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# Wavelet based Multi-scale Auto-Regressive (MAR) Model: An Application for Prediction of Coconut Price in Kerala

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## SUMMARY

In recent times, forecasting of agricultural commodity price becomes a major issue. But in the context of forecasting of time series data exhibiting Long-Range Dependence (LRD) becomes more complex with the fractional differencing value. In general, Autoregressive Fractionally Integrated Moving Average (AFRIMA) model is widely used for time-series forecasting having long range dependency. It has been observed that in many cases forecasting performance with ARFIMA model is not satisfactory. Therefore, Multi-scale Autoregressive (MAR) model based on wavelets decomposition can be used as an alternative for time-series forecasting. In the present investigation, MAR model is estimated using wavelet decomposition at level 6. Here, an attempt has been made to improve the forecasting performance of MAR model by inclusion of some extra regressors (modified MAR model). Daily wholesale price data on coconut of Kerala market has been used for the illustration purpose. A comparative study has been made for ARFIMA, MAR and modified MAR model in terms of Mean Squared Error (MSE) and Root Mean Squared Error (RMSE). The empirical study reveals that forecasting ability of modified MAR model outperforms the other two methodologies in terms of lower MSE and RMSE values.

*Keywords:* ARFIMA, Long Range Dependence (LRD), Multi-scale Auto-Regressive (MAR) model, Wavelet transformation.

## 1. INTRODUCTION

In last few decades, the wavelet transform attracted the researcher community for time series analysis in many studies. For financial time series prediction Soltani *et al.* (2000) discussed the use of the wavelet transformation in the presence of Long Range Dependence (LRD) or long memory. LRD have been observed frequently in financial time series models which makes forecasting using ARMA process inefficient. In general, a process is called long memory process if its spectral density is unbounded at the origin. Conventionally, the class of Fractionally Integrated Auto-Regressive Moving Average (ARFIMA) processes (Hosking (1981) and Granger and Joyeux (1980)) is widely used for modelling of financial time series with long memory process. It is defined with the introduction of the Hurst exponent $H$, $H \in (0,1)$, which controls for the fractional behaviour of the process. The Hurst exponent is used as a measure of LRD of a time series model. It relates to the autocorrelations of the time series, and the rate at which these decrease as the lag between pairs of values increases. For $H = 1/2$, the process is called white noise, while the property of LRD is observed for $1/2 < H < 1$. The LRD process is characterized by a very slow hyperbolical decaying of Autocorrelation Function (ACF).The numerous test procedures have been developed for LRD detection. Among them non- parametric Wavelet regression for estimating LRD parameter by Jenson (2000) is widely used. Arby and Vietch (1998) carried out a comparative study and concluded that the wavelet regression based estimation technique outperforms other popular semi-parametric methods. The classical ARIMA ($p$, $d$, $q$) models are not an adequate tool for the prediction of LRD time series, as they are unable to capture the ACF persistence imposed by $H$. The $F(d)$ process, introduced by Granger and Joyeux (1980) allows for the LRD process through fractional differencing and its application leads to the famous ARFIMA ($p$, $d$, $q$) process $\{X_t$, t = 0, 1, 2, …, N-1$\}$ may be defined as,

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$$\varphi(B)(1-B)^d X_t = \theta(B) Z_t \quad (1)$$

where, $\varphi(B) = 1 - \sum_{i=1}^{p} \varphi_i B^i$, $\theta(B) = 1 - \sum_{j=1}^{q} \theta_j B^j$, *B* is the backshift operator and $Z_t$ is the discrete white noise. For a stationary time series the value of fractional differencing parameter $d$ lies between *-1/2* to *1/2*. When $d \in (0,1/2)$, the time-series is characterized as LRD process. The relationship between the Hurst exponent and fractional differencing parameter is $H = 1/2 + d$. Therefore, the closer the value of $d$ is to 1/2, the stronger the LRD present in the model. Various parametric, semiparametric and non-parametric methods for estimating LRD are available in the literature. Paul (2014) applied ARFIMA model to wholesale prices of pigeon pea in different markets of India and concluded that the model has better performance in terms of explained variability and prediction power. Paul *et al.* (2015) have applied ARFIMA model for forecasting of agricultural commodity prices. They have also compared different estimation techniques for estimating the long memory parameter by means of MCMC and concluded that wavelet based estimation outperforms other techniques.

Estimation of ARFIMA is complicated as it involves filtering the LRD influence i.e, estimated value of $d$ is required prior to model fitting. On the other hand, the wavelet analysis allows the presence of LRD, while an estimate of *d* is not required and value of *d* is automatically considered in the wavelet based model. Ghosh *et al.* (2010) used discrete wavelet transformation for testing the trend of all-India monsoon rainfall time- series. They observed that Haar wavelet at level 6 performed better than Daubechies (D4) wavelet at level 6 in terms of power of the test. Some other applications of wavelet transform in time series analysis can be found in Paul *et al.* (2013) and Paul and Birthal (2015). Multi-scale Auto-Regressive (MAR) models (Daoudi *et al.,* 1999) involves wavelet decomposition. The analysed time series is presented as a function of the lagged values of the decomposed coefficients. As a matter of fact, MAR models can be used as a flexible alternative of the ordinary ARMA models, which could be used to explain the structural property of the time-series model. Benaouda *et al.* (2006) proposed a wavelet based approach for short-term electricity price forecasting. Hence, they concluded that the proposed approach presents better forecasting accuracy with an acceptable computation time. Bogdanova and Ivanov (2015) have developed two applied procedures. The first one is an algorithm, which assesses the presence of the LRD exponent based on the wavelet regression of Arby *et al.* (2003). Second, they outlined a data-driven additional regressor selection procedure, which relies on a multi-scale extension of the ACF.

