $v_1= 2s$ and $v_2= s(s\text{-}2)$ treatments arranged in $p=2$ rows, $q=s(s+1)$ columns and in each row-column intersection (i.e. cells) there are $k= s(s\text{-}1)$ units or plots resulting in total $n= 2s^2(s^2\text{-}1)$ experimental units or observations.

The structure of the incidence matrices as per model (2.1b) of the design obtained is as follows:

$$\boldsymbol{N}_1=\begin{pmatrix}\boldsymbol{N}_{11}\\ \boldsymbol{N}_{12}\end{pmatrix}=\begin{pmatrix}(s+1)\,\boldsymbol{I}_{\mathrm{v}_1} & (s^2\text{-}1)\,\boldsymbol{I}_{\mathrm{v}_1}\\ (s+1)^2\,\boldsymbol{I}_{\mathrm{v}_2} & (s^2\text{-}1)\,\boldsymbol{I}_{\mathrm{v}_2}\end{pmatrix}$$

$$\boldsymbol{N}_2=\begin{pmatrix}\boldsymbol{N}_{21}\\ \boldsymbol{N}_{22}\end{pmatrix}=\begin{pmatrix}\boldsymbol{J}_{\mathrm{v}_1\times\mathrm{q}}\\ 2\boldsymbol{J}_{\mathrm{v}_2\times\mathrm{q}}\end{pmatrix}$$

$\boldsymbol{W}= s\boldsymbol{J}_{p\times q}$

So,

$$\boldsymbol{N}_1\boldsymbol{N}'_1=\begin{pmatrix}\boldsymbol{N}_{11}\boldsymbol{N}'_{11} & \boldsymbol{N}_{11}\boldsymbol{N}'_{12}\\ \boldsymbol{N}_{12}\boldsymbol{N}'_{11} & \boldsymbol{N}_{12}\boldsymbol{N}'_{12}\end{pmatrix}=\begin{pmatrix}(s+1)^2\left[1+(s-1)^2\right]\boldsymbol{J}_{\mathrm{v}_1\times\mathrm{v}_1} & \left[(s+1)^3+(s^2\text{-}1)^2\right]\boldsymbol{J}_{\mathrm{v}_1\times\mathrm{v}_2}\\ \left[(s+1)^3+(s^2\text{-}1)^2\right]\boldsymbol{J}_{\mathrm{v}_2\times\mathrm{v}_1} & \left[2(s+1)^2(s^2+1)\right]\boldsymbol{J}_{\mathrm{v}_2\times\mathrm{v}_2}\end{pmatrix}$$

and

$$\boldsymbol{N}_2\boldsymbol{N}'_2=\begin{pmatrix}\boldsymbol{N}_{21}\boldsymbol{N}'_{21} & \boldsymbol{N}_{21}\boldsymbol{N}'_{22}\\ \boldsymbol{N}_{22}\boldsymbol{N}'_{21} & \boldsymbol{N}_{22}\boldsymbol{N}'_{22}\end{pmatrix}=\begin{pmatrix}s(s+1)\boldsymbol{J}_{\mathrm{v}_1\times\mathrm{v}_1} & 2s(s+1)\boldsymbol{J}_{\mathrm{v}_1\times\mathrm{v}_2}\\ 2s(s+1)\boldsymbol{J}_{\mathrm{v}_2\times\mathrm{v}_1} & 4s(s+1)\boldsymbol{J}_{\mathrm{v}_2\times\mathrm{v}_2}\end{pmatrix}$$

Also, $$\boldsymbol{R}=\begin{pmatrix}s(s+1)\boldsymbol{I}_{\mathrm{v}_1\times\mathrm{v}_1} & 0\\ 0 & 2s(s+1)\boldsymbol{I}_{\mathrm{v}_2\times\mathrm{v}_2}\end{pmatrix}$$

$\boldsymbol{K}_\alpha=kq\boldsymbol{I}_p=s^2(s+1)\boldsymbol{I}_p$ and $\boldsymbol{K}_\beta=kp\boldsymbol{I}_p=2s\boldsymbol{I}_q$

The information matrix for estimating the treatment effects of BBP-GRC design is obtained as

$$\boldsymbol{C}=\begin{pmatrix} s(s+1)\boldsymbol{I}_{\mathrm{v}_1\times\mathrm{v}_1}-\dfrac{(s+1)\left[1+(s-1)^2\right]}{s^2(s\text{-}1)}\boldsymbol{J}_{\mathrm{v}_1\times\mathrm{v}_1} & -\dfrac{\left[(s+1)^3+(s^2\text{-}1)^2\right]}{s^2(s^2\text{-}1)}\boldsymbol{J}_{\mathrm{v}_1\times\mathrm{v}_2}\\ -\dfrac{\left[(s+1)^3+(s^2\text{-}1)^2\right]}{s^2(s^2\text{-}1)}\boldsymbol{J}_{\mathrm{v}_2\times\mathrm{v}_1} & 2s(s+1)\boldsymbol{I}_{\mathrm{v}_2\times\mathrm{v}_2}-\dfrac{\left[2(s+1)(s^2+1)\right]}{s^2(s\text{-}1)}\boldsymbol{J}_{\mathrm{v}_2\times\mathrm{v}_2}\end{pmatrix}$$

