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Double Stage Shrinkage Testimators for the Variance of a Normal Distribution using Asymmetric Loss Function

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SUMMARY

The present paper proposes shrinkage testimators for the variance of a Normal Distribution when both the parameters are unknown. The choice of shrinkage factor is no longer arbitrary as it is made to dependent on test statistic. The risk properties of these testimators have been studied using the asymmetric loss function proposed by Basu and Ebrahimi (1991). It is observed that the proposed testimators dominate the usual best available pooled estimator for various degrees of asymmetry and different levels of significance. Recommendations regarding its applications have been attempted.

Keywords: Normal distribution, Variance, Double stage, Shrinkage testimators, Degree of asymmetry, Level of significance.

1. INTRODUCTION

In 1962 Katti proposed the estimation of mean using double sample information; this study was extended later by many other authors in different contexts. Shah (1964) used this method in estimating the Variance of Normal distribution. Later Srivastava (1976), Pandey (1979) among others studied the problem of estimation of Variance of Normal distribution when a guess value for the same is available.

Waiker *et al.* (1984) have proposed two stage shrinkage testimator(s) for the mean of a Normal distribution when the population Variance may be known or unknown. Pandey *et al.* (1988) have studied some shrinkage testimators for the Variance of Normal distribution under Mean square error criterion.

Pandey *et al.* (2007) have studied shrinkage estimators for the Variance of Normal distribution under an asymmetric loss function for selected range of degree of asymmetry. Srivastava and Tanna (2007, 2012) studied the risk properties of mean of an exponential distribution using asymmetric loss function. In all the above mentioned studies using asymmetric loss functions the proposed estimators and testimators perform better than the usual estimators under squared error loss function criterion with this motivation the present paper tries to study some double stage shrinkage testimators for the Variance of a Normal distribution using more general asymmetric loss function. In section-2 some shrinkage testimators for the Variance of Normal distribution have been prpoposed. Section-3 deals with the derivation of risk(s) of the proposed testimators, while section-4 is devoted to the risk comparison of the proposed ones with the best available estimators under the asymmetric loss function. The paper concludes with section-5 providing conclusions and applications.

1.1 Double Stage Estimation

The first stage sample is used to test the null hypothesis about the prior information and if it is not rejected, it is suggested to incorporate this information being supported by a test, in estimating the parameter. However, if the null hypothesis is rejected we do not use this prior information and obtain a second sample of size $n_2 = (n - n_1)$ to make up the loss of prior

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knowledge and estimate the parameter using both samples.

1.2 Asymmetric Loss Function

While estimating a parameter θ by $\hat{\theta}$ the asymmetric loss function is given by

$$L(\Delta) = b(e^{a\Delta} - a\Delta - 1), \;\; a \neq 0, b > 0 \quad (1.2.1)$$

where $\Delta = \left(\frac{\hat{\theta}}{\theta} - 1\right)$ or $\Delta = (\hat{\theta} - \theta)$ depending upon whether the scale or the shape parameter is being estimated.

The sign and magnitude of a represents the direction and degree of asymmetry respectively and b is the factor of proportionality. The positive value of a is used when overestimation is more serious than underestimation, while a negative value of a is used in reverse situations. This loss function was proposed by Basu and Ebrahimi (1991) which is more suitable to estimate the scale parameters.

2. SHRINKAGE TESTIMATORS

Let X be normally distributed with mean μ and variance σ^2, both unknown. It is assumed that the prior knowledge about σ^2 is available in the form of an initial estimate σ_0^2. We are interested in constructing an estimator of σ^2 using the sample observations and possibly the guess value σ_0^2. We define a double stage shrinkage testimator of σ^2 as follows:

