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An Improved Generalized Double Sampling Difference-type and Regression-type Estimators of Population Mean Using First Two Moments About Zero of Single Auxiliary Variable

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SUMMARY

Improved generalized double sampling difference-type and regression-type estimators using single auxiliary variable about first two moments about zero are proposed, their bias and mean square error are found and then comparative studies have been made. First, generalized difference-type estimator is proposed and the optimum values minimizing its mean square error are found but the minimizing optimum values contain some unknown parameters; hence, the alternative is to replace the unknown optimum values by the estimated optimum values depending on sample observations by unbiased or consistent estimators of parameters involved in the optimum values of the generalized difference-type estimator and thus getting the enhanced practical utility estimator known as the practical generalized double sampling regression-type estimator attaining the minimum mean square error of the generalized difference-type estimator for the optimum values.

1. INTRODUCTION

Survey statisticians usually make use of the information available on an auxiliary variable which is highly correlated with the variable under study for improving the efficiency of an estimator. In order to have a better understanding of double sampling difference and regression methods of estimation, one is consulted to see the details of these given in Cochran (1977), Murthy (1967), Sukhatme *et al*. (1997) and Tikkiwal (1960).

Let U = 1, 2, ..., N be a finite population of N units with y being the study variable taking the value Y_i for the unit i of U and x being the auxiliary variable taking the value X_i for the unit i of U, $|i$ = 1, 2, ..., N.

Let $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ and $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$ be the population means of y and x respectively. Also, let,

$$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i - \bar{Y}\right)^2$$

$$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^2,\ C_y^2 = \frac{S_y^2}{\bar{Y}^2},$$

$$C_x^2 = \frac{S_x^2}{\bar{X}^2},\ \ S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i - \bar{Y}\right)\left(X_i - \bar{X}\right),$$

$$\rho = \frac{S_{yx}}{S_y S_x},\ \beta = \frac{S_{yx}}{S_x^2} = \rho\frac{S_y}{S_x}$$

(where ρ being the population correlation coefficient between y and x), $\theta_x = \frac{1}{N}\sum_{i=1}^{N} X_i^2$ and

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$$\mu_{rs} = \frac{1}{N}\sum_{i=1}^{N}\left(Y_i - \bar{Y}\right)^r\left(X_i - \bar{X}\right)^s$$

When the population mean $\bar{X}$ of the auxiliary variable x is not known, then the alternative is to take the help of double sampling in which a preliminary large first phase sample of size n' is drawn from the population of size N by simple random sampling without replacement (SRSWOR) giving the n' sample values on x and the sample mean $\bar{x}'$ as an estimator of $\bar{X}$, and then a second phase sample of size n from the first phase sample of size n' by SRSWOR is drawn and on which the study variable y is observed. Let the first phase sample of size n' has the observations on x to be $\left(x_1', x_2', ..., x_{n'}'\right)$, the second phase sample of size n has the value $\left\{\left(y_1, x_1\right), \left(y_2, x_2\right), ..., \left(y_n, x_n\right)\right\}$ on (y, x) and the sample means are given by

$$\bar{x}' = \frac{1}{n'}\sum_{i=1}^{n'} x_i', \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \quad \text{and} \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i .$$

Further, with first two moments about zero

$$\bar{x}, \bar{x}', \; \bar{\theta}_x = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \quad \text{and} \quad \bar{\theta}_x' = \frac{1}{n'}\sum_{i=1}^{n'} x_i'^2$$

of auxiliary variable x, we propose a generalized difference-type estimator to be the function

$$T_g = g\left(\bar{y}, \bar{x}' - \bar{x}, \bar{\theta}_x' - \bar{\theta}_x\right) = g\left(\bar{y}, u, v\right) \tag{1.1}$$

satisfying the validity conditions of Taylor's series expansion such that

$$\left.\begin{aligned} & g\left(\bar{y} = \bar{Y}, u = 0, v = 0\right) = \bar{Y} \\ & \left.\frac{\partial g\left(\bar{y}, u, v\right)}{\partial \bar{y}}\right]_{(\bar{Y}, 0, 0)} = 1 \end{aligned}\right\} \tag{1.2}$$

The optimum values depending on some unknown parameters minimizing the mean square error of the generalized difference-type estimator in (1.1) give some practical problem in application involving parameters; hence, the alternative is to refine the generalized difference-type estimator by replacing the parameters in optimum values by the estimated optimum values depending upon sample observations in the generalized difference-type estimator giving rise to the generalized regression-type estimator increasing the practical utility and attaining the same minimum mean square error of the generalized difference-type estimator for the optimum values.

Some particular members of $T_g = g\left(\bar{y}, \bar{x}' - \bar{x}, \bar{\theta}_x' - \bar{\theta}_x\right) = g\left(\bar{y}, u, v\right)$ are the estimators

(*i*) $\bar{y} + d_1\left(\bar{x}' - \bar{x}\right) + d_2\left(\bar{\theta}_x' - \bar{\theta}_x\right) = \bar{y} + d_1 u + d_2 v$

(*ii*) $\bar{y}e^{d_1\left(\bar{x}' - \bar{x}\right) + d_2\left(\bar{\theta}_x' - \bar{\theta}_x\right)} = \bar{y}e^{d_1 u + d_2 v}$

(*iii*) $\bar{y}e^{\left(\bar{x}' - \bar{x}\right)^{d_1} + \left(\bar{\theta}_x' - \bar{\theta}_x\right)^{d_2}} = \bar{y}e^{u^{d_1} + v^{d_2}}$

(*iv*) $\bar{y}\left\{1 + d_1\left(\bar{x}' - \bar{x}\right) + d_2\left(\bar{\theta}_x' - \bar{\theta}_x\right)\right\} = \bar{y}\left(1 + d_1 u + d_2 v\right)$

(*v*) $\bar{y} + \left(\bar{x}' - \bar{x}\right)^{d_1} + \left(\bar{\theta}_x' - \bar{\theta}_x\right)^{d_2} = \bar{y} + u^{d_1} + v^{d_2}$

(*vi*) $\bar{y}\left\{\dfrac{1 + d_1\left(\bar{x}' - \bar{x}\right)}{1 + d_2\left(\bar{\theta}_x' - \bar{\theta}_x\right)}\right\} = \bar{y}\left(1 + d_1 u\right)\left(1 + d_2 v\right)^{-1}$

where d_1 and d_2 are the characterizing scalars to be chosen suitably. It may be mentioned that all the estimators from (*i*) to (*vi*) satisfying all the conditions of the generalized estimators $T_g = g\left(\bar{y}, \bar{x}' - \bar{x}, \bar{\theta}_x' - \bar{\theta}_x\right) = g\left(\bar{y}, u, v\right)$ belong to the class of estimators T_g.

