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Efficient Row-Column Designs with Two Rows

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SUMMARY

A general method of construction of efficient row-column designs with two rows has been developed. Lower bounds to A- and D- efficiencies of the designs obtained through the proposed method of construction for $3 \le v \le 10$, $v \le b \le v(v-1)/2$, $11 \le v \le 25$, $b = v$ and $(v, b) = (11, 13), (12, 14), (13, 14)$ and $(13, 15)$ where v is the number of treatments and b is the number of columns have been computed and compared with those of best available designs in the above parametric range in the literature under a fixed effects model. The lower bounds to A- and D- efficiencies of the designs obtained have also been compared with those of best available designs in the literature under a mixed effects model (considering column effects as random). Robustness aspects of optimal/efficient designs has been investigated under a mixed effects model for different values of ρ (a function of inter and intra column variances). Several designs have been obtained that are more efficient under mixed effects model as compared to the best available designs. A computer programme using C# programming language with ASP.NET platform has been developed for generating efficient row-column designs in two rows in the above restricted parametric range. The A- and D- efficiencies of the designs are also reported.

Keywords: Row-column designs, A-efficiency, D-efficiency, Robustness.

1. INTRODUCTION

Row-column designs are useful for the experimental situations in which there are two cross classified sources of heterogeneity in the experimental material. One important class of row column designs of great practical significance is where one of the two factors causing heterogeneity in the experimental units has only two levels and as a consequence, it is not possible to allocate more than two units in a single column/row. For example, consider an experiment conducted for improving quality of products; experimental processes in the laboratory may require use of an oven divided into smaller sections in a linear fashion. In each section temperature or other conditions may vary. Further, in each section there are two positions on which treatments can be applied. Therefore, considering sections as columns and positions as rows, the experiment can be conducted using a row-column design in two rows meaning thereby that only two treatments can be accommodated in a column. Since the columns are incomplete in the sense that only two out of a total of v treatments appear in columns, the column effects may be assumed random.

Some preliminaries of row-column designs are given in Section 2. Section 3 attempts to develop a general method of construction of efficient row-column designs in two rows. The method is general in nature and one can obtain row-column designs in two rows for any v (the number of treatments) and b (the number of columns). However, for the purpose of comparing the designs obtainable from the proposed method of

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construction with those of best designs available in the literature, the designs were generated only in a restricted parametric range $3 \le v \le 10$ for $v \le b \le v(v-1)/2$, $11 \le v \le 25$ for $b = v$, and $(v, b) = (11, 13), (12, 14), (13, 14)$ and $(13, 15)$ and the lower bounds to A- and D- efficiencies of the designs obtained by the proposed method of construction have been computed for all values of $\rho = 0.0, 0.1, 0.2, 0.3, ..., 0.9$, a function of inter and intra column variances defined in Section 2. To investigate the robustness of the designs under fixed/mixed effects model, percent coefficient of variation (CV) of the lower bounds to A- and D- efficiencies were obtained for the following four situations: (i) $\rho \ge 0$ for fixed/mixed effects model, (ii) $\rho \ge 0.1$ for the mixed effects model, (iii) $\rho \ge 0.4$ for moderate to high values of ρ, and (iv) $\rho \ge 0.7$ for high values of ρ. If the percent CV of A- and D-efficiencies of the designs for different values of ρ is less than 5%, then that design is said to be robust and can be used for different values of ρ; otherwise not. If the percent CV of only A-efficiencies [only D-efficiencies] of the designs is less than 5%, then that design is said to be robust with respect to only A-efficiencies [only D-efficiencies]. Lower bound to A– and D– efficiencies and the robustness of designs obtained in terms of percent CV, as defined above, are then compared with the best available designs in the literature catalogued in Sarkar *et al*. (2010). In the catalogue of 139 best available designs prepared by Sarkar *et al*. (2010), 118 efficient designs were obtained by Sarkar *et al.* (2010) and 21 designs were obtained after rearranging the block contents of designs obtained by Nguyen and Williams (2005) for the parametric combinations $(v, b) = (7, 14), (8, 16), (8, 20), (9, 10), (9, 17), (9, 22), (9, 23), (9, 27), (9, 28), (9, 30), (9, 33), (9, 35), (10, 10), (10, 20), (10, 25), (10, 29), (10, 30), (10, 31), (10, 34), (10, 40), (11, 11)$. Further, a design which is optimal/efficient under mixed effects model for a given value of ρ may have more efficiency than the design which is optimal/efficient under fixed effects model for that value of ρ. The results are presented in Section 4. We begin with some preliminaries in Section 2.

2. PRELIMINARIES OF ROW-COLUMN DESIGNS

Consider a row-column design d with v treatments (varieties), k rows and b columns, vector of row sizes $\mathbf{k} = b\mathbf{1}$, vector of column sizes $\mathbf{b} = k\mathbf{1}$ and vector of replication numbers $\mathbf{r}' = (r_1, ..., r_v)$. Denote by $\mathbf{R} = \text{diag}(r_1, ..., r_v)$, a diagonal matrix. For the set up under consideration, $k = 2$. Here $\mathbf{1}_t$ denotes a column vector of order $t\times 1$ with all elements as one. Let $\mathbf{M} = ((m_{hi}))_{v\times k}$, $\mathbf{N} = ((n_{hj}))_{v\times b}$ and $\mathbf{W} = \mathbf{1}_k\mathbf{1}'_b$ denote treatments *vs* rows, treatments *vs* columns and rows *vs* columns incidence matrices, respectively. For $h = 1, ..., v$; $i = 1, ..., k$ and $j = 1, ..., b$, the non-negative numbers $m_{hi}(n_{hj})$ denote the number of times treatment h appears in the i^{th} row (j^{th} column). In the present investigation, we restrict to the experimental situations where rows *vs* columns classification is orthogonal with each cell of the array having exactly one experimental unit. All row sizes are equal to number of columns, b, and all column sizes are equal to number of rows, k. Under the usual additive homoscedastic three way classified linear model, the coefficient matrix of the reduced normal equations for estimating estimable linear combinations of treatment effects is

$$\mathbf{C} = \mathbf{R} - \frac{1}{k}\mathbf{N}\mathbf{N}' - \frac{1}{b}\mathbf{M}\mathbf{M}' + \frac{1}{bk}\mathbf{r}\mathbf{r}' \tag{2.1}$$

where, $\mathbf{NN}' = (\lambda_{hh'})_{v\times v}$ with $\lambda_{hh'} = \sum_{j=1}^{b} n_{hj}n_{h'j}$ and $\mathbf{MM}' = (\mu_{hh'})_{v\times v}$, with $\mu_{hh'} = \sum_{i=1}^{k} m_{hi}m_{h'i}$, $h, h' = 1, 2, ..., v$.

In a mixed effects model, the column effects may be considered as random with mean zero and variance σ_β^2. The error variance is σ^2 and the errors and the random column effects are assumed to be independently distributed. σ_β^2 and σ^2 are unknown variance parameters. Following Shah and Sinha (1989), the **C**-matrix for row-column designs under a mixed effects model when rows are orthogonal to columns but column effects are considered as random is

$$\mathbf{C} = \mathbf{R} - \frac{1}{k}\mathbf{N}\mathbf{N}' - \frac{1}{b}\mathbf{M}\mathbf{M}' + \frac{1}{bk}\mathbf{r}\mathbf{r}' + \rho\left(\frac{1}{k}\mathbf{N}\mathbf{N}' - \frac{1}{bk}\mathbf{r}\mathbf{r}'\right), \tag{2.2}$$

where $\rho = \dfrac{\sigma^2}{\sigma^2 + k\sigma_\beta^2}$, is assumed to be known or estimated from data.