Coconut is a highly valuable multipurpose crop among all the plantation crops. Not only the raw coconut, husk is also collected from the fruits which leads the production of copra, ropes, bags etc. Raw coconut is used as direct consumption as well as for the extraction of oil. Kerala contributes the highest area and production of raw coconut in India (Annual report of Coconut Development Board 2015-16) and hence daily wholesale price data obtained from AGMARKNET (http://agmarknet.gov.in) of raw coconut is considered for this present study.

## 2. LONG RANGE DEPENDENCE (LRD)

A long range dependence or long memory process (Granger and Joyeux, 1980) can be defined as $\sum_{k=0}^{\infty} |\rho_k| = \infty$ where $\rho_k$ is the coefficient of autocorrelation with lag of $k$. Fractional integration is a generalization of integer integration. For example, an autoregressive moving-average process integrated of order $d$ [denoted by ARFIMA $(p,d,q)$] can be represented using equation (1), where $(1-B)^d$ can be expressed as follows,

$$(1-B)^d = 1 + dB + \frac{B^2 d(d-1)}{2!} + \cdots \quad (2)$$

For $0 < d < 0.5$ the long memory process is stationary. For such processes, the effect of a white noise $z(t)$ on $y(t+j)$ decays as $j$ increases. But the rate of decay is much slower (hyperbolically) than for a process integrated of order zero. According to the value of $d$, long memory process can be sub-divided into 4 groups and these are,

| Value of $d$ | Names |
|---|---|
| $d \in (-1/2, 0)$ | Intermediate Memory and Anti-persistence |
| $d = 0$ | White noise(Short-Memory) |
| $d \in (0, 1/2)$ | Stationary and Persistence Long Memory |
| $d \in [\frac{1}{2}, 1)$ | Nonstationary and Persistence Long Memory |

Various methods have been developed to estimate this LRD parameter. These are some important methods listed below,

i. Parametric method- Maximum Likelihood Estimation (MLE).

ii. Semi-parametric method- Whittle, GPH, Sperio etc.

ii. Non-parametric method- R/S analysis, Wavelet etc.

## 3. WAVELET-BASED APPROACH

The term *wavelet* (Vidakovic, 1999) is used to refer to a set of basic functions with a special structure which is the key to the main fundamental properties of wavelets and their usefulness in statistics. Wavelets are fundamental building block functions, analogous to the trigonometric sine and cosine functions. As with a sine or cosine wave, a wavelet function oscillates about zero. This oscillating property makes the function a *wave*. However, the oscillations for a wavelet damp down to zero, hence the name *wavel*et. If $\psi(.)$ be a real valued function defined over the real axis $(-\infty,\infty)$ and satisfying two basic properties such as the integral of $\psi(.)$ is zero and the square of $\psi(.)$ integrates to unity, then the function $\psi(.)$ is called a wave. Various types of filters are used for transformation of the time-series model. Among them, most widely used wavelet filter is Haar filter. The simplest wavelet basis for $L^2(R)$ is the Haar basis. The Haar function is a bonafide wavelet, though not used much in practice, uses a mother wavelet given by,

$$\psi(x) = \begin{cases} 1, & 0 \le x < ½, \\ -1, & ½ \le x \le 1, \\ 0, & \text{otherwise} \end{cases}$$

Haar wavelets possess the property of *compact support*, which means that it will vanish outside of a finite interval.

### 3.1 Wavelet transformation

The maximal overlap discrete wavelet transform (MODWT) is a linear filtering operation that transforms a series into coefficients related to variations over a set of scales. It is similar to discrete wavelet transform (DWT) in that, both are linear filtering operations producing a set of time-dependent wavelet and scaling coefficients. MODWT is well-defined for all sample sizes $N$, whereas for a complete decomposition of $J$ levels. But DWT requires $N$ to be multiple of $2^J$, where J is any positive integer. MODWT also differs from DWT in the sense that it is a highly redundant, non-orthogonal transform. MODWT coefficients are obtained by applying DWT pyramid algorithm once to $\boldsymbol{X}$ and another to the circularly shifted vector $T\boldsymbol{X}$. Hence, the first application yields the usual DWT ($\boldsymbol{W}$) of the time series vector $\mathbf{X}$ computed as $\boldsymbol{W} = P\,\boldsymbol{X}$ and the second application consists of substituting $T\mathbf{X}$ for $\mathbf{X}$ obtained as $\boldsymbol{W} = PT\boldsymbol{X}$, where $\boldsymbol{W}$ and $P$ can be written as $\boldsymbol{W}= [\boldsymbol{W}_1\boldsymbol{W}_2...\boldsymbol{W}_J\boldsymbol{V}_J]'$ and $\boldsymbol{P}=[\boldsymbol{P}_1\boldsymbol{P}_2...\boldsymbol{P}_J\boldsymbol{Q}_J]'$.

The Mallat algorithm (Mallat, 1989) filters the original data series $\mathbf{X} = (X_0, X_1, \ldots , X_{N-1})$ using a pair of high-pass and low-pass filters denoted respectively as, $\mathbf{h} = (h_0, h_1, \ldots , h_{L-1})$ and $\mathbf{g} = (g_0, g_1, \ldots , g_{L-1})$, each of length $L$, $L < N$. The wavelet ($\boldsymbol{W}j$) and the scaling coefficients ($\boldsymbol{V}j$) corresponding to the $j$th level of decomposition, $j = 1, 2, \ldots , J$, $J$ is an positive integer, are obtained by,

$$W_{j,t} \equiv \sum_{l=0}^{L_j-1} h_{j,l} X_{t-l \bmod N} \text{ and}$$

$$V_{j,t} \equiv \sum_{l=0}^{L_j-1} g_{j,l} X_{t-l \bmod N}$$

$$X_t = \sum_{j=1}^{J} W_{j,t} + V_{j,t} \tag{3}$$

where $h_{j,l}$ is the $j^{th}$ level MODWT wavelet filter and $g_{j,l}$ is the $j^{th}$ level MODWT scaling filter. A time-series can be completely or partially decomposed into a number of levels $J_0 \le log_2^N$.