2. BIAS AND MEAN SQUARE ERROR

Let $e_0 = \dfrac{\left(\bar{y} - \bar{Y}\right)}{\bar{Y}}, \; e_1 = \dfrac{\left(\bar{x} - \bar{X}\right)}{\bar{X}}, \; e_1' = \dfrac{\left(\bar{x}' - \bar{X}\right)}{\bar{X}},$

$$e_2 = \frac{\left(\bar{\theta}_x - \theta_x\right)}{\theta_x} \quad \text{and} \quad e_2' = \frac{\left(\bar{\theta}_x' - \theta_x\right)}{\theta_x}$$

so that

$$E\left(e_0\right) = E\left(e_1\right) = E\left(e_1'\right) = E\left(e_2\right) = E\left(e_2'\right) = 0$$

and $E\left(e_0^2\right) = \left(\dfrac{1}{n} - \dfrac{1}{N}\right)C_y^2, \; E\left(e_1^2\right) = \left(\dfrac{1}{n} - \dfrac{1}{N}\right)C_x^2,$

$$E\left(e_1'^2\right)=\left(\frac{1}{n'}-\frac{1}{N}\right)C_x^2,$$

$$E\left(e_2^2\right)=\left(\frac{1}{n}-\frac{1}{N}\right)\frac{1}{\theta_x^2}\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right),$$

$$E\left(e_2'^2\right)=\left(\frac{1}{n'}-\frac{1}{N}\right)\frac{1}{\theta_x^2}\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right),$$

$$E\left(e_0e_1\right)=\left(\frac{1}{n}-\frac{1}{N}\right)\rho C_yC_x,$$

$$E\left(e_0e_1'\right)=\left(\frac{1}{n'}-\frac{1}{N}\right)\rho C_yC_x,$$

$$E\left(e_0e_2\right)=\left(\frac{1}{n}-\frac{1}{N}\right)\frac{1}{\bar{Y}\theta_x}\left(\mu_{12}+2\bar{X}\mu_{11}\right),$$

$$E\left(e_0e_2'\right)=\left(\frac{1}{n'}-\frac{1}{N}\right)\frac{1}{\bar{Y}\theta_x}\left(\mu_{12}+2\bar{X}\mu_{11}\right),$$

$$E\left(e_1e_2\right)=\left(\frac{1}{n}-\frac{1}{N}\right)\frac{1}{\bar{X}\theta_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right),$$

$$E\left(e_1e_2'\right)=\left(\frac{1}{n'}-\frac{1}{N}\right)\frac{1}{\bar{X}\theta_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right),$$

$$E\left(e_1'e_1\right)=\left(\frac{1}{n'}-\frac{1}{N}\right)C_x^2,$$

$$E\left(e_1'e_2\right)=\left(\frac{1}{n'}-\frac{1}{N}\right)\frac{1}{\bar{X}\theta_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right),$$

$$E\left(e_1'e_2'\right)=\left(\frac{1}{n'}-\frac{1}{N}\right)\frac{1}{\bar{X}\theta_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right),$$

$$E\left(e_2e_2'\right)=\left(\frac{1}{n'}-\frac{1}{N}\right)\frac{1}{\theta_x^2}\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right).$$

Expanding $T_g=g\left(\bar{y},u,v\right)$ in Taylor's series about the point $T=\left(\bar{Y},0,0\right)$, we have

$$T_g=g\left(\bar{y},u,v\right)=g\left(\bar{Y},0,0\right)+\left(\bar{y}-\bar{Y}\right)g_0+\left(u-0\right)g_1+\left(v-0\right)g_2+\frac{1}{2!}\Big\{\left(\bar{y}-\bar{Y}\right)^2g_{00}+\left(u-0\right)^2g_{11}+\left(v-0\right)^2g_{22}+2\left(\bar{y}-\bar{Y}\right)\left(u-0\right)g_{01}+2\left(\bar{y}-\bar{Y}\right)\left(v-0\right)g_{02}+2\left(u-0\right)\left(v-0\right)g_{12}\Big\}+\ldots \quad (2.1)$$

where $g\left(\bar{Y},0,0\right)=\bar{Y}$, $g_0=\left.\frac{\partial g\left(\bar{y},u,v\right)}{\partial\bar{y}}\right]_T=1$,

$$g_1=\left.\frac{\partial g\left(\bar{y},u,v\right)}{\partial u}\right]_T,\quad g_2=\left.\frac{\partial g\left(\bar{y},u,v\right)}{\partial v}\right]_T,$$

$$g_{00}=\left.\frac{\partial^2 g\left(\bar{y},u,v\right)}{\partial\bar{y}^2}\right]_T=0,\quad g_{11}=\left.\frac{\partial^2 g\left(\bar{y},u,v\right)}{\partial u^2}\right]_T,$$

$$g_{22}=\left.\frac{\partial^2 g\left(\bar{y},u,v\right)}{\partial v^2}\right]_T,\quad g_{01}=\left.\frac{\partial^2 g\left(\bar{y},u,v\right)}{\partial\bar{y}\partial u}\right]_T,$$

$$g_{02}=\left.\frac{\partial^2 g\left(\bar{y},u,v\right)}{\partial\bar{y}\partial v}\right]_T \text{ and } g_{12}=\left.\frac{\partial^2 g\left(\bar{y},u,v\right)}{\partial u\partial v}\right]_T.$$

Putting the required values in (2.1),

$$T_g=\bar{Y}+\bar{Y}e_0(1)+\bar{X}\left(e_1'-e_1\right)g_1+\theta_x\left(e_2'-e_2\right)g_2+\frac{1}{2}\Big\{\bar{Y}^2e_0^2\left(0\right)+\bar{X}^2\left(e_1'-e_1\right)^2g_{11}+\theta_x^2\left(e_2'-e_2\right)^2g_{22}+2\bar{Y}\bar{X}e_0\left(e_1'-e_1\right)g_{01}+2\bar{Y}\theta_xe_0\left(e_2'-e_2\right)g_{02}+2\bar{X}\theta_x\left(e_1'-e_1\right)\left(e_2'-e_2\right)g_{12}\Big\}+\ldots$$

or

$$T_g-\bar{Y}=\bar{Y}e_0+\bar{X}\left(e_1'-e_1\right)g_1+\theta_x\left(e_2'-e_2\right)g_2+\frac{1}{2}\Big\{\bar{X}^2\left(e_1'-e_1\right)^2g_{11}+\theta_x^2\left(e_2'-e_2\right)^2g_{22}+2\bar{Y}\bar{X}e_0\left(e_1'-e_1\right)g_{01}+2\bar{Y}\theta_xe_0\left(e_2'-e_2\right)g_{02}+2\bar{X}\theta_x\left(e_1'-e_1\right)\left(e_2'-e_2\right)g_{12}\Big\}+\ldots \quad (2.2)$$

Taking expectation on both sides of (2.2), retaining terms up to degree two in powers of $\left(e_i,e_i'\right)$, i = 0, 1, 2, {that is, up to terms of order $O\left(\frac{1}{n}\right)$}, the Bias ($T_g$) up to first degree of approximation is

$$E(T_g)-\bar{Y}=E\Big[\bar{Y}e_0+\bar{X}\left(e_1'-e_1\right)g_1+\theta_x\left(e_2'-e_2\right)g_2+\frac{1}{2}\Big\{\bar{X}^2\left(e_1'-e_1\right)^2g_{11}+\theta_x^2\left(e_2'-e_2\right)^2g_{22}+2\bar{Y}\bar{X}e_0\left(e_1'-e_1\right)g_{01}+2\bar{Y}\theta_xe_0\left(e_2'-e_2\right)g_{02}+2\bar{X}\theta_x\left(e_1'-e_1\right)\left(e_2'-e_2\right)g_{12}\Big\}\Big] \quad (2.3)$$

Or,

$$\text{Bias}\,(T_g) = \left(\frac{1}{n} - \frac{1}{n'}\right)\Bigg[\frac{S_x^2}{2} g_{11} - S_{yx} g_{01}$$

$$+ \frac{\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)}{2} g_{22} - 2\left(\mu_{12} + 2\bar{X}\mu_{11}\right) g_{02}$$

$$+ 2\left(\mu_{03} + 2\bar{X}\mu_{02}\right) g_{12}\Bigg] \quad (2.4)$$

which shows that bias of T_g is of order $O\left(\frac{1}{n}\right)$; hence, for sufficiently large value of n, the bias is negligible.