The **C**-matrix in (2.1) or in (2.2) is symmetric, positive semi-definite and has row sums equal to zero. For a connected design, Rank $(\mathbf{C}) = v - 1$. Henceforth, we shall deal with connected designs only. Let $\mathbf{C}^-$ be a generalized inverse of matrix $\mathbf{C}$ *i.e.* $\mathbf{CC}^-\mathbf{C} = \mathbf{C}$. Let $\mathbf{p}'\hat{\boldsymbol{\tau}}$ be a treatment contrast, $\mathbf{p}'\mathbf{1} = 0$. Here τ is a v-component vector of treatment effects. Let $\mathbf{p}'\hat{\tau}$ be its best linear unbiased estimator (BLUE) and var($\mathbf{p}'\hat{\tau}$) = $\mathbf{p}'\mathbf{C}^{-}\mathbf{p}\sigma^2$. In general, let, $\mathbf{p}_1'\tau, \mathbf{p}_2'\tau, ..., \mathbf{p}'_s\tau$ be s treatment contrasts represented as $\mathbf{P}'\tau$, the s rows of $\mathbf{P}'$ being $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_s)'$. The BLUE of $\mathbf{P}'\tau$ is $\mathbf{P}'\hat{\tau}$ and var($\mathbf{P}'\hat{\tau}$) = $\mathbf{P}'\mathbf{C}^{-}\mathbf{P}\sigma^2$. If we are interested in all possible pairwise treatment comparisons, then $\mathbf{P}'$ will be a matrix of order $\begin{bmatrix} v \\ 2 \end{bmatrix} \times v$. In this case $\mathbf{PP}' = v\mathbf{I}_v - \mathbf{1}_v\mathbf{1}'_v$.

2.1 Lower Bounds to A- and D-efficiency

Suppose that the interest of the experimenter is in making all the possible pairwise treatment comparisons. Then a design d is said to be A-optimal if it minimizes the average variance of all possible pair wise treatment comparisons, *i.e*, if it minimizes

trace ($\mathbf{P}'\mathbf{C}^{-}\mathbf{P}$) or equivalently trace($\mathbf{C}^{+}\mathbf{PP}'$),

where $\mathbf{C}^{+}$ is the Moore-Penrose inverse of $\mathbf{C}$ and for, $\mathbf{PP}' = v\mathbf{I}-\mathbf{11}'$, trace($\mathbf{C}^{+}\mathbf{PP}'$) = trace($v\mathbf{C}^{+}$), because $\mathbf{C}^{+}\mathbf{1} = \mathbf{0}$. If the rows of $\mathbf{P}'$ contain $v\times 1$ orthonormalized contrasts, then $\mathbf{P}'\mathbf{P} = \mathbf{I}_{v-1} - \frac{1}{v}\mathbf{11}'$.

Let $\mathbf{D}(v, b, k, \rho)$ denote the class of all connected row-column designs with v treatments arranged in k rows, b columns with rows *vs* columns classification orthogonal, column effects as random and $\rho = \sigma^2/(\sigma^2 + k\sigma_\beta^2)$. For a treatment connected row-column design d, let $\theta_1, \theta_2, ..., \theta_{v-1}$ be the non-zero eigen values of $\mathbf{C}$. Define $\phi_A(d) = \sum_{i=1}^{v-1}\theta_i^{-1}$ and $\phi_D(d) = \prod_{i=1}^{v-1}\theta_i^{-1}$. For inferring on a complete set of orthonormalized treatment contrasts, a design is said to be A- [D-] optimal if it minimizes $\phi_A(d)[\phi_D(d)]$ over $\mathbf{D}(v, b, k, \rho)$.

For $\rho = 0$, the $\mathbf{C}$-matrix in (2.2) reduces to the $\mathbf{C}$-matrix in (2.1). Therefore, in the subsequent sections, lower bounds to A- and D-efficiencies have been computed only under a mixed effects model. For, $\rho = 0$ these reduce to lower bounds to A- and D-efficiencies under fixed effects model.

Making an appeal to Sarkar *et al.* (2010), the *A*-efficiency $\{e_A(d)\}$ and D-efficiency $\{e_D(d)\}$ of any design d over $\mathbf{D}(v, b, k, \rho)$ and their lower bounds are given below

$$e_A(d) = \left[\frac{\phi_A(d_A^*)}{\phi_A(d)}\right] \text{ and}$$

$$e_A(d) \le \frac{(v-1)^2}{\left\{b(k-1) + \rho b\left(1-\frac{k}{v}\right)\right\}\phi_A(d)} \tag{2.3}$$

and

$$e_D(d) = \left[\frac{\phi_D(d_D^*)}{\phi_D(d)}\right]^{1/(v-1)} \text{ and}$$

$$e_D(d) \le \frac{(v-1)}{\left\{b(k-1) + \rho b\left(1-\frac{k}{v}\right)\right\}\{\phi_D(d)\}^{\frac{1}{v-1}}} \tag{2.4}$$

where, d_A^* and d_D^* are the hypothetical A-optimal and D-optimal designs over $\mathbf{D}(v, b, k, \rho)$, respectively. A design attaining the lower bound on $e_A(d)$ $[e_D(d)]$ is A- (D-) optimal.

3. METHOD OF CONSTRUCTION

The purpose of this section is to give a general method of construction of efficient row-column designs with two rows, *i.e.*, $k = 2$. A design with $b = v$ and $k = 2$ is a minimally connected row-column design. In this design, error degree of freedom is zero. With $b = \begin{bmatrix} v \\ 2 \end{bmatrix}$ the design becomes balanced with respect to rows for even number of columns and partially balanced for odd number of columns. Thus, we are dealing with $v \le b \le \begin{bmatrix} v \\ 2 \end{bmatrix}$ which gives a wide choice of designs depending upon the availability of resources with the experimenter. The method is general in nature and can produce any row-column design with two rows for $v \le b \le v(v-1)/2$.

Method 3.1: The method is given for two situations, *viz*. when, (a) v is even, and (b) v is odd.

Case 1: v is even, $v \le b \le 2v - 1$

Step 1: Generation of first v columns of the design (*i.e.* design for $b = v$)

Column	**1**	**2**	**3**	**…**	***v***
Row 1	1	2	3	…	v
Row 2	2	3	4	…	1

Step 2: Generation of next $v/2$ columns of the design, *i.e.* from $v + 1$ to $v + v/2$

For $i = 1, 2, ..., v/2$, the contents of columns are

- columns with odd number $\{v - (i - 1), v/2 - (i - 1)\}$ mod(v)
- columns with even number $\{v/2 - (i - 1), v - (i - 1)\}$ mod(v)

Step 3: Generation of next $v/2 - 1$ columns of the design, *i.e.* from $v + v/2 + 1$ to $2v - 1$

For $j = 1, 2 ..., v/2-1$, the contents of columns are

- columns with odd number $\{v/2 - 1 - (j - 1), v - (j - 1)\}$ mod(v)
- columns with even number $\{v - (j - 1), v/2 - 1 - (j - 1)\}$ mod(v)

Example 3.1a: For $v = 6$; $b = 6, 7, 8, 9, 10, 11$, the design is

Column	1	2	3	4	5	6	7	8	9	10	11
Row1	1	2	3	4	5	6	6	2	4	2	5
Row2	2	3	4	5	6	1	3	5	1	6	1

Case 2: v is even, $b = 2v$

Step 1: Generation of first v columns of the design (*i.e.* design for $b = v$)

Column	1	2	3	…	v
Row 1	1	2	3	…	v
Row 2	2	3	4	…	1

Step 2: Generation of last v columns from ($v + 1$ to $2v$) of the design, = (*i.e.* design with $b = 2v$)

- When $v/2$ is odd

Columns	Row 1	Row 2
$v + 1$	v	$v/2 - 1$
$v + 2$	$v/2 - 1$	$v - 2$
$v + 3$	$v - 2$	$v/2 - 3$
⋮	⋮	⋮
$v + v/2$	$v/2 + 1$	v
$v + v/2 + 1$	$v - 1$	$v/2 - 2$
$v + v/2 + 2$	$v/2 - 2$	$v - 3$
$v + v/2 + 3$	$v - 3$	$v/2 - 4$
⋮	⋮	⋮
$2v$	$v/2$	$v - 1$