### 3.2 Wavelet-based estimator of the LRD parameter

The wavelet regression relies on the Mallat decomposition and it utilizes the fact that there is a linear relationship between the variable $s_j = \log_2(\text{var}(w_j))$ and the scale $j$, $j \in [j_1, j_2]$, $j_1$ and $j_2$ are integers, referred to as upper and lower cut-off, respectively. Intuitively, one would run linear regression in order to estimate the slope coefficient $\gamma$, where the fractional differencing parameter is expressed as $d = \gamma/2$. And $\gamma$ can be estimated by the equation,

$$\hat{\gamma} = \frac{\sum_{j=j_1}^{j_2} y_j (jS - S_1)/\sigma_j^2}{SS_2 - S_1^2}, \tag{4}$$

where $y_j = log_2((1/N_j)\sum_{t=0}^{N_j-1} | w_{j,t} |^2) - g(j)$, $g(j) = (-1/N_j ln2)$, $\sigma_j^2 \sim 2/N_j ln^2 2$, $S = \sum_{j=j_1}^{j_2} 1/\sigma_j^2$,

$S_1 = \sum_{j=j_1}^{j_2} j/\sigma_j^2, S_2 = \sum_{j=j_1}^{j_2} j^2/\sigma_j^2.$

Bogdanova and Ivanov (2015) have proposed an algorithm based on the simulation studies. It involves the following steps:

*Step* 1: The Mallat algorithm is applied over a window of the first 500 observations.

*Step* 2: An estimate of the slope coefficient $\hat{\gamma}$ is obtained through application of equation (4) and an estimate of *d* is derived as $\hat{d} = \hat{\gamma}/2$.

*Step* 3: The window is slid forward by 1 observation (i.e. the first observation is dropped and the 501st observation is included). The first two steps are performed again, thus obtaining another estimate of $\hat{d}$. The estimating procedure is repeated until the last observation is included in the window.

### 3.3 Wavelet-based prediction: MAR model

The Haar átrous (Shensa, 1992) decomposition of the time series $\mathbf{X} = (X_0, X_1, \ldots, X_{N-1})$ and one-step-ahead forecast of a MAR ($A_j$, $j = 1, \ldots, J + 1$) model can be expressed as,

$$\hat{X}_N = \hat{a}_0 + \sum_{j=1}^{J} \sum_{k=1}^{A_j} \hat{a}_{j,k} w_{j,N-1-2^j(k-1)} + \sum_{k=1}^{A_{J+1}} \hat{a}_{J+1,k} c_{J,N-1-2^J(k-1)}, \quad (5)$$

where $A_j$, $j = 1, \ldots, J$ is the number of lagged values of the wavelet details $\mathbf{W}_j$, $j = 1, \ldots, J$ and $A_{J+1}$ is the number of lagged values of the scaling sequence $\mathbf{C}_J$. The estimates $\{\hat{a}_{j,k}$, $k = 1, \ldots, A_j$, $j = 1, \ldots, J + 1\}$ of the unknown parameters in equation (5) are obtained by Ordinary Least-Squares (OLS) regression. Furthermore, the MAR models are very flexible to the term structure of the analyzed time-series and the presence of LRD might be modelled by the inclusion of just few additional lagged values at the lower frequencies. A matter of applied significance is to decide on how many additional lags to include and more importantly, at which frequencies. Bogdanova and Ivanov (2015) proposed a data-driven solution of the problem. Throughout the manuscript this model is represented as new model. For this purpose, they first introduce the correlation matrix *R* between original series *X* and the lagged values of wavelet coefficients, defined in equation (6):

$$R = \begin{pmatrix} corr(X_t, w_{1,t-1}) \; corr(X_t, w_{2,t-1}) \ldots corr(X_t, w_{J,t-1}) \; corr(X_t, c_{J,t-1}) \\ corr(X_t, w_{1,t-2}) \; corr(X_t, w_{2,t-2}) \ldots corr(X_t, w_{J,t-2}) \; corr(X_t, c_{J,t-2}) \\ . \\ . \\ . \\ corr(X_t, w_{1,t-S}) \; corr(X_t, w_{2,t-S}) \ldots corr(X_t, w_{J,t-S}) \; corr(X_t, c_{J,t-S}) \end{pmatrix} \quad (6)$$

Based on the LRD definition, it might be hypothesized that *R* will be dominated by significant coefficients at the lower frequencies for long-memory time-series. Two common features of *R* for both long and short-memory time-series might be noted. First, the number of significant coefficients depend on the values of the parameters controlling for the memory of the time- series. Second, the higher the value of these parameters, the stronger the influence of the smooth component on the behaviour of time-series. As mentioned earlier, in case of long-memory, MAR specification requires identification of the lagged details to be incorporated in the model. Based on the matrix *R*, the following specification procedure can be followed (Bogdanova and Ivanov, 2015),

*Step* 1: The fractional differencing exponent is estimated through the algorithm, proposed in section 3.2. If significant LRD is identified, then steps 2 and 3 are applied.

*Step* 2: Equation (5) is estimated for $A_j = 1$, $j = 1, 2, \ldots, J, J + 1$.

*Step* 3: The ACF $\{\hat{\rho}_s$, $s = 1, 2, \ldots, S\}$ of the estimated residuals is inspected for the presence of significant coefficients. If significant autocorrelation is detected, then step 4 is performed.

*Step* 4: Let $\{\hat{\rho}_m, m \leq S\}$ be statistically significant, then the MAR(1) model is augmented with the inclusion of $w_{j,(t-m)}$ as an additional regressor, where *j* represents the level of decomposition at which the highest correlation is observed for the $m^{th}$ row of the matrix *R*. This step is performed until the augmented model residuals exhibit no significant autocorrelation.