**Example 3.1.1**: Consider a BIB design with parameters as $v^*= 9$, $b^*= 12$, $r^*= 4$, $k^*= 3$, $\lambda^*=1$. Arrange the blocks of this BIB design in the first row

| Rows | Columns | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | I | II | III | IV | V | VI | VII | VIII | IX | X | XI | XII |
| I | 1 2 3<br>7 8 9 | 4 5 6<br>7 8 9 | 7 8 9<br>7 8 9 | 1 4 7<br>7 8 9 | 2 5 8<br>7 8 9 | 3 6 9<br>7 8 9 | 1 6 8<br>7 8 9 | 2 4 9<br>7 8 9 | 3 5 7<br>7 8 9 | 1 5 9<br>7 8 9 | 2 6 7<br>7 8 9 | 3 4 8<br>7 8 9 |
| II | 4 5 6<br>7 8 9 | 1 2 3<br>7 8 9 | 1 2 3<br>4 5 6 | 2 3 5<br>6 8 9 | 1 3 4<br>6 7 9 | 1 2 4<br>5 7 8 | 2 3 4<br>5 7 9 | 1 3 5<br>6 7 8 | 1 2 4<br>6 8 9 | 2 3 4<br>6 7 8 | 1 3 4<br>5 8 9 | 1 2 5<br>6 7 9 |

and its complementary in the second row. Augment 3 treatments (7, 8, 9) to all the cells of the first row. The resulting design will be a BBP-GRC design with parameters $v_1=6$ (numbered as 1,2,3,4,5,6), $v_2=3$ (numbered as 7,8,9), $p=2$, $q=12$, $r_1=12$, $r_2=24$ and $k=6$.

The information matrix for estimating treatment effects of first and second set is obtained as follows

$$\boldsymbol{C}=\begin{pmatrix} 12\boldsymbol{I}_{6\times 6}\text{-}1.111\boldsymbol{J}_{6\times 6} & -1.778\boldsymbol{J}_{6\times 3} \\ -1.778\boldsymbol{J}_{3\times 6} & 24\boldsymbol{I}_{3\times 3}\text{-}4.444\boldsymbol{J}_{3\times 3} \end{pmatrix}$$

$$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'})=0.1667\ \sigma^2,\ s\neq s',\ s,\ s'=1,2,\ldots,v_1$$

$$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'})=0.1285\ \sigma^2,\ s\neq s',\ s=1,2,\ldots,v_1, s'=\mathrm{v}_1+1,\ldots,v_2$$

Average variance is $0.1406\ \sigma^2$.

**Particular Case IB**: Consider a BIB design of the form $v^*$, $b^*={}^{v^*}C_2=\dfrac{v^*(v^*\text{-}1)}{2}$, $r^*=v^*\text{-}1$, $k^*=2$, $\lambda^*=1$. A BBP-GRC design with $v=v_1+v_2$, where $v_1=4$ and $v_2=v^*\text{-}4$ treatments arranged in $p=2$ rows, $q=\dfrac{v^*(v^*\text{-}1)}{2}$ columns and in each row-column intersection there are $k=v^*\text{-}2$ units or plots resulting in total $n=v^*(v^*\text{-}1)(v^*\text{-}2)$ experimental units or observations.

The structure of the incidence matrices as per model (2.1b) of the design obtained is as follows:

$$\boldsymbol{N}_1=\begin{pmatrix}\boldsymbol{N}_{11}\\ \boldsymbol{N}_{12}\end{pmatrix}=\begin{pmatrix} (v^*\text{-}1)\boldsymbol{I}_{\mathrm{v}_1} & \dfrac{(v^*\text{-}1)(v^*\text{-}2)}{2}\boldsymbol{I}_{\mathrm{v}_1} \\ \dfrac{(v^*\text{-}1)(v^*+2)}{2}\boldsymbol{I}_{\mathrm{v}_2} & \dfrac{(v^*\text{-}1)(v^*\text{-}2)}{2}\boldsymbol{I}_{\mathrm{v}_2} \end{pmatrix}$$

$$\boldsymbol{N}_2=\begin{pmatrix}\boldsymbol{N}_{21}\\ \boldsymbol{N}_{22}\end{pmatrix}=\begin{pmatrix}\boldsymbol{J}_{\mathrm{v}_1\times \mathrm{q}}\\ 2\boldsymbol{J}_{\mathrm{v}_2\times \mathrm{q}}\end{pmatrix}$$

$$\boldsymbol{W}=(v^*\text{-}2)\boldsymbol{J}_{\mathrm{p}\times \mathrm{q}}$$

So,

$$\boldsymbol{N}_1\boldsymbol{N}_1'=\begin{pmatrix}\boldsymbol{N}_{11}\boldsymbol{N}_{11}' & \boldsymbol{N}_{11}\boldsymbol{N}_{12}' \\ \boldsymbol{N}_{12}\boldsymbol{N}_{11}' & \boldsymbol{N}_{12}\boldsymbol{N}_{12}'\end{pmatrix}$$
$$=\begin{pmatrix} \dfrac{(v^*\text{-}1)^2\left[4+(v^*-2)^2\right]}{4}\boldsymbol{J}_{\mathrm{v}_1\times \mathrm{v}_1} & \dfrac{(v^*\text{-}1)^2(v^{*2}\text{-}2v^*+8)}{4}\boldsymbol{J}_{\mathrm{v}_1\times \mathrm{v}_2} \\ \dfrac{(v^*\text{-}1)^2(v^{*2}\text{-}2v^*+8)}{4}\boldsymbol{J}_{\mathrm{v}_2\times \mathrm{v}_1} & \dfrac{(v^*\text{-}1)^2(v^{*2}+4)}{2}\boldsymbol{J}_{\mathrm{v}_2\times \mathrm{v}_2} \end{pmatrix}$$