Elements are reduced mod(v)

- When $v/2$ is even

Columns	Row 1	Row 2
$v + 1$	v	$v/2$
$v + 2$	$v/2$	$v - 2$
$v + 3$	$v - 2$	$v/2 - 2$
⋮	⋮	⋮
$v + v/2$	2	v
$v + v/2 + 1$	$v - 1$	$v/2 - 1$
$v + v/2 + 2$	$v/2 - 1$	$v - 3$
$v + v/2 + 3$	$v - 3$	$v/2 - 3$
⋮	⋮	⋮
$2v$	1	$v - 1$

Elements are reduced mod(v)

Example 3.1b: For $v = 6$; $b = 12$, the design is

Column	1	2	3	4	5	6	7	8	9	10	11	12
Row1	1	2	3	4	5	6	6	2	4	5	1	3
Row2	2	3	4	5	6	1	2	4	6	1	3	5

Case 3: v is even, $2v + 1 \le b \le v(v - 1)/2$

Step 1: Generation of first v columns of the design (*i.e.* design for $b = v$)

Column	1	2	3	…	v
Row 1	1	2	3	…	v
Row 2	2	3	4	…	1

Step 2: Generation of next v columns from ($v + 1$ to $2v$) of the design (*i.e.* design with $b = 2v$)

- When $v/2$ is odd

Column	Row 1	Row 2
$v + 1$	v	$v/2 - 1$
$v + 2$	$v/2 - 1$	$v - 2$
$v + 3$	$v - 2$	$v/2 - 3$
⋮	⋮	⋮
$v + v/2$	$v/2 + 1$	v
$v + v/2 + 1$	$v - 1$	$v/2 - 2$
$v + v/2 + 2$	$v/2 - 2$	$v - 3$
$v + v/2 + 3$	$v - 3$	$v/2 - 4$
⋮	⋮	⋮
$2v$	$v/2$	$v - 1$

Elements are reduced mod(v)

- When $v/2$ is even

Columns	Row 1	Row 2
$v + 1$	v	$v/2$
$v + 2$	$v/2$	$v - 2$
$v + 3$	$v - 2$	$v/2 - 2$
⋮	⋮	⋮
$v + v/2$	2	v
$v + v/2 + 1$	$v - 1$	$v/2 - 1$
$v + v/2 + 2$	$v/2 - 1$	$v - 3$
$v + v/2 + 3$	$v - 3$	$v/2 - 3$
⋮	⋮	⋮
$2v$	1	$v - 1$

Elements are reduced mod(v)

Step 3: Generation of columns with numbers $2v + 1$ to $v(v - 1)/2$ of the design

For $j = 1, 2, ..., v(v - 1)/2 - 2v$, the contents of the column are

when $v/2$ is odd

- columns with odd number $\{ v - (j - 1), v/2 - (j - 1)\}$ mod(v)
- columns with even number $\{ v/2 - (j - 1), v - (j - 1)\}$ mod(v)

when $v/2$ is even and $b = (2 + m)v + 1$ to $(3 + m)v$, $m = 0, 1, 2…$

- columns with odd number $\{ v - (j - 1), v/2 + m + 1 - (j - 1)\}$ mod(v)
- columns with even number $\{ v/2 + \mathrm{m} + 1 - (j - 1), v - (j - 1)\}$ mod(v)

Example 3.1c: For $v = 6$, $b = 13, 14, 15$ the design can be obtained as

Column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Row1	1	2	3	4	5	6	6	2	4	5	1	3	6	2	4
Row2	2	3	4	5	6	1	2	4	6	1	3	5	3	5	1

Case 4: v is odd, $v \le b \le 2v - 1$

Step 1: Generation of first v columns of the design (*i.e.* design for $b = v$)

Column	1	2	3	…	v
Row 1	1	2	3	…	v
Row 2	2	3	4	…	1

Step 2: Generation of next $[v/2]$ columns of the design, *i.e.* from $v + 1$ to $v + [v/2]$

For $i = 1, 2, ..., [v/2]$, the contents of the column are

- column with odd number $\{v - (i - 1), [v/2] - (i - 1)\}$ mod(v)
- column with even number $\{[v/2] - (i - 1), v - (i - 1)\}$ mod(v)

Step 3: Generation of $v + [v/2] + 1$ to $2v - 1$ columns of the design

For $j = 1, 2, ..., [v/2]$, the contents of the column are

- column with odd number $\{ [v/2] + 1 - (j - 1), v - (j - 1)\}$ mod(v)
- column with even number $\{ v - (j - 1), [v/2] + 1 - (j - 1)\}$ mod(v)

Example 3.2a: For $v = 7$ and for any value of $7 \le b \le 13$, the design is

Column	1	2	3	4	5	6	7	8	9	10	11	12	13
Row1	1	2	3	4	5	6	7	7	2	5	4	6	2
Row2	2	3	4	5	6	7	1	3	6	1	7	3	5

Case 5: v is odd, $b = 2v$

Step 1: Generation of first v columns of the design (*i.e.* design for $b = v$)

Column	1	2	3	…	v
Row 1	1	2	3	…	v
Row 2	2	3	4	…	1

Step 2: Generation of v columns ($v + 1$ to $2v$) of the design (*i.e.* design with $b = 2v$)

Column	$v + 1$	$v + 2$	$v + 3$	…	$2v$
Row 1	v	$[v/2]$	$v - 1$	...	$[v/2] + 1$
Row 2	$[v/2]$	$v - 1$	$[v/2] - 1$	...	v

Example 3.2b: For $v = 7$; $b = 14$, the design is

Column	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Row1	1	2	3	4	5	6	7	7	3	6	2	5	1	4
Row2	2	3	4	5	6	7	1	3	6	2	5	1	4	7

Case 6: v is odd, $2v + 1 \leq b \leq v(v - 1)/2$

Step 1: Generation of first v columns of the design (*i.e.* design for $b = v$)

Column	1	2	3	…	v
Row 1	1	2	3	…	v
Row 2	2	3	4	…	1

Step 2: Generation of v columns ($v + 1$ to $2v$) of the design (*i.e.* design with $b = 2v$)

Column	$v + 1$	$v + 2$	$v + 3$	…	$2v$
Row 1	v	$[v/2]$	$v - 1$	...	$[v/2] + 1$
Row 2	$[v/2]$	v 61	$[v/2] - 1$	...	v

Step 3: Generation of columns with numbers $2v + 1$ to $v(v - 1)/2$ of the design

For $j = 1, 2, .., v(v - 1)/2 - 2v$, the contents of the column are

- columns with odd number $\{ v - (j - 1), [v/2] + 2 - (j - 1)\} \bmod(v)$
- columns with even number $\{[v/2] + 2 - (j - 1), v - (j - 1)\} \bmod(v)$

Example 3.2c: For $v = 7$; $b = 15, 16, 17, 18, 19, 20, 21$ the design is

Column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Row1	1	2	3	4	5	6	7	7	3	6	2	5	1	4	7	4	5	2	3	7	1
Row2	2	3	4	5	6	7	1	3	6	2	5	1	4	7	5	6	3	4	1	2	6

Remark 3.1: In relation to efficient row column designs in two rows, Yang (2003) showed that interwoven loop design is the best design in terms of A-[D-] efficiency. The interwoven loop design, however, has the number of arrays as multiple of number of varieties. This is a serious limitation of interwoven loop design. To overcome this limitation of the interwoven loop designs, Sarkar *et al.* (2010) developed an algorithm based on interchange and exchange of treatments for obtaining efficient row-column designs in two rows for given number of treatments and columns. Further, a web based solution of designs for microarray experiments has been provided through Microarray Designer by Bioinformatics Research Group of Ohio State University at http://bio.cse.ohio-state.edu/Microarray Designer/. These methods, however, are highly computer intensive and for every parametric combination, a new computer aided search is required. Therefore, in the present investigation emphasis has been laid on development of general method of construction of row column designs with two rows for any parametric combination, in which number of arrays may or may not be a multiple of number of varieties and number of varieties can be any number. The designs also have high lower bounds to A-[D-efficiency].