Mean square error of T_g to the first degree of approximation is

$$MSE\left(T_g\right) = E\left[T_g - \bar{Y}\right]^2$$

$$= E\left[\bar{Y}e_0 + \bar{X}\left(e_1' - e_1\right) g_1 + \theta_x\left(e_2' - e_2\right) g_2\right]^2$$

$$= \bar{Y}^2 E\left(e_0^2\right) + \bar{X}^2\left\{E\left(e_1^2\right) + E\left(e_1'^2\right) - 2E\left(e_1'e_1\right)\right\} g_1^2$$

$$+ \theta_x^2\left\{E\left(e_2^2\right) + E\left(e_2'^2\right) - 2E\left(e_2'e_2\right)\right\} g_2^2$$

$$+ 2\overline{YX}\left\{E\left(e_0e_1'\right) - E\left(e_0e_1\right)\right\} g_1$$

$$+ 2\bar{Y}\theta_x\left\{E\left(e_0e_2'\right) - E\left(e_0e_2\right)\right\} g_2$$

$$+ 2\bar{X}\theta_x\left\{E\left(e_1'e_2'\right) - E\left(e_1'e_2\right) - E\left(e_1e_2'\right) + E\left(e_1e_2\right)\right\} g_1g_2$$

$$= \bar{Y}^2\left(\frac{1}{n} - \frac{1}{N}\right) C_y^2 + \bar{X}^2\left\{\left(\frac{1}{n} - \frac{1}{N}\right) - \left(\frac{1}{n'} - \frac{1}{N}\right)\right\} C_x^2 g_1^2$$

$$+ \theta_x^2\left\{\left(\frac{1}{n} - \frac{1}{N}\right) - \left(\frac{1}{n'} - \frac{1}{N}\right)\right\}$$

$$\frac{1}{\theta_x^2}\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right) g_2^2$$

$$+ 2\overline{YX}\left\{\left(\frac{1}{n'} - \frac{1}{N}\right) - \left(\frac{1}{n} - \frac{1}{N}\right)\right\} \rho C_y C_x g_1$$

$$+ 2\bar{Y}\theta_x\left\{\left(\frac{1}{n'} - \frac{1}{N}\right) - \left(\frac{1}{n} - \frac{1}{N}\right)\right\} \frac{1}{\bar{Y}\theta_x}\left(\mu_{12} + 2\bar{X}\mu_{11}\right) g_2$$

$$+ 2\bar{X}\theta_x\left\{\left(\frac{1}{n} - \frac{1}{N}\right) - \left(\frac{1}{n'} - \frac{1}{N}\right)\right\} \frac{1}{\bar{X}\theta_x}\left(\mu_{03} + 2\bar{X}\mu_{02}\right) g_1g_2$$

$$= MSE\left(\bar{y}\right) + \left(\frac{1}{n} - \frac{1}{n'}\right)\Big[S_x^2 g_1^2 - 2S_{yx} g_1$$

$$+ \left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right) g_2^2$$

$$- 2\left(\mu_{12} + 2\bar{X}\mu_{11}\right) g_2 + 2\left(\mu_{03} + 2\bar{X}\mu_{02}\right) g_1g_2\Big] \quad (2.5)$$

The optimum values of g_1 and g_2 minimizing the mean square error in (2.5) is given by

$g_1 =$

$$\frac{S_{yx}\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right) - \left(\mu_{03} + 2\bar{X}\mu_{02}\right)\left(\mu_{12} + 2\bar{X}\mu_{11}\right)}{S_x^2\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right) - \left(\mu_{03} + 2\bar{X}\mu_{02}\right)^2}$$

$$= G_1 \quad (2.6)$$

$$g_2 = \frac{S_x^2\left(\mu_{12} + 2\bar{X}\mu_{11}\right) - S_{yx}\left(\mu_{03} + 2\bar{X}\mu_{02}\right)}{S_x^2\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right) - \left(\mu_{03} + 2\bar{X}\mu_{02}\right)^2}$$

$$= G_2 \quad (2.7)$$

and the minimum mean square error of T_g is

$$MSE\left(T_g\right)_{\min} = MSE\left(\bar{y}\right) + \left(\frac{1}{n} - \frac{1}{n'}\right)$$

$$\frac{1}{\left\{S_x^2\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right) - \left(\mu_{03} + 2\bar{X}\mu_{02}\right)^2\right\}^2}$$

$$\Big[S_x^2 S_{yx}^2\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)^2$$

$$+ S_x^2\left(\mu_{03} + 2\bar{X}\mu_{02}\right)^2\left(\mu_{12} + 2\bar{X}\mu_{11}\right)^2$$

$$- 2S_{yx}S_x^2\left(\mu_{12} + 2\bar{X}\mu_{11}\right)\left(\mu_{03} + 2\bar{X}\mu_{02}\right)$$

$$\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)$$

$$- 2S_{yx}^2 S_x^2\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)^2$$

$$+ 2S_{yx}^2\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)$$

$$\left(\mu_{03} + 2\bar{X}\mu_{02}\right)^2 + 2S_{yx}S_x^2\left(\mu_{12} + 2\bar{X}\mu_{11}\right)$$

$$\left(\mu_{03} + 2\bar{X}\mu_{02}\right)\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)$$

$$- 2S_{yx}\left(\mu_{03} + 2\bar{X}\mu_{02}\right)^3\left(\mu_{12} + 2\bar{X}\mu_{11}\right)$$

$$+ S_x^4\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)$$

$$\left(\mu_{12} + 2\bar{X}\mu_{11}\right)^2 + S_{yx}^2\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)$$

$$\left(\mu_{03} + 2\bar{X}\mu_{02}\right)^2$$

$$- 2S_{yx}S_x^2\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)$$

$$\left(\mu_{03} + 2\bar{X}\mu_{02}\right)\left(\mu_{12} + 2\bar{X}\mu_{11}\right)$$

$$- 2S_x^4\left(\mu_{12} + 2\bar{X}\mu_{11}\right)^2\left(\mu_{04} + 4\bar{X}\mu_{03} + 4\bar{X}^2\mu_{02} - \mu_{02}^2\right)$$