and

$$\boldsymbol{N}_2\boldsymbol{N}_2'=\begin{pmatrix}\boldsymbol{N}_{21}\boldsymbol{N}_{21}' & \boldsymbol{N}_{21}\boldsymbol{N}_{22}' \\ \boldsymbol{N}_{22}\boldsymbol{N}_{21}' & \boldsymbol{N}_{22}\boldsymbol{N}_{22}'\end{pmatrix}$$
$$=\begin{pmatrix} \dfrac{v^*(v^*\text{-}1)}{2}\boldsymbol{J}_{\mathrm{v}_1\times \mathrm{v}_1} & v^*(v^*\text{-}1)\boldsymbol{J}_{\mathrm{v}_1\times \mathrm{v}_2} \\ v^*(v^*\text{-}1)\boldsymbol{J}_{\mathrm{v}_1\times \mathrm{v}_2} & 2v^*(v^*\text{-}1)\boldsymbol{J}_{\mathrm{v}_2\times \mathrm{v}_2} \end{pmatrix}$$

Here, $\boldsymbol{R}=\begin{pmatrix} \dfrac{v^*(v^*\text{-}1)}{2}\boldsymbol{I}_{\mathrm{v}_1\times \mathrm{v}_1} & 0 \\ 0 & v^*(v^*\text{-}1)\boldsymbol{I}_{\mathrm{v}_2\times \mathrm{v}_2} \end{pmatrix}$

$$\boldsymbol{K}_{\acute{a}}=kq\boldsymbol{I}_{\boldsymbol{p}}=\frac{v^*(v^*\text{-}1)(v^*\text{-}2)}{2}\boldsymbol{I}_{\boldsymbol{p}}\ \text{ and } \boldsymbol{K}_{\hat{a}}=kp\boldsymbol{I}_{\boldsymbol{p}}=2(v^*\text{-}2)\boldsymbol{I}_{\boldsymbol{q}}$$

Thus, the information matrix for BBP-GRC design obtained is

$$\boldsymbol{C}=\begin{pmatrix} \dfrac{v^*(v^*\text{-}1)}{2}\boldsymbol{I}_{\mathrm{v}_1\times \mathrm{v}_1}\text{-}\dfrac{(v^*\text{-}1)\left[4+(v^*-2)^2\right]}{2v^*(v^*\text{-}2)}\boldsymbol{J}_{\mathrm{v}_1\times \mathrm{v}_1} & \text{-}\dfrac{(v^*\text{-}1)(v^*\text{-}2v^*+8)}{2v^*(v^*\text{-}2)}\boldsymbol{J}_{\mathrm{v}_1\times \mathrm{v}_2} \\ \text{-}\dfrac{(v^*\text{-}1)(v^*\text{-}2v^*+8)}{2v^*(v^*\text{-}2)}\boldsymbol{J}_{\mathrm{v}_2\times \mathrm{v}_1} & v^*(v^*\text{-}1)\boldsymbol{I}_{\mathrm{v}_2\times \mathrm{v}_2}-\dfrac{(v^*\text{-}1)(v^{*2}+4)}{v^*(v^*\text{-}2)}\boldsymbol{J}_{\mathrm{v}_2\times \mathrm{v}_2} \end{pmatrix}$$

**Example 3.1.2**: Consider a BIB design with parameters as $v^*=6$, $b^*=15$, $r^*=5$, $k^*=2$, $\lambda^*=1$. Arrange the blocks of the BIB design as per the above mentioned method. The resulting design is a BBP-GRC design with parameters $v_1=4$ (1,2,3,4), $v_2=2$ (5,6), $p=2$, $q=15$, $k=4$, $r_1=15$, $r_2=30$.

The information matrix for estimating treatment effects of first and second set is obtained as follows

$$\boldsymbol{C}=\begin{pmatrix} 15\boldsymbol{I}_{4\times 4}\text{-}2.083\boldsymbol{J}_{4\times 4} & -3.333\boldsymbol{J}_{4\times 2} \\ -3.333\boldsymbol{J}_{2\times 4} & 30\boldsymbol{I}_{2\times 2}\text{-}8.333\boldsymbol{J}_{2\times 2} \end{pmatrix}$$

| **Rows** | **Columns** | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **I** | **II** | **III** | **IV** | **V** | **VI** | **VII** | **VIII** | **IX** | **X** | **XI** | **XII** | **XIII** | **XIV** | **XV** |
| I | 1 2 5 6 | 1 3 5 6 | 1 4 5 6 | 1 5 5 6 | 1 6 5 6 | 2 3 5 6 | 2 4 5 6 | 2 5 5 6 | 2 6 5 6 | 3 4 5 6 | 3 5 5 6 | 3 6 5 6 | 4 5 5 6 | 4 6 5 6 | 5 6 5 6 |
| II | 3 4 5 6 | 2 4 5 6 | 2 3 5 6 | 2 3 4 6 | 2 3 4 5 | 1 4 5 6 | 1 3 5 6 | 1 3 4 6 | 1 3 4 5 | 1 2 5 6 | 1 2 4 6 | 1 2 4 5 | 1 2 3 6 | 1 2 3 5 | 1 2 3 4 |

$$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.1333\sigma^2,\ s \neq s',\ s,\ s' = 1,2,...,\ v_1$$

$$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.1042\sigma^2,\ s \neq s',\ s = 1,2,...,\ v_1, s' = v_1 + 1,...,v_2$$

Average variance is $0.1133\sigma^2$.