4. LOWER BOUNDS TO A- AND D- EFFICIENCIES AND COMPARISON WITH BEST AVAILABLE DESIGNS

Using the proposed method of construction, given in Section 3, 152 row-column designs for the parametric combinations, and $3 \leq v \leq 10$, $v \leq b \leq v(v - 1)/2$, $11 \leq v \leq 35$, $b = v$ and $(v, b) = (11, 12), (11, 13), (12, 13), (12, 14), (13, 14), (13, 15), (13, 16)$ have been constructed. Out of these 152 designs obtained by the proposed method in the above restricted parametric range, 139 designs in the parametric range $3 \leq v \leq 10$ for $v \leq b \leq v(v - 1)/2$, $11 \leq v \leq 25$ for $b = v$, and $(v, b) = (11, 13), (12, 14), (13, 14)$ and $(13, 15)$ were already available in the literature as catalogued by Srakar *et al.* (2010). Out of 13 new designs for the parameters $26 \leq v \leq 35$, $b = v$ and $(v, b) = (11, 12), (12, 13)$ and $(13, 16)$ generated by the proposed method, 10 were available as alternate loop designs for block design set up and 3 row-column designs with two rows for $(v, b) = (11, 12), (12, 13)$ and $(13, 16)$ may be obtained by rearranging the block contents of best available block designs given by Nguyen and Williams (2005) but none of these were studied for row-column design set up in the literature.

4.1 Comparison of Designs Obtained with the Existing Designs

It would be of interest to see how the 139 designs obtained by the proposed method perform in terms of A- and D-efficiencies in comparison to the best existing designs with same parameters available in the literature given in Srakar *et al.* (2010).

Lower bounds to A- and D-efficiencies of the 152 row-column designs under a fixed/mixed effects model have been computed using the expression (2.4) for $\rho = 0.0, 0.1, 0.2, 0.3, \ldots, 0.9$ for the designs with $3 \le v \le 10$, $v \le b \le v(v-1)/2$, $11 \le v \le 35$, and $(v, b) = (11, 12), (11, 13), (12, 13), (12, 14), (13, 14), (13, 15), (13, 16)$ obtained from the general method of construction given in Section 3.

After comparing 139 designs obtained by the proposed method and the designs already available in the literature and catalogued as explained above, it was found that 30 designs obtained by the proposed method with the following parametric combinations have the same efficiencies as those of the best available designs, catalogued above, for all values of $\rho = 0, 0.1, 0.2, 0.3, \ldots, 0.9$.

(*v*, *b*) = (3, 3), (4, 4), (4, 5), (4, 6), (5, 5), (5, 7), (5, 8), (5, 9), (6, 6), (6, 9), (6, 11), (6, 12), (6, 13), (6, 14), (6, 15), (7, 7), (7, 10), (7, 19), (8, 8), (8, 11), (8, 12), (8, 24), (8, 25), (8, 26), (8, 27), (8, 28), (9, 9), (9, 35), (10, 20), (10, 45).

Further in the parametric range $3 \le v \le 10$, $v \le b \le v(v-1)/2$, $11 \le v \le 25$ $b = v$, $(v, b) = (11, 12), (11, 13), (12, 13), (12, 14), (13, 14), (13, 15)$ and $(13, 16)$ using the proposed method three designs obtained with parametric combinations $(v, b) = (5, 10), (7, 21), (9, 36)$, were actually Youden Square designs. It is known that a Youden Square design, whenever existent, is universally optimal in $D(v, b, k)$, the class of connected row-column designs in which v treatments are arranged in b columns and k rows. Therefore, these three designs have also not been considered for comparison with the row-column designs with two rows available in the literature. In the remaining 106 designs obtained from the general method of construction, the designs found to be more A- and D-efficient for (i) $\rho \ge 0.1$, under a mixed effects model (ii) $\rho \ge 0.4$, for moderate to high values of ρ and (iii) $\rho \ge 0.7$, for high values of ρ. Seventeen designs are more A- and D-efficient under a mixed effects model, *i.e.* for $\rho \ge 0.1$, 14 designs are more A- and D-efficient for $\rho \ge 0.4$ and 9 designs are more A- and D-efficient for $\rho \ge 0.7$. The parametric combinations of these designs are given in Appendix A.1.

4.2 Robustness of Proposed Designs

To investigate the robustness of the design under fixed/mixed effects model, percent coefficient of variation (CV) of the lower bounds to A- and D-efficiencies are obtained for the following four situations: (i) $\rho \ge 0$ for investigating robustness under a fixed/mixed effects model, (ii) $\rho \ge 0.1$ for investigating robustness under a mixed effects model, (iii) $\rho \ge 0.4$ for investigating robustness under a mixed effects model for moderate to high values of ρ, and (iv) $\rho \ge 0.7$ for investigating robustness under a mixed effects model for high values of ρ. If CV of A- and D-efficiencies for different values of ρ is less than 5% for a given design, then that design is said to be robust and can be used for different values of ρ, otherwise not. If the CV of only A-efficiencies [only D-Efficiencies] is less than 5%, then that design is said to be robust with respect to A-efficiencies [D-efficiencies]. The results obtained have been compared with the best available design as described above.

From Appendix A.1, it is clear that design with $v = 6$ and $b = 8$ is more A- and D-efficient and robust under a mixed effects model (*i.e.* there is little variation in A- and D-efficiency under a mixed effects model) for any value of $\rho \ge 0.1$ and 16 designs have more A- and D-efficiency than the best available designs for any value of $\rho \ge 0.1$, although percent CV in efficiencies is more than 5%. The complete details of lower bounds to A- and D-efficiencies of these 16 designs are given in Appendix A.2. The difference in the efficiencies as compared to best available design is substantial for all values of $\rho \ge 0.1$ in case of A-efficiency and for all values of $\rho \ge 0$ in case of D-efficiency, therefore, these can be used for practical purposes.

One can easily see that 27 (11 as given in Appendix A.1 and 16 given in Appendix A.2) designs are more A- and D-efficient and robust for moderate to high values of $\rho \ge 0.4$. It can also be seen that 3 designs are more A- and D-efficient and having percent CV, more than 5% which is less than the best available designs. 8 designs are more efficient and robust for high values of $\rho \ge 0.7$ and one design is more A- and D-efficient and having percent CV more than 5% which is less than the best available designs.

Since the designs marked with asterisk (*) in Appendix A.1 are more A- and D-efficient and have CV less than the best available designs, therefore, these can be preferred over the best available designs inspite of being non-robust (CV more than 5%).

While comparing only with respect to lower bounds to A-efficiency alone, it was found that the design (10, 35) is more A-efficient for any value of $\rho \geq 0$ and have less CV (which is less than 5%) than the best available design and is also robust under a fixed /mixed effects model. The designs marked with ($) in Appendix A.1 are robust with respect to A-efficiency for moderate to high values of $\rho \geq 0.4$. The designs marked with (**) in Appendix A.1 are also robust for A-efficiency for any value of $\rho \geq 0.7$. Further, the designs (7, 12) and (10, 12) are more A-efficient and have less CV (which is less than 5%) than the best available design for any value of $\rho \geq 0.4$. The design (10, 13) is more A-efficient and have less CV (which is less than 5%) than the best available design for any value of $\rho \geq 0.7$.