1. Take a random sample x_{1i} $(i = 1,2,...,n_1)$ of size n_1 from $N(\mu, \sigma^2)$ and compute $\bar{x}_1 = \frac{1}{n_1}\Sigma x_{1i},\ s_1^2 = \frac{1}{n_1 - 1}\Sigma(x_{1i} - \bar{x}_1)^2$.
2. Test the hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 \neq \sigma_0^2$ at level α using the test statistic $\frac{v_1 s_1^2}{\sigma_0^2}$, which is distributed as χ_2 with ν1 = $(n_1 - 1)$ degrees of freedom.
3. If H_0 is accepted at α level of significance i.e. $x_1^2 < \frac{v_1 s_1^2}{\sigma_0^2} < x_2^2$, where x_1^2 and x_2^2 refer to lower and upper critical points of the unbiased portioning of the test statistic at a given level of significance α, take $k_1 s_1^2 + (1 - k_1)\sigma_0^2$ as the shrinkage estimator of σ^2 with shrinkage factor k_1 dependent on the test statistic.
4. If H_0 is rejected, take a second sample x_{2j} $(j = 1,2,...,n_2)$ of size $n_2 = (n - n_1)$ compute $\bar{x}_2 = \frac{1}{n_2}\Sigma x_{2j}, s_2^2 = \frac{1}{n_2 - 1}\Sigma(x_{2j} - \bar{x}_2)^2$ and take $(v_1 s_1^2 + v_2 s_2^2)/(v_1 + v_2)$ where $v_2 = (n_2 - 1)$ as the estimator of σ^2.

To summarize, we define the double-stage shrinkage testimator $\hat{\sigma}^2_{DST1}$ of σ^2 as follows:

$$\sigma^2_{DST1} = \begin{cases} k_1 s_1^2 + (1 - k_1)\sigma_0^2, & \text{if } H_0 \text{ is accepted} \\ s_p^2 = \dfrac{(v_1 s_1^2 + v_2 s_2^2)}{(v_1 + v_2)}, & \text{if } H_0 \text{ is rejected} \end{cases}$$

where $k_1 = \frac{v_1 s_1^2}{\sigma_0^2 \chi^2}$

Estimators of this type with k arbitrary and lying between 0 and 1 have been proposed by Katti (1962), Shah (1964), Arnold and Al-Bayyati (1970), Waiker and Katti (1971), Srivastava (1976), Pandey (1979) among others. Later on the arbitrariness in the choice of shrinkage factor was removed by making it dependent on the test statistics by Waiker *et al.* (1984) for the mean in Normal distribution. Here we propose an estimator for the Variance when the choice of shrinkage factor is no longer arbitrary as above. Further it is observed in many studies that the testimators perform better when the shrinkage factor approached to zero more rapidly, so we define another double stage shrinkage testimator $\hat{\sigma}^2_{DST2}$ of σ^2 by taking square of the shrinkage factor as $k_2 = k_1^2 = \left(\frac{v_1 s_1^2}{\sigma_0^2 \chi^2}\right)^2$ which tends to zero more rapidly than k_1 as follows

$$\sigma^2_{DST2} = \begin{cases} \left(\dfrac{v_1 s_1^2}{\sigma_0^2 \chi^2}\right) s_1^2 + \left(1 - \left(\dfrac{v_1 s_1^2}{\sigma_0^2 \chi^2}\right)^2\right)\sigma_0^2 & \text{, if } H_0 \text{ is accepted} \\ s_p^2 & \text{, if } H_0 \text{ is rejected} \end{cases}$$

3. RISK OF TESTIMATORS

In this section we derive risk of the two proposed testimators which are defined in the previous section.

3.1 Risk of $\hat{\sigma}^2_{DST1}$

The risk of $\hat{\sigma}^2_{DST1}$ under L (Δ) is defined by

$$R(\hat{\sigma}^2_{DST1}) = E[\hat{\sigma}^2_{DST1} \mid L(\Delta)]$$

$$= E\left[k_1 s_1^2 + (1-k_1)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon_1 s_1^2}{\sigma_0^2} < \chi_2^2\right]$$

$$\cdot p\left[\chi_1^2 < \frac{\upsilon_1 s_1^2}{\sigma_0^2} < \chi_2^2\right] + E\left[s_p^2 \left| \frac{\upsilon_1 s_1^2}{\sigma_0^2} < \right.\right.$$

$$\left.\chi_1^2 \cup \frac{\upsilon_1 s_1^2}{\sigma_0^2} > \chi_2^2\right] \cdot p\left[\frac{\upsilon_1 s_1^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1^2}{\sigma_0^2} > \chi_2^2\right] \qquad (3.1.1)$$

$$= e^{-a} \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} e^{a\left[\frac{\frac{\upsilon_1 s_1^2}{\sigma_0^2\chi^2}\left(s_1^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]} f(s_1^2)\, d s_1^2$$

$$- a \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} \left[\frac{\frac{\upsilon_1 s_1^2}{\sigma_0^2\chi^2}\left(s_1^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2} - 1\right] f(s_1^2)\, d s_1^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} f(s_1^2)\, ds_1^2$$