$$+2S_x^2\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2$$

$$+2S_{yx}S_x^2\left(\mu_{03}+2\bar{X}\mu_{02}\right)$$

$$\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)$$

$$\left(\mu_{12}+2\bar{X}\mu_{11}\right)-2S_{yx}\left(\mu_{12}+2\bar{X}\mu_{11}\right)\left(\mu_{03}+2\bar{X}\mu_{02}\right)^3$$

$$+2S_{yx}S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)$$

$$\left(\mu_{12}+2\bar{X}\mu_{11}\right)\left(\mu_{03}+2\bar{X}\mu_{02}\right)$$

$$-2S_{yx}^2\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)$$

$$-2S_x^2\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2$$

$$+2S_{yx}\left(\mu_{03}+2\bar{X}\mu_{02}\right)^3\left(\mu_{12}+2\bar{X}\mu_{11}\right)\Big]$$

$$=MSE\left(\bar{y}\right)+\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\frac{1}{\left\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right\}^2}$$

$$\Big[-S_x^2S_{yx}^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)^2$$

$$+S_x^2\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2$$

$$+2S_{yx}S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)$$

$$\left(\mu_{03}+2\bar{X}\mu_{02}\right)\left(\mu_{12}+2\bar{X}\mu_{11}\right)$$

$$+S_{yx}^2\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)$$

$$-2S_{yx}\left(\mu_{12}+2\bar{X}\mu_{11}\right)\left(\mu_{03}+2\bar{X}\mu_{02}\right)^3$$

$$-S_x^4\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)\Big]$$

$$=MSE\left(\bar{y}\right)+\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\frac{1}{\left\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right\}^2}$$

$$\Big[S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)$$

$$\Big\{-S_{yx}^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)$$

$$+2S_{yx}\left(\mu_{03}+2\bar{X}\mu_{02}\right)\left(\mu_{12}+2\bar{X}\mu_{11}\right)$$

$$-S_x^2\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2\Big\}-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2$$

$$\Big\{-S_x^2\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2-S_{yx}^2$$

$$\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)+2S_{yx}$$

$$\left(\mu_{03}+2\bar{X}\mu_{02}\right)\left(\mu_{12}+2\bar{X}\mu_{11}\right)\Big\}\Big]$$

$$=MSE\left(\bar{y}\right)+\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\frac{1}{\left\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right\}^2}$$

$$\Big[\Big\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)$$

$$-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\Big\}\Big\{-S_{yx}^2\left(\mu_{04}+4\bar{X}\mu_{03}\right.$$

$$\left.+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)+2S_{yx}\left(\mu_{03}+2\bar{X}\mu_{02}\right)\left(\mu_{12}+2\bar{X}\mu_{11}\right)$$

$$-S_x^2\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2\Big\}\Big]$$

$$=MSE\left(\bar{y}\right)-\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\frac{1}{\left\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right\}}$$

$$\Big[\frac{S_{yx}^2}{S_x^2}\Big\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)$$

$$-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\Big\}+\frac{S_{yx}^2}{S_x^2}\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2$$

$$-2S_{yx}\left(\mu_{03}+2\bar{X}\mu_{02}\right)\left(\mu_{12}+2\bar{X}\mu_{11}\right)$$

$$+S_x^2\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2\Big]$$

$$=\left(\frac{1}{n}-\frac{1}{N}\right)S_y^2-\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\frac{1}{\left\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right\}}$$

$$\left[\frac{S_{yx}^2}{S_x^2}+\frac{S_{yx}^2}{S_x^2}\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2-2S_{yx}\left(\mu_{03}+2\bar{X}\mu_{02}\right)\right.$$

$$\left.\left(\mu_{12}+2\bar{X}\mu_{11}\right)+S_x^2\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2\right]$$

$$=\left(\frac{1}{n'}-\frac{1}{N}\right)S_y^2+\left(\frac{1}{n}-\frac{1}{n'}\right)\left(S_y^2-\rho^2S_y^2\right)-\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\frac{\left\{\frac{S_{yx}}{S_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right)-S_x\left(\mu_{12}+2\bar{X}\mu_{11}\right)\right\}^2}{\left\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right\}}$$

$$=MSE(\bar{y}_{ld})-\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\frac{\left\{\frac{S_{yx}}{S_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right)-S_x\left(\mu_{12}+2\bar{X}\mu_{11}\right)\right\}^2}{\left\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right\}} \tag{2.8}$$

where $MSE\left(\bar{y}_{ld}\right)=\left(\frac{1}{n}-\frac{1}{n'}\right)\left(1-\rho^2\right)S_y^2+\left(\frac{1}{n'}-\frac{1}{N}\right)S_y^2$ is the mean square error of the double sampling linear regression estimator of population mean $\bar{Y}$ and

$$\left\{S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right\}$$

$$=\left\{\mu_{02}\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right\}$$

$$=\mu_{02}^3\left(\frac{\mu_{04}}{\mu_{02}^2}-\frac{\mu_{03}^2}{\mu_{02}^3}-1\right)=\mu_{02}^3\left(\beta_2-\beta_1-1\right)>0$$

3. A GENERALIZED REGRESSION-TYPE ESTIMATOR

The optimum values $g_1 = G_1$ and $g_2 = G_2$ in (2.6) and (2.7) minimizing the mean square error of the generalized difference-type estimator T_g in (1.1) contain the unknown parameters which may not be known in practice making the generalized difference-type estimator T_g to be impracticable; hence in such situations, the alternative is to replace the optimum values by their estimated optimum values based on sample observations which give rise to practicable generalized regression-type estimator T_{ge} defined as involving also estimated optimum values to be

$$T_{ge}=g\left(\bar{y},\bar{x}'-\bar{x},\bar{\theta}'_x-\bar{\theta}_x,\hat{g}_1,\hat{g}_2\right)$$

$$=g\left(\bar{y},u,v,\hat{g}_1,\hat{g}_2\right)$$

$$=g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right) \tag{3.1}$$

satisfying the validity conditions of Taylors' series expansion such that at the point

$$T_e=\left(\bar{Y},0,0,G_1,G_2\right)$$

$$\left.\begin{aligned}
&(i)\ g\left(\bar{Y},0,0,G_1,G_2\right)=\bar{Y}\\
&(ii)\ \left.\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial\bar{y}}\right]_{T_e}=1\\
&(iii)\ g_{1e}=\left.\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial u}\right]_{T_e}=G_1\\
&(iv)\ g_{2e}=\left.\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial v}\right]_{T_e}=G_2\\
&(v)\ g_{3e}=\left.\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial\hat{g}_1}\right]_{T_e}=0\\
&(vi)\ g_{4e}=\left.\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial\hat{g}_2}\right]_{T_e}=0
\end{aligned}\right\} \tag{3.2}$$

where $\hat{g}_1 =$

$$\frac{s_{yx}\left(\hat{\mu}_{04}+4\overline{x}\hat{\mu}_{03}+4\overline{x}^2\hat{\mu}_{02}-\hat{\mu}_{02}^2\right)-\left(\hat{\mu}_{03}+2\overline{x}\hat{\mu}_{02}\right)\left(\hat{\mu}_{12}+2\overline{x}\hat{\mu}_{11}\right)}{s_x^2\left(\hat{\mu}_{04}+4\overline{x}\hat{\mu}_{03}+4\overline{x}^2\hat{\mu}_{02}-\hat{\mu}_{02}^2\right)-\left(\hat{\mu}_{03}+2\overline{x}\hat{\mu}_{02}\right)^2}$$