While comparing only with respect to lower bounds to D-efficiency alone, it was found that the 16 designs (11, 11), (12, 12), (13, 13), (13, 14), (14, 14), (15, 15), (16, 16), (17, 17), (18, 18), (19, 19), (20, 20), (21, 21), (22, 22), (23, 23), (24, 24), (25, 25) {given in Appendix A.1} are more D-efficient than the best available design under a fixed/mixed effects model as well, *i.e.* for $\rho \geq 0$. Further, the designs (6, 10), (7, 11), (8, 13) and (9, 14) are more D-efficient for any value of $\rho \geq 0$ than the best available designs and robust under a fixed/mixed effects model. The designs (7, 9), (9, 13), (10, 12) and (13, 14) are more D-efficient for any value of $\rho \geq 0.1$ than the best available designs and robust under a mixed effects model. The designs marked with (&) in Appendix A.1 are robust with respect to D-efficiency under a mixed effect model for any value of $\rho \geq 0.1$. The designs (5, 6), (6, 7), (8, 10) and (9, 11) are more D-efficient for any value of $\rho \geq 0.4$ than the best available designs and have CV less than 5%. The designs (9, 15) and (13, 15) are more D-efficient for any value of $\rho \geq 0.7$ than the best available designs and have CV less than 5%.

Thirteen designs for the parameters $26 \leq v \leq 35$, $b = v$ and (v, b) = (11, 12), (12, 13) and (13, 16) were generated by the proposed method. It is interesting to note that D-efficiencies of all the 13 designs for the parameters and (v, b) = (11, 12), (12, 13) and (13, 16) are more than A-efficiencies for any value of $\rho \geq 0$. Further, lower bounds to both A- [D-] efficiencies are more than 0.9000 except for design with v = 13 and = 16. The CV of A- [D-] efficiencies of the 10 designs for the parameters $26 \leq v \leq 35$, $b = v$ is less than 5% for any value of $\rho \geq 0.4$. Therefore, these designs are robust under a mixed effects model for moderate to high values of $\rho \geq 0.4$. Further for 3 designs (v, b) = (11, 12), (12, 13) and (13, 16), CV of [D-] efficiencies is less than 5% for any value of $\rho \geq 0.4$. and the designs (11, 12) and (12,13) are more D-efficient for any value of $\rho \geq 0.4$ than the best available designs. The designs are given in Appendix A.3.

5. AN APPLICATION

An interesting application of row-column designs with two rows is in 2-colour microarray experiments conducted to study the expression level of many genes in a particular cell population or tissue (different types of tissues, drug treatments or time points of a biological process known as factor/treatment) simultaneously. Designing of 2-colour microarray experiments is an important issue to get precise comparisons of variety×gene interactions (see *e.g.* Gupta *et al.* 1999, Kerr *et al*. 2000, Kerr and Churchill 2001, Churchill 2002, Yang and Speed 2002 and Datta 2003). In a 2-colour microarray experiment, if we consider arrays as columns, dyes as rows and varieties as treatments, it can be designed using row-column designs in two rows. Further, in a 2-colour microarray experiment only two varieties can be accommodated on a single array. In that sense, arrays are incomplete and the varieties *vs* arrays classifications are non-orthogonal. In view of this, Kerr and Churchill (2001a), Wolfinger *et al.* (2001) and Lee (2004) emphasized that array effect should be taken as random and consequently the fixed effects model becomes a linear mixed effects model. In some literature on designs for 2-colour microarray experiments, it has been assumed that dye effects are orthogonal to variety effects and efforts have been made to obtain efficient designs under a block design set up (see *e.g.* Sarkar *et al.* 2007). There, however, may be situations where row effects may not be orthogonal to variety effects and the block design may not remain efficient when row effects are also included in the model. Inclusion of row effects in the model renders the design to be a row-column design. Sarkar *et al*. (2010) obtained efficient row – column designs under a fixed effects model and studied the robustness of these designs against different values of ρ under a mixed effects model. The designs were obtained through a computer aided search using an exchange–interchange of treatments algorithm. The designs obtained in the present investigation would, therefore, be helpful in designing 2-colour single factor microarray experiments.

6. DISCUSSION

Several designs obtained by the proposed method of construction, as described in Section 4, were found to have more A- and D-efficiency, more A-efficiency alone or D-efficiency alone than the best available designs in the literature. It is indeed possible to improve the A- [D-] efficiency of the designs obtained by the proposed method by rearrangement of row and column contents. Through such rearrangement done heuristically, it has been possible to obtain 16 more designs with parameters (v, b) = (7, 14), (7, 15), (7, 16), (7, 17), (7, 18), (8, 14), (8, 15), (8, 17), (8, 18), (8, 19), (8, 21), (8, 23), (10, 37), (10, 40), (10, 43), (10, 44), which have same efficiencies as that of best available design.

It was also found that the design with parameters (8, 20) obtained through rearrangement has more A- and D-efficiency for fixed/ mixed effects model and is robust under a mixed effects model.

For the experimental situation in which $\rho \geq 0.4$, after the arrangement it was found that the design (7, 13) was more A- and D-efficient than the best available design and is robust for $\rho \geq 0.4$ and the designs (8, 16), (8, 22) were more A- and D-efficient than the best available designs and were robust for $\rho \geq 0.7$.

After comparing designs for all parametric combinations a catalogue of 152 most efficient designs has been prepared. 46 designs have equal A- and D-efficiencies for all values of $\rho \geq 0$, 3 designs are universally optimal under fixed/mixed effects model, 13 new designs have high A- and D-efficiencies as given in Appendix A.2. A design with parameters (8, 20) is more A-and D-efficient than the best available design for any value of $\rho \geq 0$ and is robust under fixed/mixed effects model. The design (10, 35) is more A-efficient than the best available design for any value of $\rho \geq 0$ and is robust for A-efficiency under a fixed/mixed effects model. The design (6, 8) is more A- and D-efficient than the best available design for any value of $\rho \geq 0.1$ and is robust under a mixed effects model. 12 designs are more A- and D-efficient than the best available designs for any value of $\rho \geq 0.4$ and have CV less than 5%. 10 designs are more A- and D-efficient than the best available design for any value of $\rho \geq 0.7$ and have CV less than 5%.

Web application has been developed using C# programming language with ASP.NET platform and made available at Design Resources Server (www.iasri.res.in/drs). Through this web application, one can generate any of the 152 most efficient row-column designs in two rows along with lower bounds to A- and D-efficiencies for the parametric range, $3 \leq v \leq 10$, $v \leq b \leq v(v-1)/2$, $11 \leq v \leq 35$, $b = v$ and (v, b) = (11, 12), (11, 13), (12, 13), (12, 14), (13, 14), (13, 15), (13, 16).

The proposed method is confined to generate A- [D-] efficient row-column designs with two rows. These designs have application in two-colour microarray experiments besides many other applications. A natural question that comes to mind is as to if this procedure of generating designs can be extended with suitable modifications to generate row-column designs with three or more rows. Intuitively, it may appear possible, but the problem will altogether be different. It would be worthwhile examining the problem of generating row-column designs with three or more rows by extending the proposed method with suitable modifications.

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REFERENCES

Churchill, G. (2002). Fundamentals of experimental design for cDNA microarrays. *Nature Rev. Genet.*, **32**, 490-495.

Datta, S. (2003). Statistical techniques for microarray data: a partial overview. *Comm. Statist.: Theory Methods*, **32**, 263-280.

Gupta, P.K., Roy, J.K. and Prasad, M. (1999). DNA chips, microarray and genomics. *Current Sci.*, **77(7)**, 875-884.

Kerr, M.K. (2003). Design considerations for efficient and effective microarray studies. *Biometrics*, **59**, 822-828.

Kerr, M.K. and Churchill, G. A. (2001a). Experimental design for gene expression microarrays. *Biostatistics*, **2**, 183-201.

Kerr, M.K. and Churchill, G. A. (2001b). Statistical design and the analysis of gene expression. *Genet. Res.*, **77**, 123-128.

Kerr, M.K., Martin, M. and Churchill, G.A. (2000). Analysis of variance for gene expression microarray data. *J. Comput. Biol.*, **7**, 819-837.