So, after inclusion of $w_{j,(t-m)}$ as an additional regressor, the new model would be,

$$\hat{X}_N = \hat{a}_0 + \sum_{j=1}^{J} \hat{a}_{j,1} w_{j,N-1} + \hat{a}_{J,1} c_{J,N-1} + \hat{a}_{j,2} w_{j,N-m}. \quad (7)$$

where k is the strongest correlation corresponds to the wavelet coefficient series of $m^{th}$ lag.

## 4. EMPIRICAL STUDY

For the present study the daily financial time series data on "raw coconut" of the market Thodupuze of Kerala for minimum and maximum price has been collected from AGMARKNET over the time period from August, 2014 to December, 2017. The data set consist of 872 data points. The descriptive statistics is presented in the following table:

**Table 1:** Descriptive Statistics

| Series | Min | Max | Mean | SD | CV | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| Minimum | 1100 | 3400 | 1654 | 407.23 | 24.62 | 2.047 | 8.434 |
| Maximum | 1200 | 3500 | 1860 | 422.8 | 22.73 | 1.17 | 5.79 |

The above table shows that standard deviation is high for maximum series whereas the coefficient of variation (CV) is higher for minimum series. Moreover, both the series is positively skewed and leptokurtic in nature.

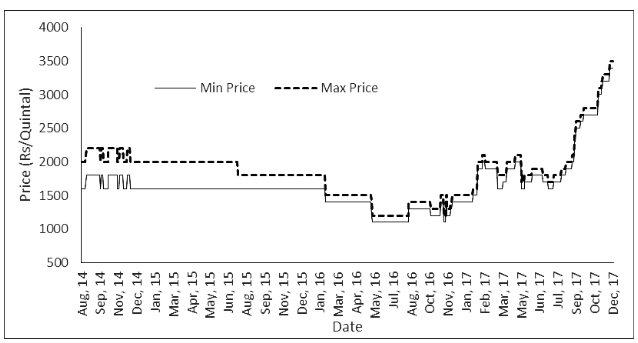


**Fig. 1.** Time plot of minimum and maximum price (Rs. /Quintal) of Coconut

### 4.1 Estimation of LRD parameter

Estimation procedure mentioned in section 3.2 has been performed for both the series and output of the test is displayed in Table 2. It is clearly observed that maximum and minimum series exhibit strong long-range dependence. The ACF plots of two series displayed in Figure 2. It indicates the hyperbolical decaying of auto-correlation near about 200 lags. This indicates the presence of stationary long-memory property of the corresponding series.

**Table 2.** Estimated value of d

| Price Series | d | t-Value |
|---|---|---|
| Minimum | 0.39* | 7.61 |
| Maximum | 0.41* | 8.36 |

*denotes the significance at 5% level.

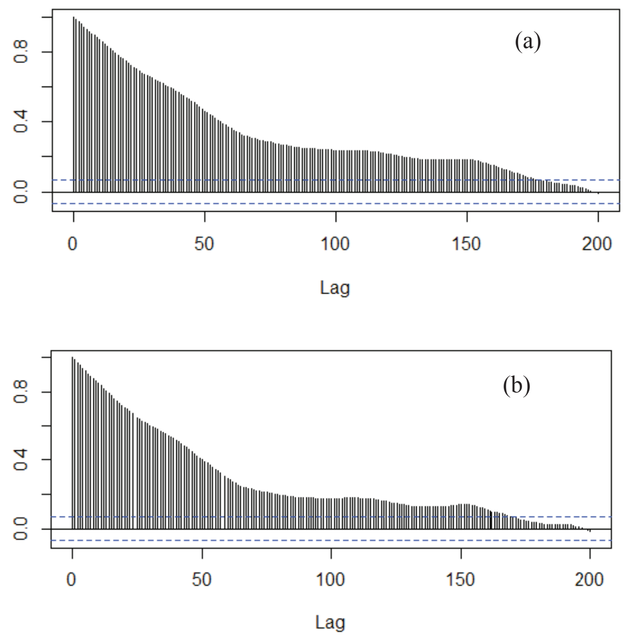


**Fig. 2.** ACF plots of (a) Maximum and (b) Minimum series

### 4.2 Wavelet decomposition of maximum and minimum series

MODWT is computed using "Haar" wavelet filter at level 6. The MODWT coefficients are shown in Figure 3 and Figure 4. The wavelet coefficients are denoted by $W_1$, $W_2$, $W_3$, $W_4$, $W_5$, $W_6$ and the scaling coefficient is denoted by $V_6$. The coefficients at the top (below) are "low-frequency" ("high frequency") information. The wavelet coefficients do not remain constant over time and reflects the changes of the data at various time-epochs. The locations of abrupt jumps can be spotted by looking for vertical (between levels) clustering of relatively large coefficients. Smoothness of the plot increases as we move from high frequency to low frequency components. Here, $V_6$, scaling coefficient is considered as the actual signal, hidden in the noisy time series data.