$$=\hat{G}_1 \tag{3.3}$$

$$\hat{g}_2=\frac{s_x^2\left(\hat{\mu}_{12}+2\overline{x}\hat{\mu}_{11}\right)-s_{yx}\left(\hat{\mu}_{03}+2\overline{x}\hat{\mu}_{02}\right)}{s_x^2\left(\hat{\mu}_{04}+4\overline{x}\hat{\mu}_{03}+4\overline{x}^2\hat{\mu}_{02}-\hat{\mu}_{02}^2\right)-\left(\hat{\mu}_{03}+2\overline{x}\hat{\mu}_{02}\right)^2}$$

$$=\hat{G}_2 \tag{3.4}$$

for $\hat{\mu}_{04}=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^4$

$$\hat{\mu}_{03}=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^3$$

$$\hat{\mu}_{02}=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2$$

$$\hat{\mu}_{12}=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\overline{y}\right)\left(x_i-\overline{x}\right)^2,$$

$$\hat{\mu}_{11}=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\overline{y}\right)\left(x_i-\overline{x}\right),$$

$$\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i,\ s_{yx}=\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i-\overline{y}\right)\left(x_i-\overline{x}\right),$$

$$s_x^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2.$$

Under the condition (3.2), we may prove that T_{ge} attains the minimum mean square error of T_g in (2.8) as follows:

Expanding in second order Taylor's series of T_{ge} about the point $T_e=\left(\overline{Y},0,0,G_1,G_2\right)$, we have

$$T_{ge}=g\left(\overline{Y},0,0,G_1,G_2\right)+\left(\overline{y}-\overline{Y}\right)$$

$$\left.\frac{\partial g(\overline{y},u,v,\hat{G}_1,\hat{G}_2)}{\partial\overline{y}}\right]_{T_e}+\left(u-0\right)\left.\frac{\partial g(\overline{y},u,v,\hat{G}_1,\hat{G}_2)}{\partial u}\right]_{T_e}$$

$$+\left(v-0\right)\left.\frac{\partial g(\overline{y},u,v,\hat{G}_1,\hat{G}_2)}{\partial v}\right]_{T_e}$$

$$+\left(\hat{G}_1-G_1\right)\left.\frac{\partial g(\overline{y},u,v,\hat{G}_1,\hat{G}_2)}{\partial\hat{G}_1}\right]_{T_e}$$

$$+\left(\hat{G}_2-G_2\right)\left.\frac{\partial g(\overline{y},u,v,\hat{G}_1,\hat{G}_2)}{\partial\hat{G}_2}\right]_{T_e}$$

$$+\frac{1}{2!}\left\{\left(\overline{y}-\overline{Y}\right)\frac{\partial}{\partial\overline{y}}+\left(u-0\right)\frac{\partial}{\partial u}+\left(v-0\right)\frac{\partial}{\partial v}\right.$$

$$\left.+\left(\hat{G}_1-G_1\right)\frac{\partial}{\partial\hat{G}_1}+\left(\hat{G}_2-G_2\right)\frac{\partial}{\partial\hat{G}_2}\right\}^2$$

$$g(\overline{y}_*,u_*,v_*,\hat{G}_{1*},\hat{G}_{2*}) \tag{3.5}$$

where $\overline{y}_*=\overline{Y}+\theta\left(\overline{y}-\overline{Y}\right)$, $u_*=0+\theta\left(u-0\right)$,

$v_*=0+\theta\left(v-0\right)$, $\hat{G}_{1*}=G_1+\theta\left(\hat{G}_1-G_1\right)$,

$\hat{G}_{2*}=G_2+\theta\left(\hat{G}_2-G_2\right)$, $0<\theta<1$.

Now substituting in (3.5) from (3.2), we get

$$T_{ge}-\overline{Y}=\overline{Y}e_0+\overline{X}\left(e_1'-e_1\right)g_{1e}+\theta_x\left(e_2'-e_2\right)g_{2e}$$

$$+e_3g_{3e}+e_4g_{4e}+\frac{1}{2!}\left\{\overline{Y}e_0\frac{\partial}{\partial\overline{y}}+\overline{X}\left(e_1'-e_1\right)\frac{\partial}{\partial u}\right.$$

$$\left.+\theta_x\left(e_2'-e_2\right)\frac{\partial}{\partial v}+e_3\frac{\partial}{\partial\hat{G}_1}+e_4\frac{\partial}{\partial\hat{G}_2}\right\}^2$$

$$g(\overline{y}_*,u_*,v_*,\hat{G}_{1*},\hat{G}_{2*}) \tag{3.6}$$

where $\hat{G}_1-G_1=e_3$ and $\hat{G}_2-G_2=e_4$.

Squaring both sides of (3.6), taking expectations and ignoring terms greater than two in $\left(e_i,e_{i'}'\right),\left(i=0,1,2,3,4;i'=1,2\right)$, we have the mean square error of T_{ge} to the first degree of approximation to be

$$MSE\left(T_{ge}\right)=E\left[\overline{Y}e_0+\overline{X}\left(e_1'-e_1\right)g_{1e}\right.$$

$$\left.+\theta_x\left(e_2'-e_2\right)g_{2e}+e_3g_{3e}+e_4g_{4e}\right]^2$$

$$=E\left[\overline{Y}e_0+\overline{X}\left(e_1'-e_1\right)g_{1e}+\theta_x\left(e_2'-e_2\right)g_{2e}+0+0\right]^2$$

$$=\left(\frac{1}{n}-\frac{1}{N}\right)S_y^2-\left(\frac{1}{n}-\frac{1}{n'}\right)\left[\frac{S_{yx}^2}{S_x^2}+\frac{S_{yx}^2}{S_x^2}\left(\mu_{03}+2\overline{X}\mu_{02}\right)^2\right.$$

$$-2S_{yx}\left(\mu_{03}+2\bar{X}\mu_{02}\right)\left(\mu_{12}+2\bar{X}\mu_{11}\right)$$

$$+S_x^2\left(\mu_{12}+2\bar{X}\mu_{11}\right)^2\Big]\Big/\Big[S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}\right.$$

$$\left.+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\Big]$$

$$=\left(\frac{1}{n'}-\frac{1}{N}\right)S_y^2+\left(\frac{1}{n}-\frac{1}{n'}\right)\left(S_y^2-\rho^2S_y^2\right)-\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\left[\left\{\frac{S_{yx}}{S_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right)-S_x\left(\mu_{12}+2\bar{X}\mu_{11}\right)\right\}^2\right]$$

$$\left[S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right]$$

$$=MSE\left(\bar{y}_{ld}\right)-\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\left[\left\{\frac{S_{yx}}{S_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right)-S_x\left(\mu_{12}+2\bar{X}\mu_{11}\right)\right\}^2\right]\Big/$$