Lee, M. -L.T. (2004). *Analysis of Microarray Gene Expression Data*. Kluwer Academic Publishers, USA.

Nguyen, N.K. and Williams, E.R. (2005). Experimental designs for 2-colour cDNA microarray experiments. *Proc. of the 4th International Symposium on Business and Industrial Statistics*, Palm Cove Resort (Tropical North Queensland).

Sarkar, A. and Parsad, R. (2007). Block designs for 2-colour micro-array experiments - An overview. *Pusa AgriSci.,* 38-54.

Sarkar, A., Parsad, R., Rathore, A. and Gupta, V.K. (2007). Efficient block designs for microarray experiments. *J. Ind. Soc. Agril. Statist.*, **61**(**2**), 227-248.

Sarkar, A., Parsad, R., Gupta, V.K., Chatterjee, K. and Rathore, A. (2010). Efficient row-column designs for microarray experiments. *J. Ind. Soc. Agril. Statist.*, **64**(**1**), 89-117.

Shah, K.R. and Sinha, B.K. (1989). *Theory of Optimal Designs*. Lecture Notes in Statistics: Springer-Verlag, New York, **54,** 171.

Wit, E., Nobile, A. and Khanin, R. (2005). Near optimal designs for dual channel microarray studies. *J. Roy. Statist. Soc.,* **C54**(**5**), 817-830.

Wolfinger, R.D., Gibson, G., Wolfinger, E.D., Bennett, L., Hamadeh, H., Busherl, P., Afshari, C. and Paules, R. (2001). Assessing gene significance from cDNA microarray expression data via mixed models. *J. Comput. Biol.*, **8**, 625-637.

Yang, X. (2003). *Optimal design of single factor cDNA microarray experiments and mixed models for gene expression data*. Ph. D. Thesis, Virginia Polytechnic Institute and State University.

Yang, Y.H. and Speed, T. (2002). Design issues for cDNA microarray experiments. *Nature Rev. Genet.*, **3**, 579-588.

http://www.iasri.res.in/drs

http://bio.cse.ohio-state.edu/Microarray Designer

Appendix A.1

Row-column designs that have more A- and D-Efficiency than the best designs existing in the literature

Range of ρ	**Designs for (v, b) that are more efficient and having CV less than 5% (More efficient and Robust)**	**Designs for (v, b) that are more efficient than the best available designs and have CV more than 5% (More efficient and non-robust)**
$\rho \geq 0.1$	(6,8)	(11, 11), (12, 12), (13, 13), (13, 14), (14, 14), (15, 15), (16, 16), (17, 17), (18, 18), (19, 19), (20, 20), (21, 21), (22, 22), (23, 23), (24, 24), (25, 25)
$\rho \geq 0.4$	(6, 10), (7, 9)**, (7, 11), (8, 9), (8, 13), (9, 14), (9, 15), (9, 24), (10, 13), (10, 34), (10, 35)	(9, 13)*$^{\$}$, (10, 14)*$^{\&}$, (10, 15)*$^{\$\&}$
$\rho \geq 0.7$	(5, 6), (6, 7), (6, 8), (7, 8), (8, 9), (9, 10), (9, 12), (10, 11)	(13,15)*

$^{\$}$ indicates that the design is robust with respect to A-efficiency alone for moderate to high values of $\rho \geq 0.4$ and $^{\&}$ indicates that the design is robust with respect to D-efficiency alone for any value of $\rho \geq 0.1$

Appendix A.2

Row-Column Designs for Microarray Experiments for $11 \leq v \leq 25$ and $(v, b) = (13, 14)$

Sl.No.	v	b	Eff	ρ = 0	ρ = 0.1	ρ = 0.2	ρ = 0.3	ρ = 0.4	ρ = 0.5	ρ = 0.6	ρ = 0.7	ρ = 0.8	ρ = 0.9	CV(Eff) (ρ = 0.1–0.9)	CV(Eff) (ρ = 0.4–0.9)	CV(Eff) (ρ = 0.7–0.9)
1	**11**	**11**	**AEff**	**0.4545**	**0.6767**	**0.7973**	**0.8716**	**0.9198**	**0.9518**	**0.9729**	**0.9865**	**0.9946**	**0.9988**	**11.3329**	**2.8410**	**0.5139**
			AEff	0.5025	0.5369	0.5574	0.5712	0.5810	0.5882	0.5937	0.5980	0.6012	0.6037	3.6372	1.3108	0.3882
			DEff	**0.7343**	**0.8427**	**0.9008**	**0.9368**	**0.9604**	**0.9761**	**0.9866**	**0.9933**	**0.9973**	**0.9994**	**5.2459**	**1.3819**	**0.2539**
			DEff	0.5998	0.6186	0.6308	0.6391	0.6451	0.6494	0.6526	0.6548	0.6564	0.6575	1.9289	0.6554	0.1689
2	**12**	**12**	**AEff**	**0.4231**	**0.6654**	**0.7917**	**0.8685**	**0.9181**	**0.9509**	**0.9725**	**0.9863**	**0.9945**	**0.9988**	**11.7511**	**2.9031**	**0.5221**
			AEff	0.5000	0.5321	0.5509	0.5634	0.5723	0.5789	0.5838	0.5877	0.5906	0.5929	3.3573	1.2072	0.3604
			DEff	**0.7201**	**0.8374**	**0.8980**	**0.9353**	**0.9595**	**0.9757**	**0.9863**	**0.9932**	**0.9973**	**0.9994**	**5.4263**	**1.4137**	**0.2583**
			DEff	0.5897	0.6071	0.6183	0.6260	0.6314	0.6353	0.6382	0.6402	0.6417	0.6427	1.8014	0.6106	0.1601
3	**13**	**13**	**AEff**	**0.3956**	**0.6562**	**0.7872**	**0.8661**	**0.9167**	**0.9501**	**0.9721**	**0.9861**	**0.9945**	**0.9988**	**12.0916**	**2.9559**	**0.5311**
			AEff	0.4954	0.5123	0.5237	0.5321	0.5384	0.5433	0.5472	0.5503	0.5528	0.5548	2.5076	1.0251	0.3331
			DEff	**0.7077**	**0.8330**	**0.8958**	**0.9340**	**0.9588**	**0.9753**	**0.9861**	**0.9931**	**0.9972**	**0.9994**	**5.5742**	**1.4379**	**0.2620**
			DEff	0.5543	0.5641	0.5709	0.5758	0.5795	0.5822	0.5843	0.5859	0.5870	0.5879	1.3142	0.4944	0.1393
4	**14**	**14**	**AEff**	**0.3714**	**0.6487**	**0.7835**	**0.8640**	**0.9156**	**0.9495**	**0.9717**	**0.9859**	**0.9944**	**0.9987**	**12.3695**	**2.9935**	**0.5356**
			AEff	0.4951	0.5109	0.5216	0.5293	0.5351	0.5397	0.5432	0.5461	0.5484	0.5503	2.3339	0.9542	0.3132
			DEff	**0.6968**	**0.8293**	**0.8939**	**0.9330**	**0.9583**	**0.9749**	**0.9859**	**0.9930**	**0.9972**	**0.9994**	**5.6992**	**1.4572**	**0.2664**
			DEff	0.5498	0.5589	0.5653	0.5698	0.5732	0.5758	0.5777	0.5792	0.5802	0.5811	1.2361	0.4673	0.1338
5	**15**	**15**	**AEff**	**0.3500**	**0.6423**	**0.7804**	**0.8623**	**0.9146**	**0.9490**	**0.9715**	**0.9858**	**0.9944**	**0.9987**	**12.6092**	**3.0308**	**0.5401**
			AEff	0.4949	0.5098	0.5197	0.5269	0.5324	0.5366	0.5399	0.5426	0.5447	0.5464	2.1848	0.8874	0.2854
			DEff	**0.6871**	**0.8261**	**0.8923**	**0.9321**	**0.9578**	**0.9747**	**0.9858**	**0.9929**	**0.9972**	**0.9994**	**5.8082**	**1.4743**	**0.2709**
			DEff	0.5460	0.5545	0.5605	0.5647	0.5679	0.5703	0.5721	0.5734	0.5744	0.5752	1.1637	0.4362	0.1282
6	**16**	**16**	**AEff**	**0.3309**	**0.6370**	**0.7777**	**0.8608**	**0.9138**	**0.9485**	**0.9712**	**0.9857**	**0.9943**	**0.9987**	**12.8086**	**3.0602**	**0.5437**
			AEff	0.4949	0.5088	0.5182	0.5249	0.5300	0.5339	0.5370	0.5395	0.5415	0.5431	2.0521	0.8349	0.2720
			DEff	**0.6784**	**0.8234**	**0.8909**	**0.9313**	**0.9573**	**0.9744**	**0.9857**	**0.9929**	**0.9972**	**0.9994**	**5.9011**	**1.4935**	**0.2709**
			DEff	0.5427	0.5507	0.5563	0.5603	0.5633	0.5655	0.5672	0.5684	0.5694	0.5701	1.1004	0.4116	0.1225
7	**17**	**17**	**AEff**	**0.3137**	**0.6324**	**0.7754**	**0.8596**	**0.9131**	**0.9481**	**0.9710**	**0.9856**	**0.9943**	**0.9987**	**12.9821**	**3.0867**	**0.5482**
			AEff	0.4948	0.5081	0.5169	0.5232	0.5280	0.5317	0.5346	0.5369	0.5388	0.5403	1.9349	0.7862	0.2583
			DEff	**0.6706**	**0.8210**	**0.8897**	**0.9307**	**0.9570**	**0.9742**	**0.9856**	**0.9928**	**0.9972**	**0.9994**	**5.9822**	**1.5043**	**0.2754**
			DEff	0.5399	0.5475	0.5527	0.5565	0.5593	0.5613	0.5629	0.5641	0.5650	0.5656	1.0366	0.3875	0.1091
`8	**18**	**18**	**AEff**	**0.2982**	**0.6285**	**0.7735**	**0.8585**	**0.9125**	**0.9478**	**0.9708**	**0.9855**	**0.9943**	**0.9987**	**13.1292**	**3.1087**	**0.5527**
			AEff	0.4949	0.5074	0.5157	0.5217	0.5262	0.5297	0.5324	0.5346	0.5363	0.5378	1.8312	0.7421	0.2438
			DEff	**0.6635**	**0.8190**	**0.8887**	**0.9301**	**0.9566**	**0.9741**	**0.9855**	**0.9928**	**0.9971**	**0.9994**	**6.0501**	**1.5166**	**0.2745**
			DEff	0.5374	0.5446	0.5495	0.5531	0.5557	0.5577	0.5592	0.5603	0.5611	0.5617	0.9866	0.3688	0.1022
9	**19**	**19**	**AEff**	**0.2842**	**0.6250**	**0.7717**	**0.8575**	**0.9119**	**0.9475**	**0.9707**	**0.9854**	**0.9942**	**0.9987**	**13.2623**	**3.1293**	**0.5564**
			AEff	0.4949	0.5068	0.5147	0.5204	0.5246	0.5279	0.5305	0.5325	0.5342	0.5355	1.7357	0.7025	0.2300
			DEff	**0.6570**	**0.8171**	**0.8877**	**0.9296**	**0.9563**	**0.9739**	**0.9854**	**0.9927**	**0.9971**	**0.9994**	**6.1147**	**1.5273**	**0.2790**
			DEff	0.5352	0.5420	0.5467	0.5501	0.5526	0.5544	0.5558	0.5569	0.5577	0.5582	0.9394	0.3497	0.0960