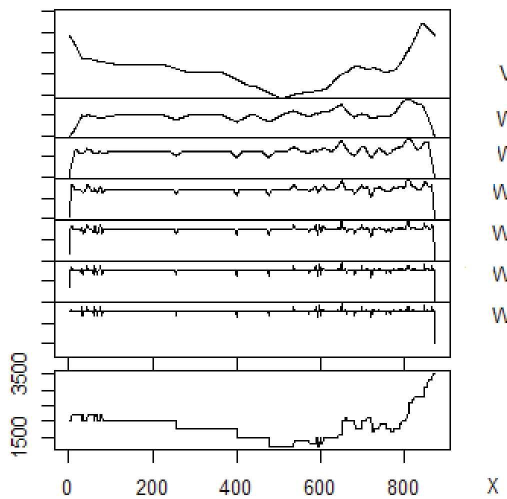


**Fig. 3.** MODWT plot of maximum series

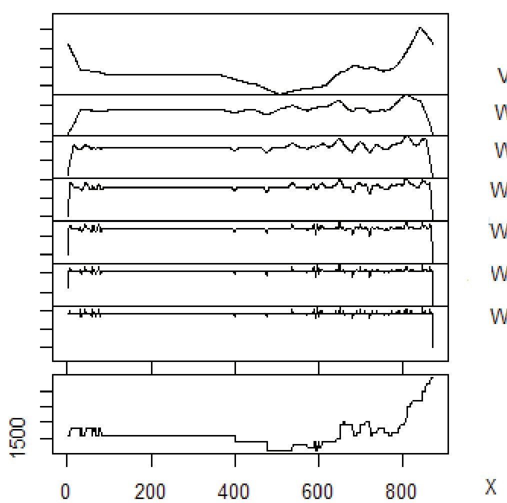


**Fig. 4.** MODWT plot of minimum series

## 4.3 Calculation of correlation matrix R

The correlation coefficients of the maximum and minimum price series with their corresponding 10 lagged wavelet coefficients have been calculated using equation 6 and shown in the below table. The correlation coefficients which are significant at 5% level is mention in that Table 3 and 4.

## 4.4 ARFIMA model fitting

ARFIMA model is fitted for the maximum and minimum series with the help of Equation 1. All the parameters along with the test statistic are provided in the Table 5. All the parameters are significant at 1% level for both the series. Model validation is provided in the section 4.6.

**Table 3.** Correlation coefficients calculated for the series Maximum price with the lagged values of wavelet coefficients and scaling coefficient for decomposition level 6

| Lag | $W_1$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ | $W_6$ | $V_6$ |
|---|---|---|---|---|---|---|---|
| t-1 | 0.127* | 0.224* | 0.307* | 0.335* | 0.318* | 0.384* | 0.814* |
| t-2 | 0.129* | 0.224* | 0.299* | 0.299* | 0.309* | 0.382* | 0.805* |
| t-3 | 0.162* | 0.196* | 0.289* | 0.263* | 0.3* | 0.38* | 0.796* |
| t-4 | 0.097* | 0.188* | 0.271* | 0.24* | 0.292* | 0.379* | 0.787* |
| t-5 | 0.123* | 0.196* | 0.243* | 0.226* | 0.282* | 0.38* | 0.77* |
| t-6 | 0.118* | 0.189* | 0.202* | 0.216* | 0.271* | 0.382* | 0.769* |
| t-7 | 0.121* | 0.176* | 0.16* | 0.21* | 0.262* | 0.384* | 0.76* |
| t-8 | 0.1* | 0.146* | 0.142* | 0.206* | 0.253* | 0.385* | 0.751* |
| t-9 | 0.1* | 0.098* | 0.135* | 0.203* | 0.244* | 0.387* | 0.741* |
| t-10 | 0.053 | 0.087* | 0.134* | 0.201* | 0.236* | 0.388* | 0.732* |

*denotes the significance at 5% level.

**Table 4.** Correlation coefficients calculated for the series Minimum price with the lagged values of wavelet coefficients and scaling coefficient for decomposition level 6

| Lag | $W_1$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ | $W_6$ | $V_6$ |
|---|---|---|---|---|---|---|---|
| t-1 | 0.060 | 0.104* | 0.153* | 0.247* | 0.364* | 0.462* | 0.805* |
| t-2 | 0.065 | 0.106* | 0.144* | 0.251* | 0.362* | 0.462* | 0.794* |
| t-3 | 0.066* | 0.106* | 0.131* | 0.254* | 0.360* | 0.461* | 0.782* |
| t-4 | 0.067* | 0.099* | 0.131* | 0.260* | 0.357* | 0.461* | 0.771* |
| t-5 | 0.063 | 0.090* | 0.138* | 0.267* | 0.353* | 0.460* | 0.759* |
| t-6 | 0.053 | 0.078* | 0.153* | 0.273* | 0.349* | 0.458* | 0.747* |
| t-7 | 0.055 | 0.060* | 0.170* | 0.277* | 0.344* | 0.456* | 0.735* |
| t-8 | 0.031 | 0.087* | 0.187* | 0.278* | 0.338* | 0.455* | 0.722* |
| t-9 | 0.034 | 0.121* | 0.200* | 0.277* | 0.330* | 0.452* | 0.709* |
| t-10 | 0.087 | 0.142* | 0.207* | 0.272* | 0.322* | 0.449* | 0.696* |

*denotes the significance at 5% level.

## 4.5 MAR model fitting

MAR model is fitted for the maximum and minimum series with the help of equation 5. It is seen that the intercept term is not significant at 5% level while rest of the coefficients are significant at same level of significance for both the series. Residuals are calculated for the fitted model for calculating ACF plot. This ACF plot will lead to the selection of the new regressors.

ACF plot of residuals of maximum series given in the Figure5, showed that lag 4 and lag 9 have significant auto-correlation function. Now from table 3, it can be easily found out that $V_6$ has the strongest correlation coefficient with X in both the lags. So according to the suggested algorithm, we have to include $V_6$ of lag 4 and 9 in the MAR model for maximum series.

**Table 5.** Parameter estimation of ARFIMA model for the maximum and minimum series

| Parameters | Maximum | | | | Minimum | | | |
|---|---|---|---|---|---|---|---|---|
| | Estimate | St. Error | Z-value | p-Value | Estimate | St. Error | Z-value | p-Value |
| d | 0.499* | 0.002 | 256.063 | < 0.001 | 0.499* | 0.002 | 250.571 | < 0.001 |
| MA(1) | -0.645* | 0.033 | -19.526 | < 0.001 | -0.649* | 0.033 | -19.548 | < 0.001 |
| MA(2) | -0.515* | 0.042 | -12.211 | < 0.001 | -0.533* | 0.036 | -14.849 | < 0.001 |
| MA(3) | -0.479* | 0.034 | -14.084 | < 0.001 | -0.466* | 0.041 | -11.441 | < 0.001 |
| MA(4) | -0.293* | 0.039 | -7.514 | < 0.001 | -0.284* | 0.039 | -7.241 | < 0.001 |
| MA(5) | -0.140* | 0.039 | -3.534 | < 0.001 | -0.147* | 0.040 | -3.712 | < 0.001 |

*denotes the significance at 1% level.