$$\left[S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right]$$

$$=MSE\left(T_g\right)_{\min} \qquad (3.7)$$

showing that the mean square error depending on estimated optimum value estimator T_{ge} attains the minimum mean square error (3.7) or (2.8) if the following conditions for T_{ge} are satisfied

$$\left.\begin{aligned}
&(i)\ T_{ge}\ at\ T_e=g\left(\bar{Y},0,0,G_1,G_2\right)=\bar{Y}\\
&(ii)\ g_{0e}=\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial\bar{y}}\Bigg]_{T_e}=1\\
&(iii)\ g_{1e}=\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial u}\Bigg]_{T_e}=G_1\\
&(iv)\ g_{2e}=\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial v}\Bigg]_{T_e}=G_2\\
&(v)\ g_{3e}=\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial\hat{G}_1}\Bigg]_{T_e}=0\\
&(vi)\ g_{4e}=\frac{\partial g\left(\bar{y},u,v,\hat{G}_1,\hat{G}_2\right)}{\partial\hat{G}_2}\Bigg]_{T_e}=0
\end{aligned}\right\} \qquad (3.8)$$

Taking example of T_g for characterizing scalars d_1 and d_2 to be

$$\bar{y}+d_1\left(\bar{x}'-\bar{x}\right)+d_2\left(\bar{\theta}_x'-\bar{\theta}_x\right)=\bar{y}+d_1u+d_2v \qquad (3.9)$$

having $g_1=d_1=G_1$ and $g_2=d_2=G_2$ which have from (2.6) and (2.7) the estimators $\hat{d}_1=\hat{G}_1$ and $\hat{d}_2=\hat{G}_2$ and thus, we get the estimator depending on estimated optimum values from (3.3) and (3.4) to be $\bar{y}+\hat{G}_1u+\hat{G}_2v$ satisfying the conditions of (3.8)

$$(i)\ \bar{y}+\hat{G}_1u+\hat{G}_2v\Big]_{T_e}=\bar{Y}$$

$$(ii)\ g_{0e}=\frac{\partial\left(\bar{y}+\hat{G}_1u+\hat{G}_2v\right)}{\partial\bar{y}}\Bigg]_{T_e}=1$$

$$(iii)\ g_{1e}=\frac{\partial\left(\bar{y}+\hat{G}_1u+\hat{G}_2v\right)}{\partial u}\Bigg]_{T_e}=G_1$$

$$(iv)\ g_{2e}=\frac{\partial\left(\bar{y}+\hat{G}_1u+\hat{G}_2v\right)}{\partial v}\Bigg]_{T_e}=G_2$$

$$(v)\ g_{3e}=\frac{\partial\left(\bar{y}+\hat{G}_1u+\hat{G}_2v\right)}{\partial\hat{G}_1}\Bigg]_{T_e}=0$$

$$(vi)\ g_{4e}=\frac{\partial\left(\bar{y}+\hat{G}_1u+\hat{G}_2v\right)}{\partial\hat{G}_2}\Bigg]_{T_e}=0$$

establishing the fact that the estimator $\bar{y}+\hat{G}_1u+\hat{G}_2v$ belonging to T_{ge} depending on estimated optimum values attains the minimum mean square error in (3.7).

4. CONCLUDING REMARKS

(1) We see that

$$MSE\left(T_{ge}\right)=MSE\left(T_g\right)$$

$$=\left(\frac{1}{n'}-\frac{1}{N}\right)S_y^2+\left(\frac{1}{n}-\frac{1}{n'}\right)\left(1-\rho^2\right)S_y^2-\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\left[\left\{\frac{S_{yx}}{S_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right)-S_x\left(\mu_{12}+2\bar{X}\mu_{11}\right)\right\}^2\right]\Big/$$

$$\left[S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right]$$

$$=MSE\left(\bar{y}_{ld}\right)-\left(\frac{1}{n}-\frac{1}{n'}\right)$$

$$\left[\left\{\frac{S_{yx}}{S_x}\left(\mu_{03}+2\bar{X}\mu_{02}\right)-S_x\left(\mu_{12}+2\bar{X}\mu_{11}\right)\right\}^2\right]\Big/$$

$$\left[S_x^2\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right)-\left(\mu_{03}+2\bar{X}\mu_{02}\right)^2\right] \quad (4.1)$$

Showing that the proposed generalized double sampling difference-type estimator T_g and the regression-type estimator T_{ge} are better than the double sampling linear regression estimator $\bar{y}_{ld}$ in the sense of having lesser mean square error of T_{ge} or T_g than that of $\bar{y}_{ld}$.

(2) We Consider the Estimator

$$\bar{y}e^{d_1\left(\bar{x}'-\bar{x}\right)+d_2\left(\bar{\theta}'_x-\bar{\theta}_x\right)}=\bar{y}e^{d_1u+d_2v} \quad (4.2)$$

belonging to the generalized estimator T_g with the optimum values

$$g_1=\bar{Y}d_1=G_1 \text{ or } d_1=\frac{G_1}{\bar{Y}} \quad (4.3)$$

$$\text{and } g_2=\bar{Y}d_2=G_2 \text{ or } d_2=\frac{G_2}{\bar{Y}} \quad (4.4)$$

and further, the estimated optimum values (4.3) and (4.4) are

$$\hat{d}_1=\frac{\hat{G}_1}{\bar{y}} \text{ and } \hat{d}_2=\frac{\hat{G}_2}{\bar{y}} \quad (4.5)$$

giving the estimator depending on estimated optimum values $\hat{d}_1$ and $\hat{d}_2$ to be

$$\bar{y}e^{\left(\frac{\hat{G}_1}{\bar{y}}\right)u+\left(\frac{\hat{G}_2}{\bar{y}}\right)v} \quad (4.6)$$

satisfying the conditions

$$(i)\ T_{ge}\ at\ T_e=(\bar{Y},0,0,G_1,G_2)=\bar{Y}$$

$$(ii)\ g_{0e}=\frac{\partial}{\partial\bar{y}}\left[\left(\bar{y}e^{\left(\frac{\hat{G}_1}{\bar{y}}\right)u+\left(\frac{\hat{G}_2}{\bar{y}}\right)v}\right)\right]_{T_e}=1$$

$$(iii)\ g_{1e}=\frac{\partial}{\partial u}\left[\left(\bar{y}e^{\left(\frac{\hat{G}_1}{\bar{y}}\right)u+\left(\frac{\hat{G}_2}{\bar{y}}\right)v}\right)\right]_{T_e}=G_1$$

$$(iv)\ g_{2e}=\frac{\partial}{\partial v}\left[\left(\bar{y}e^{\left(\frac{\hat{G}_1}{\bar{y}}\right)u+\left(\frac{\hat{G}_2}{\bar{y}}\right)v}\right)\right]_{T_e}=G_2$$

$$(v)\ g_{3e}=\frac{\partial}{\partial\hat{G}_1}\left[\left(\bar{y}e^{\left(\frac{\hat{G}_1}{\bar{y}}\right)u+\left(\frac{\hat{G}_2}{\bar{y}}\right)v}\right)\right]_{T_e}=0$$

$$(vi)\ g_{4e}=\frac{\partial}{\partial\hat{G}_2}\left[\left(\bar{y}e^{\left(\frac{\hat{G}_1}{\bar{y}}\right)u+\left(\frac{\hat{G}_2}{\bar{y}}\right)v}\right)\right]_{T_e}=0 \text{ of (3.8);}$$

hence, the regression-type estimator

$$\left(\bar{y}e^{\left(\frac{\hat{G}_1}{\bar{y}}\right)u+\left(\frac{\hat{G}_2}{\bar{y}}\right)v}\right)$$

attains the minimum mean square error in (3.7).