10	**20**	**20**	**AEff**	**0.2714**	**0.6220**	**0.7702**	**0.8567**	**0.9115**	**0.9472**	**0.9705**	**0.9854**	**0.9942**	**0.9987**	**13.3762**	**3.1458**	**0.5564**
			AEff	0.4950	0.5063	0.5139	0.5192	0.5232	0.5263	0.5288	0.5307	0.5323	0.5336	1.6508	0.6716	0.2229
			DEff	**0.6511**	**0.8155**	**0.8869**	**0.9291**	**0.9561**	**0.9738**	**0.9853**	**0.9927**	**0.9971**	**0.9993**	**6.1689**	**1.5330**	**0.2754**
			DEff	0.5332	0.5397	0.5442	0.5474	0.5498	0.5515	0.5528	0.5538	0.5546	0.5551	0.8952	0.3311	0.0966
11	**21**	**21**	**AEff**	**0.2597**	**0.6193**	**0.7688**	**0.8559**	**0.9110**	**0.9470**	**0.9704**	**0.9853**	**0.9942**	**0.9987**	**13.4801**	**3.1635**	**0.5609**
			AEff	0.4950	0.5059	0.5131	0.5182	0.5220	0.5249	0.5273	0.5291	0.5306	0.5318	1.5718	0.6363	0.2082
			DEff	**0.6457**	**0.8141**	**0.8862**	**0.9287**	**0.9559**	**0.9736**	**0.9853**	**0.9927**	**0.9971**	**0.9993**	**6.2172**	**1.5417**	**0.2754**
			DEff	0.5315	0.5377	0.5419	0.5450	0.5472	0.5489	0.5502	0.5511	0.5518	0.5524	0.8575	0.3223	0.0963
12	**22**	**22**	**AEff**	**0.2490**	**0.6169**	**0.7676**	**0.8552**	**0.9106**	**0.9468**	**0.9703**	**0.9853**	**0.9942**	**0.9987**	**13.5723**	**3.1788**	**0.5609**
			AEff	0.4951	0.5055	0.5124	0.5172	0.5209	0.5237	0.5259	0.5277	0.5291	0.5302	1.5043	0.6063	0.1934
			DEff	**0.6406**	**0.8127**	**0.8855**	**0.9284**	**0.9557**	**0.9735**	**0.9852**	**0.9927**	**0.9971**	**0.9993**	**6.2647**	**1.5491**	**0.2754**
			DEff	0.5299	0.5358	0.5399	0.5428	0.5450	0.5466	0.5478	0.5487	0.5493	0.5498	0.8229	0.3010	0.0819
13	**23**	**23**	**AEff**	**0.2391**	**0.6148**	**0.7665**	**0.8546**	**0.9103**	**0.9466**	**0.9702**	**0.9852**	**0.9941**	**0.9987**	**13.6520**	**3.1885**	**0.5645**
			AEff	0.4952	0.5052	0.5117	0.5164	0.5199	0.5226	0.5247	0.5263	0.5277	0.5288	1.4395	0.5782	0.1939
			DEff	**0.6360**	**0.8116**	**0.8849**	**0.9280**	**0.9555**	**0.9734**	**0.9852**	**0.9926**	**0.9971**	**0.9993**	**6.3031**	**1.5557**	**0.2799**
			DEff	0.5285	0.5342	0.5381	0.5408	0.5429	0.5444	0.5456	0.5464	0.5471	0.5475	0.7853	0.2918	0.0831
14	**24**	**24**	**AEff**	**0.2300**	**0.6128**	**0.7655**	**0.8540**	**0.9100**	**0.9464**	**0.9701**	**0.9852**	**0.9941**	**0.9987**	**13.7289**	**3.2006**	**0.5645**
			AEff	0.4953	0.5049	0.5112	0.5156	0.5190	0.5215	0.5235	0.5251	0.5264	0.5275	1.3770	0.5541	0.1864
			DEff	**0.6317**	**0.8105**	**0.8843**	**0.9277**	**0.9553**	**0.9733**	**0.9851**	**0.9926**	**0.9971**	**0.9993**	**6.3412**	**1.5631**	**0.2799**
			DEff	0.5272	0.5327	0.5364	0.5390	0.5410	0.5425	0.5436	0.5444	0.5450	0.5454	0.7541	0.2789	0.0754
15	**25**	**25**	**AEff**	**0.2215**	**0.6111**	**0.7646**	**0.8535**	**0.9097**	**0.9462**	**0.9700**	**0.9851**	**0.9941**	**0.9987**	**13.7941**	**3.2118**	**0.5690**
			AEff	0.4954	0.5046	0.5107	0.5149	0.5181	0.5206	0.5225	0.5241	0.5253	0.5263	1.3267	0.5365	0.1712
			DEff	**0.6277**	**0.8095**	**0.8838**	**0.9274**	**0.9551**	**0.9732**	**0.9851**	**0.9926**	**0.9971**	**0.9993**	**6.3760**	**1.5706**	**0.2799**
			DEff	0.5260	0.5313	0.5348	0.5374	0.5393	0.5407	0.5417	0.5425	0.5431	0.5435	0.7256	0.2668	0.0757
16	**13**	**14**	**AEff**	**0.4571**	**0.6761**	**0.7880**	**0.8570**	**0.9020**	**0.9319**	**0.9517**	**0.9645**	**0.9721**	**0.9761**	**10.7546**	**2.7234**	**0.4955**
			AEff	0.5256	0.5716	0.5980	0.6152	0.6270	0.6356	0.6418	0.6463	0.6496	0.6520	4.0693	1.3372	0.3599
			DEff	**0.7377**	**0.8381**	**0.8920**	**0.9257**	**0.9479**	**0.9627**	**0.9725**	**0.9789**	**0.9828**	**0.9848**	**4.9757**	**1.3252**	**0.2494**
			DEff	0.6467	0.6755	0.6935	0.7057	0.7141	0.7201	0.7243	0.7272	0.7292	0.7304	2.4660	0.7796	0.1811