$$+ e^{-a} \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} e^{a\left(s_p^2/\sigma^2\right)} f(s_1^2) f(s_2^2) ds_1^2 ds_2^2$$

$$- a \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} \left(\frac{s_p^2}{\sigma^2} - 1\right) f(s_1^2) f(s_2^2) ds_1^2 ds_2^2$$

$$- \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} f(s_1^2) f(s_2^2) ds_1^2 ds_2^2$$

$$- \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} f(s_1^2) f(s_2^2) ds_1^2 ds_2^2 \qquad (3.1.2)$$

Where

$$f(s_1^2) = \frac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)} \left(s_1^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^2}{\sigma^2}\right)} ds_1^2$$

$$f(s_2^2) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)} \left(s_2^2\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)} ds_2^2$$

Straight forward integration of (3.1.2) gives

$$R(\hat{\sigma}^2_{DST1}) = \left(\frac{\sigma^2}{\nu_1}\right)^{\nu_1/2} \left(\frac{\sigma^2}{\nu_2}\right)^{\nu_2/2}$$

$$\left[\begin{array}{c} I_1^* - \frac{2a}{\lambda\,\chi^2}\left(\frac{\nu_1}{2}+1\right) \\ \left\{I\left(\chi_2^2\lambda, \frac{\nu_1}{2}+2\right) - I\left(\chi_1^2\lambda, \frac{\nu_1}{2}+2\right)\right\} \\ + \quad \frac{a\nu_1}{\chi^2}\left\{I\left(\chi_2^2\lambda, \frac{\nu_1}{2}+1\right) - I\left(\chi_1^2\lambda, \frac{\nu_1}{2}+1\right)\right\} \\ -(a\lambda - a + 1)\left\{I\left(\chi_2^2\lambda, \frac{\nu_1}{2}\right) - I\left(\chi_1^2\lambda, \frac{\nu_1}{2}\right)\right\} \\ -\frac{a\nu_1}{\nu_1+\nu_2}\left\{I\left(\chi_1^2\lambda, \frac{\nu_1}{2}+1\right) - I\left(\chi_2^2\lambda, \frac{\nu_1}{2}+1\right)\right\} \\ + \frac{a\nu_1}{\nu_1+\nu_2}\left\{I\left(\chi_1^2\lambda, \frac{\nu_1}{2}\right) - I\left(\chi_2^2\lambda, \frac{\nu_1}{2}\right)\right\} \\ -\left\{I\left(\chi_1^2\lambda, \frac{\nu_1}{2}\right) - I\left(\chi_2^2\lambda, \frac{\nu_1}{2}\right) + 1\right\} + I_2^* \end{array}\right] \qquad (3.1.3)$$

where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x} x^{p-1} dx$ refers to the standard incomplete gamma function, $\lambda = \frac{\sigma_0^2}{\sigma^2}$ and

$$I_1^* = \frac{e^{a(\lambda-1)}}{2^{\nu_1/2}\,\Gamma\left(\frac{\nu_1}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} e^{\left[\frac{at_1^2}{\lambda\nu_1 x^2} - \frac{at_1}{x^2}\right]}$$

$$e^{-\frac{1}{2}t_1}\,(t_1)^{\frac{\nu_1}{2}-1}\,dt_1$$

$$I_2^* = \frac{e^{-a}}{2^{\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}\right)}\left(\frac{1}{2}-\frac{a}{\nu_1+\nu_2}\right)^{\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}\right)}}$$

$$\left[I\left(\chi_1^2\lambda, \frac{\nu_1}{2}\right) - I\left(\chi_2^2\lambda, \frac{\nu_1}{2}\right) + 1\right]$$