**Example 3.1.3**: Consider a BIB design with parameters as $v^*=7,\ b^*=7,\ r^*=4,\ k^*=4,\ \lambda^*=2$. Here, $v^*<2k^*$ i.e. Case II of the method given. Arrange the blocks of this BIB design in the first row and its complementary in the second row. Augment 1 treatment (number 7) to all the cells of the second row. The resulting design will be a BBP-GRC design with parameters $v_1=6$ (numbered as 1,2,3,4,5,6), $v_2=1$ (numbered as 7), $p=2$, $q=7$, $r_1=7$, $r_2=14$ and $k=4$.

| Rows | Columns | | | | | | |
|---|---|---|---|---|---|---|---|
| | I | II | III | IV | V | VI | VII |
| I | 3 5 6 7 | 1 4 6 7 | 1 2 5 7 | 1 2 3 6 | 2 3 4 7 | 1 3 4 5 | 2 4 5 6 |
| II | 1 2 4 7 | 2 3 5 7 | 3 4 6 7 | 4 5 7 7 | 5 6 1 7 | 6 7 2 7 | 7 1 3 7 |

The information matrix for estimating treatment effects of first set and single control is obtained as follows:

$$C = \begin{bmatrix} 7I_6 - 0.89J_{6\times3} & -1.64\ J_{6\times1} \\ -1.64\ J_{1\times6} & 9.86 \end{bmatrix}$$

The variance factor of estimate of contrasts pertaining to test treatments is 0.286 whereas the variance factor of estimate of contrasts pertaining to test treatments versus control is 0.220.

**Remark:** If we consider a Partially Balanced Incomplete Block (PBIB) design with parameters $v^*$, $b^*$, $r^*$, $k^*$, $\lambda_i$ $(i=1,2,...)$ and its complement and use the same method as given above, the resulting design will be a BBP-GRC design.

**Example 3.1.4:** Consider a group divisible (GD) design with parameters $v^*=12$, $b^*=9$, $r^*=3$, $k^*=4$, $\lambda_1=0$, $\lambda_2=1$. Arrange the blocks of the GD design and its complement design as described in the above method. The resulting design will be a BBP-GRC design with parameters $v_1=8$ (1,2,3,4,5,6,7,8), $v_2=4$ (9,10,11,12), $p=2$, $q=9$, $r_1=9$, $r_2=18$ and $k^*=8$.

| Rows | Columns | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | I | II | III | IV | V | VI | VII | VIII | IX |
| I | 1 4<br>7 10<br>9 10<br>11 12 | 1 5<br>8 11<br>9 10<br>11 12 | 1 6<br>9 12<br>9 10<br>11 12 | 2 4<br>8 12<br>9 10<br>11 12 | 2 5<br>9 10<br>9 10<br>11 12 | 2 6<br>7 11<br>9 10<br>11 12 | 3 4<br>9 11<br>9 10<br>11 12 | 3 5<br>7 12<br>9 10<br>11 12 | 3 6<br>8 10<br>9 10<br>11 12 |
| II | 2 3<br>5 6<br>8 9<br>11 12 | 6 2<br>12 7<br>3 4<br>9 10 | 2 3<br>7 8<br>4 5<br>10 11 | 1 3<br>7 9<br>5 6<br>10 11 | 1 3<br>7 8<br>4 6<br>11 12 | 1 3<br>8 9<br>4 5<br>10 12 | 1 2<br>7 8<br>5 6<br>10 12 | 1 2<br>8 9<br>4 6<br>10 11 | 1 2<br>4 5<br>7 9<br>11 12 |

The information matrix for estimating treatment effects of first and second set is obtained as follows

$$C = \begin{pmatrix} 9I_{4\times4} - 0.625J_{4\times4} & -J_{4\times4} \\ -J_{4\times4} & 18I_{4\times4} - 2.5J_{4\times4} \end{pmatrix}$$

$$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.222\sigma^2,\ s \neq s',\ s,\ s' = 1,2,...,\ v_1$$

$$V(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.017\sigma^2,\ s \neq s',\ s = 1,2,...,\ v_1, s' = v_1 + 1,...,v_2$$

Average variance is $0.187\sigma^2$.

**Method 3.2**: **Case I:** Consider a two-class association scheme for $v^*$ treatments with number of first associates as $n_1$ and number of second associates as $n_2$. Arrange the first associates along with the corresponding treatment in the first column. The second associates are arranged in the second column.

i. If $|n_1+1-n_2|$ is even then augment $v_2=(n_1+1-n_2)/2$ new treatments in each cell of column which has lesser cell size. The resulting design will be a BBP-GRC design with parameter $v_1=v^*$, $v_2=(n_1+1-n_2)/2$, $p=v^*$, $q=2$, $r_1=v^*$, $r_2=v^*v_2$ and $k=n_1+1$.

ii. If $|n_1+1-n_2|$ is odd then augment one new treatments $(n_1+1-n_2)$ number of times in each cell of column which has lesser cell size. The resulting design will be a BBP-GRC design with parameter $v_1=v^*$, $v_2=1$, $p=v^*$, $q=2$, $r_1=v^*$, $r_2=v^*(n_1+1-n_2)$ and $k=n_1+1$.