(3) For all the other Estimators

$$\bar{y}+\left(\bar{x}'-\bar{x}\right)^{d_1}+\left(\bar{\theta}'_x-\bar{\theta}_x\right)^{d_2},\ \bar{y}+u^{d_1}+v^{d_2},$$

$$\bar{y}e^{\left(\bar{x}'-\bar{x}\right)^{d_1}+\left(\bar{\theta}'_x-\bar{\theta}_x\right)^{d_2}}=\bar{y}e^{u^{d_1}+v^{d_2}},$$

$$\bar{y}\left\{1+d_1\left(\bar{x}'-\bar{x}\right)+d_2\left(\bar{\theta}'_x-\bar{\theta}_x\right)\right\}=\bar{y}\left\{1+d_1u+d_2v\right\},$$

$$\bar{y}\left\{1+\left(\bar{x}'-\bar{x}\right)^{d_1}+\left(\bar{\theta}'_x-\bar{\theta}_x\right)^{d_2}\right\}=\bar{y}\left\{1+u^{d_1}+v^{d_2}\right\}$$

$$\text{and } \bar{y}\left\{\frac{1+d_1\left(\bar{x}'-\bar{x}\right)}{1+d_2\left(\bar{\theta}'_x-\bar{\theta}_x\right)}\right\}=\bar{y}\left(1+d_1u\right)\left(1+d_2v\right)^{-1},$$

we may easily found the optimum values of d_1 and d_2 minimizing the mean square error to be equal to that of mean square error of (2.8). Now, taking these estimators depending on estimated optimum values, we can easily prove that these satisfy all the six conditions of (3.8) owing to which these estimators will also attain the minimum mean square error in (3.7).

(4) Single sampling results of generalized difference-type and regression-type estimators will be easily obtained as special case of $n' = N$.

5. EMPIRICAL STUDY

The theoretical results obtained in the study are illustrated here numerically, using the following data sets respectively.

Population 1: The data set summarized below is obtained from Singh and Chaudhary (1986) as given on page 194.

$\bar{y} = 568.58;\ \bar{x} = 568.25;\ \bar{\theta}_x = 590126.8;$

$s_y = 499.37;\ s_x = 528.05;\ \hat{\mu}_{04} = 326506150560.52;$

$\hat{\mu}_{03} = 215375912.2;\ \hat{\mu}_{02} = 267218.67;$

$\hat{\mu}_{12} = 219770487.1;\ \hat{\mu}_{11} = 235912.60;\ s_{yx} = 246169.7;$

$N = 1238;\ n' = 550;\ n = 24.$

Population 2: The data set summarized below is obtained from Murthy (1967) as given on page 178.

$\bar{y} = 144.84;\ \bar{x} = 434.04;\ \bar{\theta}_x = 288785.8;$

$s_y = 101.59;\ s_x = 323.38;\ \hat{\mu}_{04} = 56292830839;$

$\hat{\mu}_{03} = 48614868.15;\ \hat{\mu}_{02} = 100395.08;$

$\hat{\mu}_{12} = 692079.80;\ \hat{\mu}_{11} = 17448.41;\ s_{yx} = 18175.43;$

$N = 108;\ n' = 60;\ n = 25.$

Population 3: The data set summarized below is obtained from Sukhatme and Sukhatme (1997) as given on page 51.

$\bar{y} = 735.8;\ \bar{x} = 927.36;\ \bar{\theta}_x = 1183725;$

$s_y = 522.17;\ s_x = 580.70;$

$\hat{\mu}_{04} = 471425371327.51;\ \hat{\mu}_{03} = 196887322.7;$

$\hat{\mu}_{02} = 323728.87;\ \hat{\mu}_{12} = 190116364.6;$

$\hat{\mu}_{11} = 266761.40;\ s_{yx} = 277876.5;\ N = 892;$

$n' = 395;\ n = 25$

Population 4: The data set summarized below is obtained from Sukhatme and Sukhatme (1997) as given on page 185.

$\bar{y} = 199.44;\ \bar{x} = 856.41;\ \bar{\theta}_x = 1255128;$

$s_y = 166.08;\ s_x = 733.14;\ \hat{\mu}_{04} = 3637832748576.77;$

$\hat{\mu}_{03} = 1062758624;\ \hat{\mu}_{02} = 521686.59;$

$\hat{\mu}_{12} = -17880653.6;\ \hat{\mu}_{11} = 52941.43;$

$s_{yx} = 54545.72;\ N = 170;\ n' = 75;\ n = 34.$

Table 1. Mean Square Error (MSE) and Percent Relative Efficiency (PRE) of the proposed class and all the other estimators considered with respect to $\bar{y}_{ld}$

Estimator	Population I		Population II		Population III		Population IV	
	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
$\bar{y}_{ld}$	1528.60	100	243.54	100	2021.59	100	560.04	100
T_g	1245.44	123	196.76	124	1932.13	105	380.59	147
T_{ge}	1245.44	123	196.76	124	1932.13	105	380.59	147
T_1	1245.44	123	196.76	124	1932.13	105	380.59	147
T_2	1245.44	123	196.76	124	1932.13	105	380.59	147
T_3	1245.44	123	196.76	124	1932.13	105	380.59	147
T_4	1245.44	123	196.76	124	1932.13	105	380.59	147
T_5	1245.44	123	196.76	124	1932.13	105	380.59	147
T_6	1245.44	123	196.76	124	1932.13	105	380.59	147
T_7	1245.44	123	196.76	124	1932.13	105	380.59	147

From the above table, it is easily observed that the proposed class of estimators T_g and T_{ge} have lesser minimizing mean square error than the mean square error for the estimator $\bar{y}_{ld}$. Hence their percent relative efficiency is more than that of $\bar{y}_{ld}$.