3.2 Risk of $\hat{\sigma}^2_{DST2}$

Again, we obtain the risk of $\hat{\sigma}^2_{DST2}$ under L(Δ), given by

$$R(\hat{\sigma}^2_{DST2}) = E[\hat{\sigma}^2_{DST2} \,|\, L(\Delta)]$$

$$= E\left[\left(\frac{\upsilon_1 s_1^2}{\sigma_0^2 \chi^2}\right)^2 s_1^2 + \left(1-\left(\frac{\upsilon_1 s_1^2}{\sigma_0^2 \chi^2}\right)^2\right)\sigma_0^2 \Big/ \chi_1^2 < \frac{\upsilon_1 s_1^2}{\sigma_0^2} < \chi_2^2\right]\cdot p\left[\chi_1^2 < \frac{\upsilon_1 s_1^2}{\sigma_0^2} < \chi_2^2\right] + E\left[s_p^2 \,\middle|\, \frac{\upsilon_1 s_1^2}{\sigma_0^2} < \chi_1^2 \;\; \frac{\upsilon_1 s_1^2}{\sigma_0^2} > \chi_2^2\right]\cdot p\left[\frac{\upsilon_1 s_1^2}{\sigma_0^2} < \chi_1^2 \cup \frac{\upsilon_1 s_1^2}{\sigma_0^2} > \chi_2^2\right] \quad (3.2.1)$$

$$= e^{-a} \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} e^{a\left[\frac{\left(\frac{\upsilon_1 s_1^2}{\sigma_0^2 \chi^2}\right)^2 \left(s_1^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2}\right]} f(s_1^2)\, ds_1^2$$

$$- a \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} \left[\frac{\left(\frac{\upsilon_1 s_1^2}{\sigma_0^2 \chi^2}\right)^2 \left(s_1^2-\sigma_0^2\right)+\sigma_0^2}{\sigma^2} - 1\right] f(s_1^2)\, ds_1^2$$

$$- \int_{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}}^{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}} f(s_1^2) ds_1^2 + e^{-a} \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} e^{a\left(s_p^2 / \sigma^2\right)} f(s_1^2) f(s_2^2) ds_1^2 ds_2^2$$

$$+ e^{-a} \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} e^{a\left(s_p^2 / \sigma^2\right)} f(s_1^2) f(s_2^2) ds_1^2 ds_2^2$$

$$- a \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} \left(\frac{s_p^2}{\sigma^2} - 1\right) f(s_1^2) f(s_2^2) ds_1^2 ds_2^2$$

$$- a \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} \left(\frac{s_p^2}{\sigma^2} - 1\right) f(s_1^2) f(s_2^2) ds_1^2 ds_2^2$$

$$- \int_0^{\frac{\chi_1^2\sigma_0^2}{\upsilon_1}} \int_0^{\infty} f(s_1^2) f(s_2^2) ds_1^2 ds_2^2$$

$$- \int_{\frac{\chi_2^2\sigma_0^2}{\upsilon_1}}^{\infty} \int_0^{\infty} f(s_1^2) f(s_2^2) ds_1^2 ds_2^2 \quad (3.2.2)$$

Where

$$f(s_1^2) = \frac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)} \left(s_1^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^2}{\sigma^2}\right)} ds_1^2$$

$$f(s_2^2) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)} \left(s_2^2\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)} ds_2^2$$

Straight forward integration of (3.2.2) gives

$$\left(\hat{\sigma}^2_{DST2}\right) = \left(\frac{\sigma^2}{v_1}\right)^{v_1/2} \left(\frac{\sigma^2}{v_2}\right)^{v_2/2}$$

$$\left[\begin{array}{c} I_1^* - \frac{4a}{\lambda^2 (x^2)^2}\left(\frac{v_1}{2}+1\right)\left(\frac{v_1}{2}+2\right) \\ \left\{I\left(\chi_2^2\lambda, \frac{v_1}{2}+3\right) - I\left(\chi_1^2\lambda, \frac{v_1}{2}+3\right)\right\} \\ + \frac{4a}{\lambda(x^2)^2}\left(\frac{v_1}{2}\right)\left(\frac{v_1}{2}+1\right) \\ \left\{I\left(\chi_2^2\lambda, \frac{v_1}{2}+2\right) - I\left(\chi_1^2\lambda, \frac{v_1}{2}+2\right)\right\} \\ -(a\lambda - a + 1)\left\{I\left(\chi_2^2\lambda, \frac{v_1}{2}\right) - I\left(\chi_1^2\lambda, \frac{v_1}{2}\right)\right\} \\ -\frac{aV_1}{v_1+v_2}\left\{I\left(\chi_1^2\lambda, \frac{v_1}{2}+1\right) - I\left(\chi_2^2\lambda, \frac{v_1}{2}+1\right)\right\} \\ + \frac{aV_1}{v_1+v_2}\left\{I\left(\chi_1^2\lambda, \frac{v_1}{2}\right) - I\left(\chi_2^2\lambda, \frac{v_1}{2}\right)\right\} \\ -\left\{I\left(\chi_1^2\lambda, \frac{v_1}{2}\right) - I\left(\chi_2^2\lambda, \frac{v_1}{2}\right) + 1\right\} + I_2^* \end{array}\right]$$