The design obtained is variance balanced with respect to the first set and second set of treatments

**Particular Case**: Consider a triangular association scheme with $v^* = \frac{n(n-1)}{2}$, $n_1 = 2(n-2)$, $n_2 = \frac{(n-2)(n-3)}{2}$. Arrange the first associates along with the corresponding treatment in the first column. The second associates are arranged in the second column. If $|n_1+1-n_2|$ is even augment $v_2$ new treatments in each cell of the second column or $|n_1+1-n_2|$ is odd augment one new treatment in each cell of the second column. The resulting design will be a

BBP-GRC design with parameters $v_1 = v^* = \frac{n(n-1)}{2}$, $v_2 = \frac{n_1 + n_2 - 1}{2} = \frac{9n - n^2 - 12}{4}$, $p = v^* = \frac{n(n-1)}{2}$, $q = 2$, $r_1 = v^* = \frac{n(n-1)}{2}$, $r_2 = pv_2 = \frac{n(n-1)(9n - n^2 + 12)}{8}$ and $k = n_1 + 1 = (2n-3)$.

Here,

$$\boldsymbol{R} = \begin{pmatrix} \frac{n(n-1)}{2}\boldsymbol{I}_{v_1 \times v_1} & 0 \\ 0 & \frac{n(n-1)(9n-n^2-12)}{8}\boldsymbol{I}_{v_2 \times v_2} \end{pmatrix}$$

$\boldsymbol{K}_\alpha = kq\boldsymbol{I}_p = 2(2n-3)\boldsymbol{I}_p$ and $\boldsymbol{K}_\beta = kp\boldsymbol{I}_p = \frac{n(n-1)(2n-3)}{2}\boldsymbol{I}_q$

$$\boldsymbol{N_1} = \begin{pmatrix} \boldsymbol{N}_{11} \\ \boldsymbol{N}_{12} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{v_1 \times p} \\ \frac{9n-n^2-12}{4}\boldsymbol{J}_{v_2 \times p} \end{pmatrix}$$

$$\boldsymbol{N_2} = \begin{pmatrix} \boldsymbol{N}_{21} \\ \boldsymbol{N}_{22} \end{pmatrix} = \begin{pmatrix} (2n-3)\boldsymbol{I}_{v_1} & \frac{(n-2)(n-3)}{2}\boldsymbol{I}_{v_1} \\ 0 & \frac{n(n-1)(9n-n^2-12)}{8}\boldsymbol{I}_{v_2} \end{pmatrix}$$

$\boldsymbol{W} = (2n-3)\,\boldsymbol{J}_{p \times q}$

So,

$$\boldsymbol{N}_1\boldsymbol{N}_1' = \begin{pmatrix} \boldsymbol{N}_{11}\boldsymbol{N}_{11}' & \boldsymbol{N}_{11}\boldsymbol{N}_{12}' \\ \boldsymbol{N}_{12}\boldsymbol{N}_{11}' & \boldsymbol{N}_{12}\boldsymbol{N}_{12}' \end{pmatrix} = \begin{pmatrix} \frac{n(n-1)}{2}\boldsymbol{J}_{v_1 \times v_1} & \frac{n(n-1)(9n-n^2-12)}{8}\boldsymbol{J}_{v_1 \times v_2} \\ \frac{n(n-1)(9n-n^2-12)}{8}\boldsymbol{J}_{v_2 \times v_1} & \frac{n(n-1)(9n-n^2-12)^2}{32}\boldsymbol{J}_{v_2 \times v_2} \end{pmatrix}$$

and

$$\boldsymbol{N}_2\boldsymbol{N}_2' = \begin{pmatrix} \boldsymbol{N}_{21}\boldsymbol{N}_{21}' & \boldsymbol{N}_{21}\boldsymbol{N}_{22}' \\ \boldsymbol{N}_{22}\boldsymbol{N}_{21}' & \boldsymbol{N}_{22}\boldsymbol{N}_{22}' \end{pmatrix} = \begin{pmatrix} \left[(2n-3)^2 + \frac{(n-2)^2(n-3)^2}{4}\right]\boldsymbol{J}_{v_1 \times v_1} & \frac{n^2(n-1)(n-2)(n-3)}{4}\boldsymbol{J}_{v_1 \times v_2} \\ \frac{n^2(n-1)(n-2)(n-3)}{4}\boldsymbol{J}_{v_1 \times v_2} & \frac{n^2(n-1)^2(9n-n^2-12)}{64}\boldsymbol{J}_{v_2 \times v_2} \end{pmatrix}$$

The information matrix for BBP-GRC design obtained is

$$\boldsymbol{C} = \begin{pmatrix} \frac{n(n-1)}{2}\boldsymbol{I}_{v_1 \times v_1} - \frac{2\left[(2n-3)^2 + \frac{(n-2)^2(n-3)^2}{4}\right]}{n(n-1)(2n-3)}\boldsymbol{J}_{v_1 \times v_1} & -\frac{(n-2)(n-3)(9n-n^2-12)}{8(2n-3)}\boldsymbol{J}_{v_1 \times v_2} \\ -\frac{(n-2)(n-3)(9n-n^2-12)}{8(2n-3)}\boldsymbol{J}_{v_2 \times v_1} & \frac{n(n-1)(9n-n^2-12)}{8}\boldsymbol{I}_{v_2 \times v_2} - \frac{n(n-1)(9n-n^2-12)}{32(2n-3)}\boldsymbol{J}_{v_2 \times v_2} \end{pmatrix}$$