**Table 6.** Estimation parameters for maximum and minimum series

| Coefficients | Maximum Series | | | | Minimum Series | | | |
|---|---|---|---|---|---|---|---|---|
| | Value | St. error | t-Statistic | p-Value | Values | St. Error | t-Statistics | p-Value |
| Intercept | -1.751 | 7.563 | -0.231 | 0.817 | -6.151 | 7.253 | -0.855 | 0.403 |
| $W_1$ | 1.009* | 0.048 | 20.934 | <0.001 | 1.021* | 0.043 | 23.792 | <0.001 |
| $W_2$ | 1.022* | 0.044 | 22.842 | <0.001 | 1.023* | 0.04 | 25.857 | <0.001 |
| $W_3$ | 0.957* | 0.034 | 27.817 | <0.001 | 0.971* | 0.03 | 31.962 | <0.001 |
| $W_4$ | 0.998* | 0.026 | 38.312 | <0.001 | 0.994* | 0.023 | 43.631 | <0.001 |
| $W_5$ | 0.993* | 0.019 | 50.102 | <0.001 | 1.005* | 0.017 | 57.734 | <0.001 |
| $W_6$ | 1.026* | 0.013 | 76.505 | <0.001 | 1.029* | 0.012 | 86.013 | <0.001 |
| $V_6$ | 1.001* | 0.003 | 250.739 | <0.001 | 1.007* | 0.004 | 233.502 | <0.001 |

*denotes the significance at 5% level.

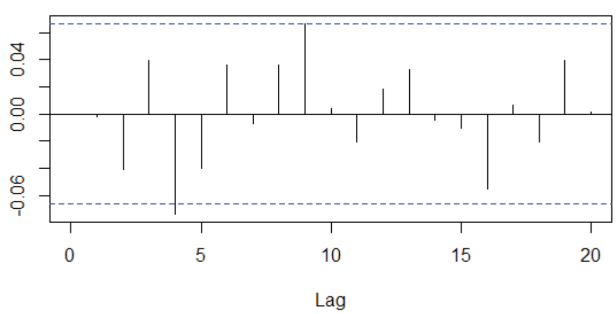


**Fig. 5.** Estimated ACF of the residuals for maximum series

After inclusion of the two extra regressors, parameters of the MAR model have been re-estimated for maximum series and given in the table 7. In this estimated model, all the coefficients are significant at 5% level of significance except the intercept term.

Similarly, residual ACF plot for minimum series is given in the Figure 6, which showed that lag 4 has significant auto-correlation function. Now from Table 4, we can identify that $V_6$ has the strongest correlation coefficient with X of lag 4. So, according to the suggested algorithm we have to include $V_6$ of lag 4 in the MAR model.

**Table 7.** Estimation output of equation 7 for maximum series

| Coefficients | Value | St. Error | t-Statistic | p-Value |
|---|---|---|---|---|
| Intercept | 8.65 | 7.61 | 1.14 | 0.26 |
| $W_1$ | 0.94* | 0.05 | 19.80 | <0.001 |
| $W_2$ | 0.95* | 0.04 | 21.50 | <0.001 |
| $W_3$ | 0.90* | 0.03 | 26.41 | <0.001 |
| $W_4$ | 0.91* | 0.03 | 33.33 | <0.001 |
| $W_5$ | 0.94* | 0.02 | 46.24 | <0.001 |
| $W_6$ | 0.83* | 0.03 | 31.11 | <0.001 |
| $V_6$ | 0.20* | 0.10 | 2.09 | <0.001 |
| $V_{6,N-4}$ | 0.79* | 0.10 | 8.17 | <0.001 |
| $V_{6,N-9}$ | 0.37* | 0.13 | 2.84 | <0.001 |

*denotes the significance at 1% level.

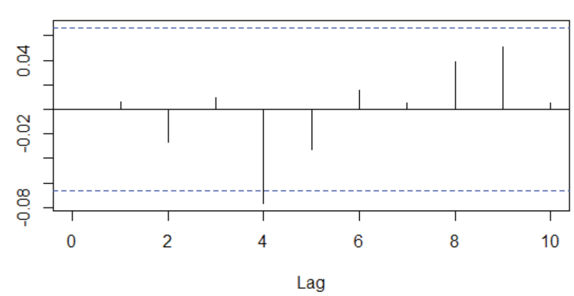


**Fig. 6.** Estimated ACF of the residuals for Min series

After including that extra regressor, parameters of the MAR model have been re-estimated and given in the Table 8. In this model all the coefficients are significant at 5% level of significance except the intercept term.

**Table 8.** Estimation output of equation 7 for minimum series

| Coefficients | Values | St. Error | t-Statistics | p-Value |
|---|---|---|---|---|
| Intercept | 5.64 | 7.53 | 0.75 | 0.45 |
| $W_1$ | 0.95* | 0.04 | 22.64 | <0.001 |
| $W_2$ | 0.96* | 0.04 | 24.47 | <0.001 |
| $W_3$ | 0.92* | 0.03 | 30.46 | <0.001 |
| $W_4$ | 0.92* | 0.02 | 38.36 | <0.001 |
| $W_5$ | 0.95* | 0.02 | 53.25 | <0.001 |
| $W_6$ | 0.87* | 0.02 | 36.59 | <0.001 |
| $V_6$ | 0.36* | 0.09 | 4.05 | <0.001 |
| $V_{6,N-4}$ | 0.64* | 0.09 | 7.41 | <0.001 |

*denotes the significance at 5% level.