(3.2.3)

where $I(x;p) = (1/\Gamma p)\int_0^x e^{-x} x^{p-1}\, dx$ refers to the standard incomplete gamma function and

$$I_1^* = \frac{e^{a(\lambda-1)}}{2^{v_1/2}\,\Gamma\left(\frac{v_1}{2}\right)} \int_{x_1^2\lambda}^{x_2^2\lambda} e^{\left[\frac{at_1^3}{\lambda^2 v_1 (x^2)^2} - \frac{at_1^2}{\lambda (x^2)^2}\right]}$$

$$e^{-\frac{1}{2}t_1} (t_1)^{\frac{v_1}{2}-1}\, dt_1$$

$$I_2^* = \frac{e^{-a}}{2^{\left(\frac{v_1}{2}+\frac{v_2}{2}\right)}\left(\frac{1}{2}-\frac{a}{v_1+v_2}\right)^{\left(\frac{v_1}{2}+\frac{v_2}{2}\right)}}$$

$$\left[I\left(\chi_1^2\lambda, \frac{v_1}{2}\right) - I\left(\chi_2^2\lambda, \frac{v_1}{2}\right) + 1\right]$$

4. RELATIVE RISK OF $\hat{\sigma}^2_{DSTi}$

A natural way of comparing the risk of the proposed testimators, is to study its performance with respect to the best available estimator s_p^2 in this case. For this purpose, we obtain the risk of s_p^2 under L(Δ) as:

$$R_E(s_p^2) = E[s_p^2 \,|\, L(\hat{\sigma}^2, \sigma^2)]$$

$$= e^{-a} \int_0^\infty \int_0^\infty e^{a\left[\frac{s_p^2}{\sigma^2}\right]} f(s_1^2) f(s_2^2)\, ds_1^2 ds_2^2$$

$$- a \int_0^\infty \int_0^\infty \left[\frac{s_p^2}{\sigma^2} - 1\right] f(s_1^2) f(s_2^2)\, ds_1^2 ds_2^2$$

$$- \int_0^\infty \int_0^\infty f(s_1^2) f(s_2^2)\, ds_1^2 ds_2^2 \tag{4.1}$$

Where

$$f(s_1^2) = \frac{1}{2^{\upsilon_1/2}\,\Gamma\left(\upsilon_1/2\right)} \left(s_1^2\right)^{\frac{\upsilon_1}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_1 s_1^2}{\sigma^2}\right)} ds_1^2$$

$$f(s_2^2) = \frac{1}{2^{\upsilon_2/2}\,\Gamma\left(\upsilon_2/2\right)} \left(s_2^2\right)^{\frac{\upsilon_2}{2}-1} e^{\left(-\frac{1}{2}\frac{\upsilon_2 s_2^2}{\sigma^2}\right)} ds_2^2$$

A straightforward integration of (4.1) gives

$$R_E(s_p^2) = \left(\frac{\sigma^2}{\nu_1}\right)^{\nu_1/2} \left(\frac{\sigma^2}{\nu_2}\right)^{\nu_2/2} \left[\frac{e^{-a}}{2^{\left(\nu_1/2 + \nu_2/2\right)} \left(\frac{1}{2} - \frac{a}{\nu_1 + \nu_2}\right)^{\nu_1/2 + \nu_2/2}} - 1\right] \tag{4.2}$$

Now, we define the Relative Risk of $\hat{\sigma}^2_{DSTi}$, $i = 1,2$ with respect to s_p^2 under $L(\hat{\sigma}^2, \sigma^2)$ as follows:

$$RR_1 = \frac{R_E(s_p^2)}{R(\hat{\sigma}^2_{DST1})} \tag{4.3}$$

Using (4.2) and (3.1.3) the expression for RR_1 given in (4.3) can be obtained; it is observed that RR_1 is a function of 'v_1', 'v_2', 'λ', 'α' and 'a'.