**Example 3.2.1**: Consider a triangular association scheme with parameters $v^* = 10$, $n_1 = 6, n_2 = 3$. Arrange the first associates along with the corresponding treatment in the first column. The second associates are arranged in the second column. Here, $|n_1 + 1 - n_2| = 4$, so augment $v_2 = 2$ new treatments two times in each cell of the second column. The resulting design will be a BBP-GRC design with parameters $v_1 = v^* = 10$, $v_2 = 2$, $p = 10$, $q = 2$, $r_1 = 10$, $r_2 = 20$ and $k = 7$.

| **Rows** | **Columns** | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **I** | | | | | | | **II** | | | | | | |
| I | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 12 | 12 |
| II | 2 | 1 | 3 | 4 | 5 | 8 | 9 | 6 | 7 | 10 | 11 | 11 | 12 | 12 |
| III | 3 | 1 | 2 | 4 | 6 | 8 | 10 | 5 | 7 | 9 | 11 | 11 | 12 | 12 |
| IV | 4 | 1 | 2 | 3 | 7 | 9 | 10 | 5 | 6 | 8 | 11 | 11 | 12 | 12 |
| V | 5 | 1 | 6 | 7 | 2 | 8 | 9 | 3 | 4 | 10 | 11 | 11 | 12 | 12 |
| VI | 6 | 1 | 5 | 7 | 3 | 8 | 10 | 2 | 4 | 9 | 11 | 11 | 12 | 12 |
| VII | 7 | 1 | 5 | 6 | 4 | 9 | 10 | 2 | 3 | 8 | 11 | 11 | 12 | 12 |
| VIII | 8 | 2 | 5 | 9 | 3 | 6 | 10 | 1 | 4 | 7 | 11 | 11 | 12 | 12 |
| IX | 9 | 2 | 5 | 8 | 4 | 7 | 10 | 3 | 1 | 6 | 11 | 11 | 12 | 12 |
| X | 10 | 3 | 6 | 8 | 4 | 7 | 9 | 1 | 2 | 5 | 11 | 11 | 12 | 12 |

The information matrix for estimating treatment effects of first and second set is obtained as follows:

$$\boldsymbol{C} = \begin{pmatrix} 10\boldsymbol{I}_{10\times10} - 0.8286\boldsymbol{J}_{10\times10} & -0.8571\boldsymbol{J}_{10\times2} \\ -0.8571\boldsymbol{J}_{2\times10} & 20\boldsymbol{I}_{2\times2} - 5.714286\boldsymbol{J}_{2\times2} \end{pmatrix}$$

$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.200\sigma^2$, $s \neq s'$, $s, s' = 1, 2, \ldots, v_1$

$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.173\sigma^2$, $s \neq s'$, $s = 1, 2, \ldots, v_1, s' = v_1 + 1, \ldots, v_2$

**Case II**: Consider a two-class association scheme ($v^*$, $n_1$, $n_2$). Arrange the first associates along with the corresponding treatment in the first column. The second associates are arranged in the second column. Then augment $v_2$ new treatments in each cell of both the columns. The resulting design will be a BBP-GRC design with parameters $v_1 = v^*$, $v_2$, $p = v^*$, $q = 2$, $r_1 = v^*$, $r_2 = 2v^*$, $k_1 = n_1 + v_2 + 1$ and $k_2 = n_2 + v_2$. The design obtained so will have unequal cell sizes and is variance balanced with respect to the first set and second set of treatments.

**Example 3.2.2**: Consider a group divisible association scheme with parameters vüüü $n_1 =$ $n_2 =$ . Arrange the first associates along with the corresponding treatment in the first column. The second associates are arranged in the second column. Augment 2 new treatments in each cell of both the columns. The resulting design will be a BBP-GRC design with parameters $v_1 = 12$, $v_2 = 2$, $p = 12$, $q = 2$, $r_1 = 12$, $r_2 = 24$, $k_1 = 6$ and $k_2 = 10$.

| Rows | Columns | |
|---|---|---|
| | I | II |
| I | 1 2 3 4 13 14 | 5 6 7 8 9 10 11 12 13 14 |
| II | 2 1 3 4 13 14 | 5 6 7 8 9 10 11 12 13 14 |
| III | 3 1 2 4 13 14 | 5 6 7 8 9 10 11 12 13 14 |
| IV | 4 1 2 3 13 14 | 5 6 7 8 9 10 11 12 13 14 |
| V | 5 6 7 8 13 14 | 1 2 3 4 9 10 11 12 13 14 |
| VI | 6 5 7 8 13 14 | 1 2 3 4 9 10 11 12 13 14 |
| VII | 7 5 6 8 13 14 | 1 2 3 4 9 10 11 12 13 14 |
| VIII | 8 5 6 7 13 14 | 1 2 3 4 9 10 11 12 13 14 |
| IX | 9 10 11 12 13 14 | 1 2 3 4 5 6 7 8 13 14 |
| X | 10 9 11 12 13 14 | 1 2 3 4 5 6 7 8 13 14 |
| XI | 11 9 10 12 13 14 | 1 2 3 4 5 6 7 8 13 14 |
| XII | 12 9 10 11 13 14 | 1 2 3 4 5 6 7 8 13 14 |

The information matrix for estimating treatment effects of first and second set is obtained as follows

$$C = \begin{pmatrix} 12\boldsymbol{I}_{12\times12} - 0.756\boldsymbol{J}_{12\times12} & -1.467\,\boldsymbol{J}_{12\times2} \\ -1.467\,\boldsymbol{J}_{2\times12} & 24\boldsymbol{I}_{2\times2} - 3.2\boldsymbol{J}_{2\times2} \end{pmatrix}$$

$$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.183\sigma^2,\ s \neq s',\ s,\ s' = 1,2,\ldots,\ v_1$$

$$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.134\sigma^2,\ s \neq s',\ s = 1,2,\ldots,\ v_1, s' = v_1 + 1,\ldots,v_2$$

Average variance is $0.155\sigma^2$.