Also for all the other estimators,

$$T_1 = \bar{y}e^{d_1(\bar{x}'-\bar{x})+d_2(\bar{\theta}'_x-\bar{\theta}_x)} = \bar{y}e^{d_1u+d_2v},$$

$$T_2 = \bar{y}+(\bar{x}'-\bar{x})^{d_1}+(\bar{\theta}'_x-\bar{\theta}_x)^{d_2},\ T_3 = \bar{y}+u^{d_1}+v^{d_2},$$

$$T_4 = \bar{y}e^{(\bar{x}'-\bar{x})^{d_1}+(\bar{\theta}'_x-\bar{\theta}_x)^{d_2}} = \bar{y}e^{u^{d_1}+v^{d_2}},$$

$$T_5 = \bar{y}\left\{1+d_1(\bar{x}'-\bar{x})+d_2(\bar{\theta}'_x-\bar{\theta}_x)\right\} = \bar{y}\{1+d_1u+d_2v\},$$

$$T_6 = \bar{y}\left\{1+(\bar{x}'-\bar{x})^{d_1}+(\bar{\theta}'_x-\bar{\theta}_x)^{d_2}\right\} = \bar{y}\{1+u^{d_1}+v^{d_2}\}$$

and $T_7 = \bar{y}\left\{\dfrac{1+d_1(\bar{x}'-\bar{x})}{1+d_2(\bar{\theta}'_x-\bar{\theta}_x)}\right\} = \bar{y}(1+d_1u)(1+d_2v)^{-1}$

which are particular members of the proposed class it may be said that their respective minimizing mean square error is equivalent to the minimizing mean square error of the proposed class T_g and T_{ge}.

APPENDIX

We have the following results regarding $E(e_2^2)$, $E(e_0e_2)$ and $E(e_1e_2)$:

(i) $$E(e_2^2) = \frac{E(\hat{\theta}_x-\theta_x)^2}{\theta_x^2}$$

$$= \frac{1}{\theta_x^2}E\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{N}\sum_{i=1}^{N}X_i^2\right)\right]^2 = \frac{1}{\theta_x^2}E\left[\bar{z}_s-\bar{Z}\right]^2$$

$$\left(\text{where } \bar{z}_s = \frac{1}{n}\sum_{i=1}^{n}z_i,\ \bar{Z} = \frac{1}{N}\sum_{i=1}^{N}Z_i,\ x_i^2 = z_i,\ Z_i = X_i^2\right)$$

$$= \frac{1}{\theta_x^2}\left[\frac{1}{n}-\frac{1}{N}\right]S_z^2$$

with $$S_z^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Z_i-\bar{Z})^2 \qquad (1)$$

$$= \frac{1}{\theta_x^2}\left[\frac{1}{n}-\frac{1}{N}\right]\left[\frac{1}{N-1}\left\{\sum_{i=1}^{N}Z_i^2-N\theta_x^2\right\}\right],$$

since $\bar{Z} = \theta_x = \dfrac{1}{\theta_x^2}\left[\dfrac{1}{n}-\dfrac{1}{N}\right]\dfrac{1}{N-1}$

$$\left[\sum_{i=1}^{N}\left\{(X_i-\bar{X})+\bar{X}\right\}^4-N(\mu_{02}+\bar{X}^2)^2\right]$$

$$= \frac{1}{\theta_x^2}\left[\frac{1}{n}-\frac{1}{N}\right]\frac{1}{N-1}$$

$$\left[\sum_{i=1}^{N}(X_i-\bar{X})^4+4\bar{X}\sum_{i=1}^{N}(X_i-\bar{X})^3+6\bar{X}^2\sum_{i=1}^{N}(X_i-\bar{X})^2\right.$$

$$\left.+4\bar{X}^3\sum_{i=1}^{N}(X_i-\bar{X})+N\bar{X}^4-N(\mu_{02}^2+\bar{X}^4+2\bar{X}^2\mu_{02})\right]$$

$$= \frac{1}{\theta_x^2}\left[\frac{1}{n}-\frac{1}{N}\right]\frac{N}{N-1}$$

$$\left(\mu_{04}+4\bar{X}\mu_{03}+6\bar{X}^2\mu_{02}+0+\bar{X}^4-\mu_{02}^2-\bar{X}^4-2\bar{X}^2\mu_{02}\right)$$

$$= \frac{1}{\theta_x^2}\left[\frac{1}{n}-\frac{1}{N}\right]\frac{N}{N-1}\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right) \qquad (2)$$

whence, for large N, $\dfrac{N}{N-1} = 1$

$$E(e_2^2) = \left(\frac{1}{n}-\frac{1}{N}\right)\frac{1}{\theta_x^2}\left(\mu_{04}+4\bar{X}\mu_{03}+4\bar{X}^2\mu_{02}-\mu_{02}^2\right).$$

(ii) $$E(e_0e_2) = \frac{1}{\bar{Y}\theta_x}E\left\{(\bar{y}_s-\bar{Y})(\hat{\theta}_x-\theta_x)\right\}$$

$$= \frac{1}{\bar{Y}\theta_x}E\left\{(\bar{y}_s-\bar{Y})(\bar{z}_s-\bar{Z})\right\};$$

since $\hat{\theta}_x = \dfrac{1}{n}\sum_{i=1}^{n}x_i^2 = \bar{z}_s$

$$= \frac{1}{\bar{Y}\theta_x}\left(\frac{1}{n}-\frac{1}{N}\right)\frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i-\bar{Y})X_i^2 \qquad (3)$$

$$= \frac{1}{\bar{Y}\theta_x}\left(\frac{1}{n}-\frac{1}{N}\right)\frac{1}{(N-1)}\sum_{i=1}^{N}\left\{(X_i-\bar{X})+\bar{X}\right\}^2(Y_i-\bar{Y})$$

$$= \frac{1}{\bar{Y}\theta_x}\left(\frac{1}{n}-\frac{1}{N}\right)\frac{1}{(N-1)}\left[\sum_{i=1}^{N}(Y_i-\bar{Y})(X_i-\bar{X})^2\right.$$

$$\left.+2\bar{X}\sum_{i=1}^{N}(Y_i-\bar{Y})(X_i-\bar{X})+\bar{X}^2\sum_{i=1}^{N}(Y_i-\bar{Y})\right]$$

$$= \frac{1}{\bar{Y}\theta_x}\left(\frac{1}{n}-\frac{1}{N}\right)\frac{N}{(N-1)}\left(\mu_{12}+2\bar{X}\mu_{11}\right) \qquad (4)$$

whence, for large N with $\dfrac{N}{N-1} = 1$,

$$E(e_0e_2) = \left(\frac{1}{n}-\frac{1}{N}\right)\frac{1}{\bar{Y}\theta_x}\left(\mu_{12}+2\bar{X}\mu_{11}\right).$$

(iii) Proceeding as for $E(e_1e_2)$, we have

$$E(e_1e_2) = \frac{1}{\bar{X}\theta_x} E\left\{\left(\bar{x}_s - \bar{X}\right)\left(\hat{\theta}_x - \theta_x\right)\right\}$$

$$= \frac{1}{\bar{X}\theta_x}\left(\frac{1}{n} - \frac{1}{N}\right)\frac{1}{(N-1)}\sum_{i=1}^{N}\left(X_i - \bar{X}\right)X_i^2$$

$$= \frac{1}{\bar{X}\theta_x}\left(\frac{1}{n} - \frac{1}{N}\right)\frac{N}{(N-1)}\left(\mu_{03} + 2\bar{X}\mu_{02}\right) \quad (5)$$

from which, for large N with $\frac{N}{N-1} = 1$,

$$E(e_1e_2) = \left(\frac{1}{n} - \frac{1}{N}\right)\frac{1}{\bar{X}\theta_x}\left(\mu_{03} + 2\bar{X}\mu_{02}\right).$$

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