Finally, we define the Relative Risk of $\hat{\sigma}^2_{DST2}$ by

$$RR_2 = \frac{R_E(s_p^2)}{R(\hat{\sigma}^2_{DST2})} \tag{4.4}$$

The expression for RR_2 is given by (4.4) which can be obtained by using (4.2) and (3.2.3). Again we observe that RR_2 is a function of 'v_1', 'v_2', 'λ', 'α' and 'a'.

4.1 Recommendations for $\hat{\sigma}^2_{DST1}$

It is observed that the above expressions (4.3) and (4.4) are functions of α, λ, v_1, v_2 and the degrees of asymmetry 'a'. For the comparing the performance of proposed testimators with the pooled Variance estimator we have considered several values for these viz. (v_1, v_2) = (6,6), (6,9), (6,12), (6,15), (6,18); (8,8), (8,12), (8,16), (8,20), (8,24) and (10,10), (10,15), (10,20), (10,25), (10,30) ; α = 1%, 5% and 10%, and a = -3, -2, -1, 1, 1.25, 1.50 and λ = 0.2 (0.1) 2.0. For all these values taken to study the risk behavior there will be several tables for the relative risks of the two testimators however we have not presented all the tables here. Some of the tables are shown below.

Our recommendations based on all the tables of relative risks are summarized as follows.

(i) The proposed testimator $\hat{\sigma}^2_{DST1}$ performs better than the pooled estimator s_p^2 for almost all the values considered as above. However some of the best performances are outlined specifically.

(ii) $\hat{\sigma}^2_{DST1}$ dominates the usual estimator when (v_1, v_2) = (6,6) ; α = 1% ; a = +1 or -1 the range for its better performance is $0.2 \le \lambda \le 2.0$ which is quite large as compared to other reported ranges in case of squared error loss function.

(iii) As 'v_2' increases the RR_1 values are still greater than unity for (6,9) set but these values decrease in magnitude also the range of 'λ' changes slightly as now it becomes $0.6 \le \lambda \le 1.8$ for negative values of 'a'. A similar pattern of relative risk values is observed when 'a' is positive for almost $0.6 \le \lambda \le 1.8$ indicating that the proposed testimator perform better in both the situations of over as well as underestimation.

Table 4.1.1. Relative Risk of $\hat{\sigma}^2_{DST1}$, $\alpha = 1\%$, $(v_1, v_2) = (6, 6)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	1.059	1.531	1.229	1.835	1.884	1.890
0.40	1.257	1.649	2.081	2.06	1.984	1.975
0.60	1.658	2.618	3.714	3.514	3.762	3.509
0.80	3.484	4.013	5.103	5.913	4.623	3.974
1.00	4.433	5.486	6.834	7.02	5.153	4.851
1.20	4.086	5.332	6.08	6.884	4.774	3.368
1.40	3.753	4.414	5.827	4.499	3.213	2.336
1.60	2.357	3.417	4.518	2.909	2.087	1.541
1.80	1.637	2.339	3.117	1.911	1.354	0.999
2.00	1.239	1.735	2.236	1.295	0.899	0.654

Table 4.1.2. Relative Risk of $\hat{\sigma}^2_{DST1}$ $\alpha = 5\%$, $(v_1, v_2) = (6, 6)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	1.379	1.831	1.813	1.49	1.692	1.057
0.40	1.972	1.912	2.436	2.592	2.54	1.536
0.60	1.339	2.021	3.939	2.855	2.711	2.676
0.80	2.271	3.074	4.42	3.909	3.568	3.334
1.00	3.593	4.462	5.081	5.001	4.111	4.67
1.20	4.153	4.172	5.051	3.563	2.842	2.267
1.40	3.549	3.736.	4.476	2.299	1.837	1.472
1.60	2.754	2.888	3.762	1.634	1.294	1.034
1.80	2.182	2.166	3.133	1.212	0.945	0.748
2.00	1.815	1.658	2.649	0.926	0.707	0.551

Table 4.1.3. Relative Risk of $\hat{\sigma}^2_{DST1}$ $\alpha = 1\%$, $(v_1, v_2) = (8, 8)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	1.486	1.74	1.481	1.775	1.792	1.851
0.40	1.989	1.839	1.861	2.693	2.606	2.56
0.60	1.617	2.722	2.195	3.211	3.123	3.249
0.80	3.301	3.041	3.476	4.793	4.437	4.311
1.00	4.105	5.296	6.005	6.403	5.446	5.405
1.20	4.077	4.315	5.212	5.968	4.572	3.507
1.40	3.886	3681	4.75	3.679	2.869	2.256
1.60	2.5	2.75	3.089	2.326	1.797	1.413
1.80	1.782	2.675	2.395	1.534	1.159	0.897
2.00	1.392	2.069	2.338	1.059	0.779	0.589