The series can also be obtained by arranging the first associates in the first column and the second associates in the second column and augmenting $v_2$ new treatments in each cell of both the columns resulting in BBP-GRC design with incomplete rows.

**Example 3.2.3**: Consider a group divisible association scheme with parameters vüüü $n_1 =$ $n_2 =$ . Arrange the first associates in the first column and the second associates in the second column and augment 2new treatments in each cell of both the columns resulting in BBP-GRC design with parameter $v_1 = 12$, $v_2 = 2$, $p = 12$, $q = 2$, $r_1 = 11$, $r_2 = 24$, $k_1 = 5$ and $k_2 = 10$:

| Rows | Columns | |
|---|---|---|
| | I | II |
| I | 2 3 4 13 14 | 5 6 7 8 9 10 11 12 13 14 |
| II | 1 3 4 13 14 | 5 6 7 8 9 10 11 12 13 14 |
| III | 1 2 4 13 14 | 5 6 7 8 9 10 11 12 13 14 |
| IV | 1 2 3 13 14 | 5 6 7 8 9 10 11 12 13 14 |
| V | 6 7 8 13 14 | 1 2 3 4 9 10 11 12 13 14 |
| VI | 5 7 8 13 14 | 1 2 3 4 9 10 11 12 13 14 |
| VII | 5 6 8 13 14 | 1 2 3 4 9 10 11 12 13 14 |
| VIII | 5 6 7 13 14 | 1 2 3 4 9 10 11 12 13 14 |
| IX | 10 11 12 13 14 | 1 2 3 4 5 6 7 8 13 14 |
| X | 9 11 12 13 14 | 1 2 3 4 5 6 7 8 13 14 |
| XI | 9 10 12 13 14 | 1 2 3 4 5 6 7 8 13 14 |
| XII | 9 10 11 13 14 | 1 2 3 4 5 6 7 8 13 14 |

The information matrix for estimating treatment effects of first and second set is obtained as follows

$$C = \begin{pmatrix} 10.93\boldsymbol{I}_{12\times12} - 0.678\boldsymbol{J}_{12\times12} & -1.41\boldsymbol{J}_{12\times2} \\ -1.41\boldsymbol{J}_{2\times10} & 24\boldsymbol{I}_{2\times2} - 3.6\boldsymbol{J}_{2\times2} \end{pmatrix}$$

$$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.183\sigma^2,\ s \neq s' = 1,2,\ldots,\ v_1$$

$$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'}) = 0.134\sigma^2,\ s \neq s',\ s = 1,2,\ldots,\ v_1, s' = v_1 + 1,\ldots,v_2$$

Average Variance $0.169\sigma^2$

**Method 3.3**: Consider any GRC design with parameters $v^*$, $p^*$, $q^*$, $r^*$ and $k^*$. Out of $v^*$ treatments, $cu$ treatments ($c > 1$, $u > 1$) such that $cu \leq (v^* - 2)$ and divide these $cu$ treatments into $c$ sets of size $u$ each. Replace all the treatments of 1st set of size $u$ with 1st control treatment, 2nd set with 2nd control treatment and so on $c$th set with $c$th control treatment. The resulting design is BBP-GRC design for comparing $v_1 = (v^* - cu)$ test treatments, $v_2 = c$ control treatments in $p = p^*$ rows, $q = q^*$ columns, $r_1 = r^*$, $r_2 = ur^*$ and $k = k^*$.

**Example 3.3.1**: Consider the following GRC design (Datta *et al.*, 2016) with parameters $v^* = 7$, $p^* = 3$, $q^* = 7$, $r^* = 6$ and $k^* = 2$:

| Rows | Columns | | | | | | |
|---|---|---|---|---|---|---|---|
| | I | II | III | IV | V | VI | VII |
| I | 1 7 | 2 1 | 3 2 | 4 3 | 5 4 | 6 4 | 7 6 |
| II | 2 6 | 3 7 | 4 1 | 5 2 | 6 3 | 7 4 | 1 5 |
| III | 3 5 | 4 6 | 5 7 | 6 1 | 7 2 | 1 3 | 2 4 |

Let $u = 2$ and $c = 2$, replace the last set of 2 treatments (6 and 7) with one control (5) and second last set of 2 treatments (4 and 5) with another control (4). The design so obtained is a BBP-GRC design for comparing a set of $v_1=3$ (1, 2, 3) treatments of first set replicated $r_1=6$ times with $v_2=2$ (4, 5) treatments of second set replicated $r_2=12$ times in $p=p^*=3$ rows, $q=q^*=7$ columns and cell size $k=2$. The design is as shown below.

| Rows | Columns | | | | | | |
|---|---|---|---|---|---|---|---|
| | I | II | III | IV | V | VI | VII |
| I | 1 5 | 2 1 | 3 2 | 4 3 | 4 4 | 5 4 | 5 5 |
| II | 2 5 | 3 5 | 4 1 | 4 2 | 5 3 | 5 4 | 1 4 |
| III | 3 4 | 4 5 | 4 5 | 5 1 | 5 2 | 1 3 | 2 4 |

The information matrix for estimating treatment effects is obtained as follows:

$$\boldsymbol{C}=\begin{pmatrix} 5.833\boldsymbol{I}_3-0.833\boldsymbol{J}_{3\times3} & -1.667\boldsymbol{J}_{3\times2} \\ -1.667\boldsymbol{J}_{2\times3} & 11.666\boldsymbol{I}_2-3.333\boldsymbol{J}_{2\times2} \end{pmatrix}$$

$$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'})=0.343\sigma^2,\ s\neq s',\ s,\ s' = 1,2,\ldots,\ v_1$$

$$\mathrm{V}(\hat{\tau}_s - \hat{\tau}_{s'})=0.257\sigma^2,\ s\neq s',\ s = 1,2,\ldots,\ v_1, s' = v_1+1,\ldots,v_2$$

Average variance is 0.274 $\sigma^2$.