## 4.6 Model validation

Last 80 observations of the corresponding series were kept before for model validation. Calculated Mean square error (MSE) and Root mean square error (RMSE) are displayed in Table 9 for maximum and minimum series. It is clearly seen that the new model has lesser MSE and RMSE as compared to other two models. The actual as well as predicted prices using the best found model are also reported in table 10. The closeness of predicted price to the actual price are clearly visible.

**Table 9.** Calculated MSE and RMSE for the Maximum and Minimum series

| Model | Max Series | | Min Series | |
|---|---|---|---|---|
| | MSE | RMSE | MSE | RMSE |
| ARFIMA | 5873.69 | 76.64 | 6007.8 | 77.51 |
| MAR | 376.74 | 19.41 | 277.36 | 16.65 |
| New Model | 248.37 | 15.76 | 252.01 | 15.87 |

In the Figure 7, actual observations and predicted values is plotted. The graph indicates that the predicted values are very close to the actual values for both the series. RMSE of New model, MAR model and ARFIMA model have been calculated over a moving window of 10 side length for 80 observations for both the series. The plots of RMSE are displayed in Figure 8 and 9 for maximum and minimum series respectively. Here we can see that in both plots, RMSE of the new

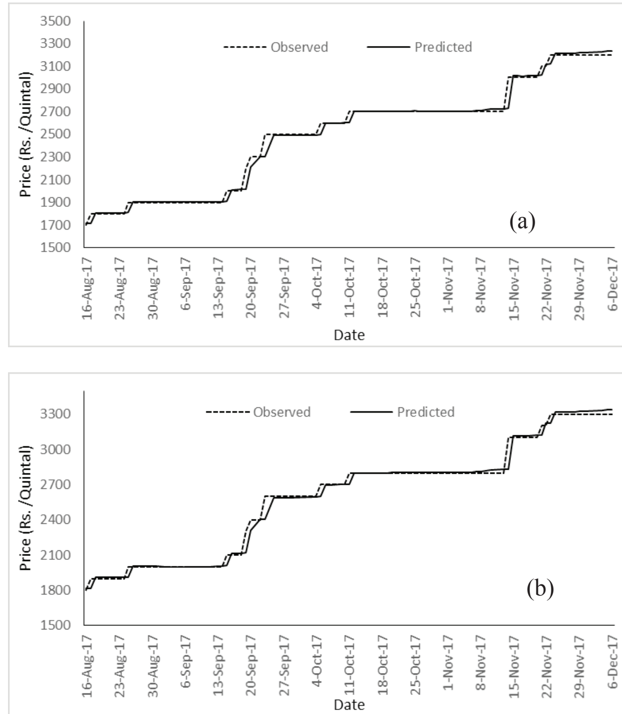


**Fig. 7.** Observed versus predicted graph for last 80 observations: (a) Minimum and (b) Maximum series

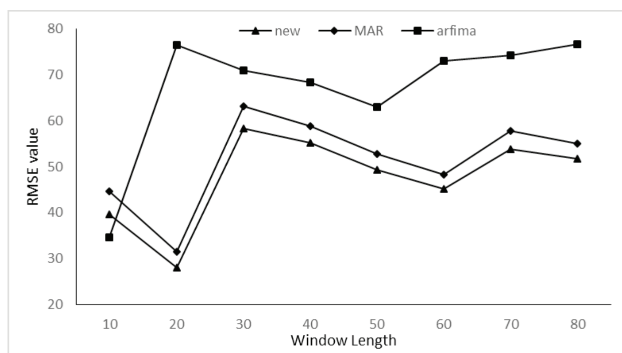


**Fig. 8.** RMSE for New, MAR and ARFIMA model over a moving window of 80 observations for maximum series

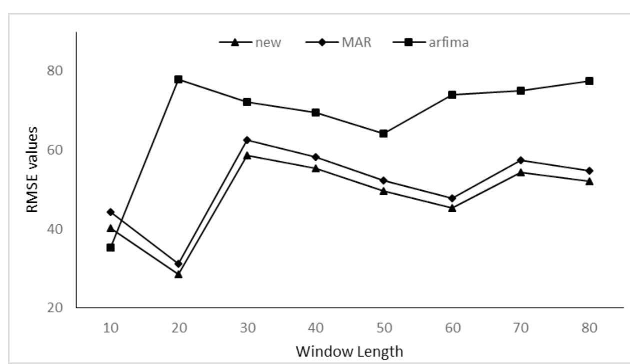


**Fig. 9.** RMSE for New, MAR and ARFIMA model over a moving window of 80 observations for minimum series

**Table 10.** Actual vs predicted price (Rs/Quintal)

| Obs. | Minimum Price | | Maximum Price | | Obs. | Minimum Price | | Maximum Price | |
|---|---|---|---|---|---|---|---|---|---|
| | Actual | Forecast | Actual | Forecast | | Actual | Forecast | Actual | Forecast |
| 1 | 1700 | 1711.68 | 1800 | 1811.78 | 41 | 2700 | 2699.82 | 2800 | 2798.45 |
| 2 | 1700 | 1713.14 | 1800 | 1814.01 | 42 | 2700 | 2700.28 | 2800 | 2799.01 |
| 3 | 1800 | 1713.80 | 1900 | 1814.78 | 43 | 2700 | 2700.99 | 2800 | 2799.75 |
| 4 | 1800 | 1808.26 | 1900 | 1909.02 | 44 | 2700 | 2702.32 | 2800 | 2801.34 |
| 5 | 1800 | 1808.34 | 1900 | 1909.24 | 45 | 2700 | 2703.66 | 2800 | 2802.92 |
| 6 | 1800 | 1807.88 | 1900 | 1908.15 | 46 | 2700 | 2704.99 | 2800 | 2805.08 |
| 7 | 1800 | 1807.22 | 1900 | 1908.34 | 47 | 2700 | 2705.53 | 2800 | 2805.77 |
| 8 | 1800 | 1808.37 | 1900 | 1909.74 | 48 | 2700 | 2706.05 | 2800 | 2806.32 |
| 9 | 1800 | 1809.53 | 1900 | 1911.14 | 49 | 2700 | 2705.56 | 2800 | 2805.73 |
| 10 | 1900 | 1810.68 | 2000 | 1913.12 | 50 | 2700 | 2705.07 | 2800 | 2805.13 |
| 11 | 1900 | 1905.62 | 2000 | 2006.52 | 51 | 2700 | 2704.40 | 2800 | 2804.35 |
| 12 | 1900 | 1905.17 | 2000 | 2006.08 | 52 | 2700 | 2703.72 | 2800 | 2803.57 |
| 13 | 1900 | 1904.44 | 2000 | 2005.24 | 53 | 2700 | 2703.05 | 2800 | 2802.79 |
| 14 | 1900 | 1902.71 | 2000 | 2003.25 | 54 | 2700 | 2702.64 | 2800 | 2801.76 |
| 15 | 1900 | 1901.72 | 2000 | 2002.14 | 55 | 2700 | 2703.02 | 2800 | 2802.20 |
| 16 | 1900 | 1901.73 | 2000 | 2002.17 | 56 | 2700 | 2703.22 | 2800 | 2802.46 |