Table 4.2.1. Relative Risk of $\hat{\sigma}^2_{DST2}$ $\alpha = 1\%$, $(v_1, v_2) = (6, 6)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	0.647	1.117	1.897	1.214	1.164	1.156
0.40	0.652	1.997	2.725	1.452	1.347	1.299
0.60	1.381	2.054	3.202	2.629	2.496	3.603
0.80	3.826	4.273	4.73	4.669	3.67	4.078
1.00	5.225	5.732	5.933	5.684	5.396	6.077
1.20	4.882	4.758	4.747	4.077	4.857	4.248
1.40	3.444	3.075	3.385	3.59	3.369	2.531
1.60	2.035	2.952	3.022	2.702	1.951	1.464
1.80	1.397	1.976	2.755	1.68	1.181	0.871
2.00	1.06	1.463	2.735	1.103	0.752	0.541

Table 4.2.2. Relative Risk of $\hat{\sigma}^2_{DST2}$ $\alpha = 1\%$, $(v_1, v_2) = (8, 8)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	0.74	1.306	1.328	1.143	1.074	1.036
0.40	0.61	1.962	2.865	1.177	1.173	1.909
0.60	1.285	2.009	4.475	3.825	3.595	3.514
0.80	3.539	4.15	5.735	4.962	4.863	3.643
1.00	6.627	7.176	7.917	6.658	5.38	4.939
1.20	4.728	5.556	5.968	4.261	4.945	3.906
1.40	3.439	3.151	4.721	3.488	2.737	2.179
1.60	2.076	2.192	3.519	2.069	1.577	1.233
1.80	1.466	1.692	3.76	1.313	0.968	0.736
2.00	1.151	1.406	2.141	0.885	0.631	0.465

Table 4.2.3. Relative Risk of $\hat{\sigma}^2_{DST2}$ $\alpha = 5\%$, $(v_1, v_2) = (6, 6)$

λ	a = -3	a = -2	a = -1	a = 1	a = 1.25	a = 1.50
0.20	0.848	1.521	1.017	1.577	1.549	1.582
0.40	0.957	1.87	2.343	1.947	1.967	1.927
0.60	1.987	1.302	2.97	2.942	2.777	2.698
0.80	3.798	2.522	4.836	4.478	3.225	3.658
1.00	4.434	4.561	5.914	5.711	4.446	5.626
1.20	3.403	3.652	4.607	3.579	2.887	2.326
1.40	2.456	3.585	3.919	2.183	1.751	1.413
1.60	1.887	2.773	2.884	1.487	1.171	0.936
1.80	1.555	2.272	2.015	1.065	0.818	0.641
2.00	1.357	1.964	1.388	0.79	0.589	0.45

(iv) As the other quantity of interest is the level of significance in addition to the degrees of asymmetry. We change 'α' to 5% and 10% and it is observed that still the proposed testimator performs better for the 'ranges' mentioned as above. i.e. when 'a' is negative the range is $0.2 \leq \lambda \leq 2.0$ and when 'a' is positive it becomes $0.2 \leq \lambda \leq 1.6$ indicating that range shrinks slightly for the overestimation case.

(v) Now, we have considered the other values of (v_1, v_2) as mentioned above and it is observed that RR1 values are still higher than unity for these different data sets, with almost the same ranges of 'λ' mentioned as above for positive as well as negative values of 'a'. Again as v_2 increases the magnitude of RR1 values decreases but does not fall below 1, indicating a better performance.

(vi) Overall recommendations are: v_1 should be small i.e. $v_1 \ngtr 10$ and $v_2 \leq 3v_1$, $\alpha = 1\%$ i.e. a smaller level of significance be taken, different degrees of asymmetry could be taken as it is observed that even 'a' could be as extreme negative as a = -3 or it could be considerably positive i.e. a = 1.5. However the suggested best values of 'a' could be a = -1 or a = +1.