It is seen that in all the methods obtained above for constructing BBP-GRC designs, the contrast for first set versus second set of treatments is estimated more precisely i.e. estimated variances pertaining to test vs control treatments is less as compared to that of test vs test comparisons.

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## REFERENCES

Abeynayake, N.R, and Jaggi S. 2009. A review of block designs for test treatments - control(s) comparisons. *J. Food Agril.* **2(1)**, 22-29.

Bailey, R.A. 1988. Semi Latin squares. *J. Statist. Plg. Inf.* **18**, 299-312.

Bailey, R.A. 1992. Efficient semi-Latin squares. *Statistica Sinica* **2**, 413-437.

Bailey, R.A., and Monod, H. 2001. Efficient semi-Latin rectangles: Designs for plant disease experiments. *Scandinavian J. Statist.* **28**, 257-270.

Bedford, D., and Whitaker, R.M. 2001. A new construction for efficient semi-Latin squares. *J. Statist. Plg. Inf.* **98**, 287-292.

Darby, L.A., and Gilbert, N. 1958. The Trojan Square. *Euphytica* **7**, 183-188.

Datta, A., Jaggi, S., Varghese, C., and Varghese, E. 2014. Structurally incomplete row-column designs with multiple units per cell. *Statistics and Applications* **12 (1&2)**, 71-79.

Datta, A., Jaggi, S., Varghese, C., and Varghese, E. 2015. Some series of row-column designs with multiple units per cell. *Cal. Statist. Assoc. Bull.* **67**, 265-266.

Datta, A., Jaggi, S., Varghese, C., and Varghese, E. 2016. Series of incomplete row-column designs with two units per cell. *Advances in Methodology and Statistics* **13(1)**, 17-25.

Dunnett, C.W. 1955. A multiple comparison procedure for comparing several treatments with a control. *J. Amer. Statist. Assoc.* **50**, 1096-1121.

Edmondson, R.N. 1998. Trojan square and incomplete Trojan square design for crop research. *J. Agril. Sci.* **31**, 135-142.

Edmondson, R.N. 2002. Generalized incomplete Trojan designs. *Biometrika* **89(4)**, 877-891.

Gupta, V.K. and Parsad, R. 2001. Block designs for comparing test treatments with control treatments-an overview. *Statistics and Applications* **3**, 133-146.

Gupta, V.K., Ramana, D.V.V and Agarwal, S.K. 1998. Weighted A-optimal row-column designs for making treatment-control and treatment-treatment comparisons. *Journal of Combinatorics, Information and System Sciences* **23(1-4)**, 333-344 (Special Issue in honour of Professor J.N. Srivastava).

Harshbarger, B. and Davis, L.L. 1952. Latinized rectangular lattices. *Biometrics* **8**, 73-84.

Hedayat, A.S., Jacrox, M. and Mazumdar, D. 1988. Optimal designs for comparing test treatments with a control. *Statistica Sinica* **3**, 363-370.

Jacroux, M. 2003. Some MV-optimal group divisible type block designs for comparing a set of test treatments with a set of standard treatments. *J. Statist. Plg. Inf.* **113(2)**, 597-615.

Jaggi, S. and Gupta, V.K. 1997. A-optimal block designs with unequal block sizes for comparing two disjoint sets of treatments. *Sankhya* B **59(2)**, 164-180.

Jaggi, S., Gupta, V.K. and Parsad, R. 1996. A-efficient block designs for comparing two disjoint sets of treatments. *Communications in Statistics: Theory & Methods* **25(5)**, 967-983.

Jaggi, S., Varghese, C., Varghese, E. and Sharma, V.K. 2010. Generalized incomplete Trojan-type designs. *Statistics and Probability Letters* **80**, 706-710.

Majumdar, D. and Tamhane, A. C. 1996. Row-column designs for comparing treatments with a control. *J. Statist. Plg. Inf.* **49**, 387-400.

Parsad, R. and Gupta, V.K. 2001. Balanced bipartite row-column designs. *Ars Combinatoria* **61**, 301-312.

Parsad, R., Gupta, V.K. and Singh, V.P.N. 1996. Trace optimal designs with unequal block sizes for comparing two disjoint sets of treatments. *Sankhya B* **58 (3)**, 414-426.

Preece, D.A. and Freeman, G.H. 1983. Semi-Latin squares and related designs. *J. R. Statist. Soc*. **B45**, 267-277.

Sarkar, K., Varghese, C., Jaggi, S. and Varghese, E. 2013. Balanced treatment-control row-column designs. *International Journal of Theoretical & Applied Sciences* **5(2)**, 60-64.

Williams, E.R. 1986. Row and column designs with contiguous replicates, *Austr. J. Statist.* **28**, 154-163.