| 17 | 1900 | 1901.74 | 2000 | 2002.21 | 57 | 2700 | 2703.95 | 2800 | 2803.37 |
| 18 | 1900 | 1901.76 | 2000 | 2002.24 | 58 | 2700 | 2707.96 | 2800 | 2808.05 |
| 19 | 1900 | 1903.76 | 2000 | 2003.85 | 59 | 2700 | 2711.96 | 2800 | 2812.73 |
| 20 | 1900 | 1907.38 | 2000 | 2007.89 | 60 | 2700 | 2716.50 | 2800 | 2818.07 |
| 21 | 2000 | 1910.99 | 2100 | 2012.51 | 61 | 2700 | 2721.04 | 2800 | 2823.41 |
| 22 | 2000 | 2010.41 | 2100 | 2110.85 | 62 | 2700 | 2724.57 | 2800 | 2827.60 |
| 23 | 2000 | 2014.71 | 2100 | 2115.82 | 63 | 3000 | 2728.10 | 3100 | 2833.53 |
| 24 | 2200 | 2018.74 | 2300 | 2121.54 | 64 | 3000 | 3014.03 | 3100 | 3114.86 |
| 25 | 2300 | 2210.36 | 2400 | 2310.69 | 65 | 3000 | 3013.81 | 3100 | 3115.12 |
| 26 | 2300 | 2306.01 | 2400 | 2405.07 | 66 | 3000 | 3013.76 | 3100 | 3115.31 |
| 27 | 2500 | 2305.54 | 2600 | 2405.92 | 67 | 3000 | 3013.98 | 3100 | 3115.25 |
| 28 | 2500 | 2492.39 | 2600 | 2589.21 | 68 | 3000 | 3018.02 | 3100 | 3120.27 |
| 29 | 2500 | 2491.48 | 2600 | 2588.67 | 69 | 3100 | 3021.05 | 3200 | 3124.72 |
| 30 | 2500 | 2491.01 | 2600 | 2588.38 | 70 | 3100 | 3117.87 | 3200 | 3220.60 |
| 31 | 2500 | 2490.55 | 2600 | 2588.09 | 71 | 3200 | 3119.32 | 3300 | 3223.35 |
| 32 | 2500 | 2493.10 | 2600 | 2591.36 | 72 | 3200 | 3214.50 | 3300 | 3317.03 |
| 33 | 2500 | 2495.61 | 2600 | 2594.34 | 73 | 3200 | 3214.30 | 3300 | 3317.02 |
| 34 | 2600 | 2499.11 | 2700 | 2598.90 | 74 | 3200 | 3214.83 | 3300 | 3317.22 |
| 35 | 2600 | 2596.41 | 2700 | 2694.89 | 75 | 3200 | 3216.15 | 3300 | 3318.90 |
| 36 | 2600 | 2598.56 | 2700 | 2697.23 | 76 | 3200 | 3218.48 | 3300 | 3321.78 |
| 37 | 2600 | 2600.44 | 2700 | 2699.74 | 77 | 3200 | 3222.82 | 3300 | 3326.95 |
| 38 | 2600 | 2601.32 | 2700 | 2700.60 | 78 | 3200 | 3227.15 | 3300 | 3331.98 |
| 39 | 2700 | 2603.21 | 2800 | 2703.24 | 79 | 3200 | 3231.47 | 3300 | 3337.01 |
| 40 | 2700 | 2698.90 | 2800 | 2797.30 | 80 | 3200 | 3234.20 | 3300 | 3340.20 |

model i.e the model proposed by Bogdanova and Ivanov (2015) has lower values throughout the 8 windows. We also observed that, short range forecast (window of 10) has the lower RMSE value than the long range forecast.

## 5. CONCLUSION

In recent times, forecasting of agricultural commodity price data having long memory property becomes important. A wavelet based approach to the analysis and modelling of financial time- series exhibiting strong LRD has been illustrated in this study, which includes wavelet decomposition of the time series using MODWT. Here, long memory parameter of the data under consideration has been estimated with the help of wavelet decomposition. After decomposition, ARFIMA and Multiscale Autoregressive (MAR) models have been fitted to the time series data and the new model suggested by Bogdanova and Ivanov, 2015 (modified MAR) is also fitted to the same data set. For testing the adequacy of the model, Mean Square Error (MSE) and Root Mean Square Error (RMSE) were calculated for the fitted models. A comparative study has been done among ARFIMA, MAR and modified MAR model and it is found that for both the series (maximum and minimum), modified MAR model has better forecasting ability as compared to the usual MAR and ARFIMA model in terms of lower MSE and RMSE values.

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