(vii) When these RR_1 values are compared with the Mean Square values of $\hat{\sigma}^2_{DST1}$ proposed by Pandey *et al.* (1988) it is observed that the magnitude of RR_1 values are HIGHER, the range of 'λ' increases considerably as it was (0.5 – 1.5) and now it becomes almost (0.2 – 2.0) earlier it was recommended that $v_2 \leq 2v_1$ now it becomes $v_2 \leq 3v_1$ a considerable increase in the choice of v_2. Implying that the use of ASL not only allows to take account for various degrees of asymmetry (i.e. choose 'a' accordingly when over / under estimation is more serious) but also facilitates increase in the range of 'λ', v_2 etc. which could be more useful for practical purposes.

4.2 Recommendations for $\hat{\sigma}^2_{DST2}$

We have also proposed $\hat{\sigma}^2_{DST2}$ which is obtained by squaring the shrinkage factor. The performance of it, is compared with respect to s_p^2 for the same data as considered for $\hat{\sigma}^2_{DST1}$. Again, similar tables of RR_2 will be generated for these data sets only few tables are given for reference purpose however recommendations based on all these computations are as follows:

(i) It is observed that the magnitude of RR2 values is higher than RR_1 values implying that taking square of shrinkage factor improves the performance of the proposed testimator.

(ii) $\hat{\sigma}^2_{DST2}$ dominates s_p^2 when $(v_1, v_2) = (6,6)$, $\boldsymbol{\alpha} = 1\%$ for a = -1, $0.2 \leq \lambda \leq 2.0$ and for a = +1, $0.2 \leq \lambda \leq 2.0$ but the magnitude of relative risk values is higher in this case, even though the data set remains the same.

(iii) As 'v_2' increases the RR_2 values decrease in their magnitude (but still above unity). Here the range of 'λ' changes little bit as it becomes now $0.6 \leq \lambda \leq 1.8$ for negative values of 'a'. However when 'a' is positive the same range of 'λ' i.e. $0.2 \leq \lambda \leq 2.0$ is observed for the better performance.

(iv) The performance of $\hat{\sigma}^2_{DST2}$ is at its best when a = ± 1. As 'v_2' increases i.e. for the other data set (6, 9), (6, 12), (6, 15) or (6, 18) the magnitude of RR_2 decreases slightly but does not go below unity. Again, if we increase v_1 i.e. (8, 8), (8, 12) etc. Similar behaviour of RR_2 values is observed but their magnitude change.

(v) Again changing the level of significance values to $\alpha = 5\%$ and $\alpha = 10\%$ it is observed that values of RR_2 obtained are 'good' in the sense of being more than unity. But it is also noticed that there is a decrease in the magnitude of RR2 values as 'α' increase. So, a higher value of the level of significance is not suggested.

(vi) We therefore recommend as that: v_1 should be small i.e. $v_1 \ngtr 10$ and $v_2 \leq 3v_1$, and choose $\alpha = 1\%$. The degree of asymmetry could chosen for different degrees of asymmetry ranging from a = -3 to a = 1.5.

(vi) Comparing these RR_2 values with those obtained by Pandey and Srivastava (1988) under the MSE criterion (or the use of

'SELF') we find that these values are 'better' than those values in terms of their magnitude showing that the application of Asymmetric Loss Function yields better result. Also in a given situation when overestimation is more serious than underestimation or vice-versa this loss function provides a choice to tailor the risk by choosing 'a' appropriately.

5. CONCLUSION

Two shrinkage testimators viz. $\hat{\sigma}^2_{DST1}$ and $\hat{\sigma}^2_{DST2}$ have been proposed for the variance of a Normal distribution. It is concluded that (i) use asymmetric loss function to study the risk properties. (ii) v_1 Should be small preferably should not exceed 10 for both the cases. (iii) $v_2 \leq 3v_1$ (iv) take $\alpha = 1\%$ and take $0.2 \leq \lambda \leq 2.0$ for negative values of 'a' and take $0.2 \leq \lambda \leq 1.8$ for positive values of 'a'. (v) Take 'SQUARE' of the shrinkage factor.

6. APPLICATIONS

The proposed method of using an asymmetric loss function can be used in many real life situations which can be modeled through a Normal distribution such as studying the variation in some physical characteristics (say) soil fertility or distribution of errors in various situations may not have equal consequences, similarly estimating the mileage under different traffic conditions may show some variation, so underestimating or overestimating it may be serious for fuel economy suggestions, these are some situations under for which the proposed testimators can be used with less risk.

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