Layout of the design (v, b) = (13, 14) is the following:

Column	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
Row 1	1	2	3	4	5	6	7	8	9	10	11	12	13	13
Row 2	2	3	4	5	6	7	8	9	10	11	12	13	1	6

Bold faced indicate the efficiencies of the designs obtained by proposed method of construction and regular face indicates the efficiencies of the corresponding best available Design

Appendix A.3

Efficient Row-Column Designs for $26 \leq v \leq 35$, $b = v$ and (v, b) = (11, 12), (12, 13), (13, 16)

Sl.No.	v	b	Eff	$\rho = 0$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$	CV(Eff) (ρ = 0.1–0.9)	CV(Eff) (ρ = 0.4-0.9)	CV(Eff) (ρ = 0.7–0.9)
1	26	26	AEff	0.2137	0.6095	0.7638	0.8531	0.9094	0.9461	0.9699	0.9851	0.9941	0.9987	13.8554	3.2227	0.5690
			DEff	0.6239	0.8086	0.8834	0.9272	0.9550	0.9732	0.9850	0.9926	0.9971	0.9993	6.4061	1.5736	0.2799
2	27	27	AEff	0.2063	0.6080	0.7630	0.8526	0.9092	0.9459	0.9699	0.9851	0.9941	0.9987	13.9146	3.2317	0.5690
			DEff	0.6204	0.8077	0.8829	0.9270	0.9549	0.9731	0.9850	0.9926	0.9971	0.9993	6.4377	1.5780	0.2799
3	28	28	AEff	0.1995	0.6066	0.7623	0.8522	0.9090	0.9458	0.9698	0.9850	0.9941	0.9987	13.9677	3.2384	0.5735
			DEff	0.6171	0.8070	0.8826	0.9267	0.9548	0.9730	0.9850	0.9925	0.9970	0.9993	6.4600	1.5799	0.2835
4	29	29	AEff	0.1931	0.6054	0.7616	0.8519	0.9087	0.9457	0.9698	0.9850	0.9941	0.9987	14.0156	3.2493	0.5735
			DEff	0.6140	0.8062	0.8822	0.9265	0.9546	0.9730	0.9849	0.9925	0.9970	0.9993	6.4874	1.5862	0.2835
5	30	30	AEff	0.1871	0.6042	0.7610	0.8515	0.9086	0.9456	0.9697	0.9850	0.9941	0.9987	14.0621	3.2537	0.5735
			DEff	0.6111	0.8056	0.8819	0.9263	0.9545	0.9729	0.9849	0.9925	0.9970	0.9993	6.5082	1.5905	0.2835
6	31	31	AEff	0.1815	0.6032	0.7605	0.8512	0.9084	0.9455	0.9696	0.9850	0.9941	0.9987	14.1004	3.2614	0.5735
			DEff	0.6084	0.8050	0.8815	0.9262	0.9544	0.9729	0.9849	0.9925	0.9970	0.9993	6.5303	1.5937	0.2835
7	32	32	AEff	0.1761	0.6022	0.7600	0.8509	0.9082	0.9454	0.9696	0.9849	0.9940	0.9987	14.1378	3.2667	0.5772
			DEff	0.6057	0.8044	0.8812	0.9260	0.9544	0.9728	0.9848	0.9925	0.9970	0.9993	6.5506	1.5947	0.2835
8	33	33	AEff	0.1711	0.6013	0.7595	0.8507	0.9081	0.9453	0.9696	0.9849	0.9940	0.9987	14.1733	3.2712	0.5772
			DEff	0.6033	0.8038	0.8810	0.9259	0.9543	0.9728	0.9848	0.9925	0.9970	0.9993	6.5701	1.5979	0.2835
9	34	34	AEff	0.1664	0.6004	0.7590	0.8504	0.9079	0.9452	0.9695	0.9849	0.9940	0.9987	14.2087	3.2788	0.5772
			DEff	0.6009	0.8033	0.8807	0.9257	0.9542	0.9727	0.9848	0.9925	0.9970	0.9993	6.5882	1.6023	0.2835
10	35	35	AEff	0.1619	0.5996	0.7586	0.8502	0.9078	0.9452	0.9695	0.9849	0.9940	0.9987	14.2400	3.2821	0.5772
			DEff	0.5987	0.8029	0.8805	0.9256	0.9541	0.9727	0.9848	0.9925	0.9970	0.9993	6.6022	1.6054	0.2835
11	11	12	AEff	0.5147	0.6979	0.7980	0.8610	0.9027	0.9307	0.9493	0.9614	0.9688	0.9726	16.4747	9.8379	2.5740
			DEff	0.7629	0.8478	0.8960	0.9268	0.9473	0.9611	0.9704	0.9765	0.9801	0.9821	7.3231	1.5836	1.2923
12	12	13	AEff	0.4853	0.6864	0.7926	0.8588	0.9023	0.9313	0.9506	0.9630	0.9706	0.9745	17.6314	10.3239	2.6558
			DEff	0.7499	0.8426	0.8938	0.9262	0.9476	0.9620	0.9716	0.9778	0.9816	0.9835	7.7957	4.7852	1.2916
13	13	16	AEff	0.5381	0.7036	0.7958	0.8539	0.8921	0.9175	0.9344	0.9452	0.9518	0.9551	15.1600	9.1161	2.3592
			DEff	0.7758	0.8512	0.8947	0.9226	0.9411	0.9535	0.9619	0.9673	0.9705	0.9722	6.6049	4.1395	1.1306

Layout of the design (v, b) = (11, 12)

Column	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Row 1	1	2	3	4	5	6	7	8	9	10	11	11
Row 2	2	3	4	5	6	7	8	9	10	11	1	5

Laqyout of the design (v, b) = (12, 13)

Column	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Row 1	1	2	3	4	5	6	7	8	9	10	11	12	12
Row 2	2	3	4	5	6	7	8	9	10	11	12	1	6

Laqyout of the design (v, b) = (13, 16)

Column	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
Row 1	1	2	3	4	5	6	7	8	9	10	11	12	13	13	5	11
Row 2	2	3	4	5	6	7	8	9	10	11	12	13	1	6	12	4