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Optimal Partial Triallel Crosses

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SUMMARY

A method of construction of block designs for partial triallel crosses for $p > 3$ lines is proposed by using the mutually orthogonal Latin squares of order p, where p is a prime or power of a prime. Optimality of these designs is discussed by using the approach of Das and Gupta (1997).

Keywords: Latin square, Mutually orthogonal Latin square, Partial triallel cross, Mating design.

1. INTRODUCTION

Two Latin squares of the same order are said to be orthogonal to each other if, when they are superimposed on one another, every ordered pair of symbol occurs once and only once in the composite Latin square. A set of $p - 1$ Latin squares of order p is called mutually orthogonal Latin squares (MOLS), if they are pair-wise orthogonal. Orthogonal Latin squares are used for the construction of balanced incomplete block designs, square lattice designs, orthogonal arrays and quasi factorial designs. A set of $p - 1$ orthogonal Latin squares of side p can always be constructed if p is a prime or power of a prime. If $p = 4t + 2$, $t > 1$, then there exist more than one mutually orthogonal Latin squares of order p [Bose *et al.* (1960)]. An exhaustive list of these squares is available in Fisher and Yates (1973) and most extensive treatment of Latin squares can also be found in D'enes and Keedwell (1991). In this article, we are also using the MOLS in construction of mating designs for partial triallel cross experiments.

Triallel crosses form an important class of mating designs, which are used for studying the genetic properties of a set of inbred lines in plant breeding experiments. For p inbred lines, the number of different crosses for a complete triallel experiment is $3\ {}^pC_3 = p\,(p-1)\,(p-2)/2$ of the type $(i \times j) \times k$, $i \neq j \neq k = 0, 1, 2, \ldots, p-1$. Rawlings and Cockerham (1962) were the first to introduce mating designs for triallel crosses.

Triallel cross experiments are generally conducted using a completely randomized design (CRD) or a randomized complete block (RCB) design as environmental design involving $3\ {}^pC_3$ crosses. Even with a moderate number of parents, say $p = 10$, in a triallel cross experiment; the number of crosses becomes unmanageable to be accommodated in homogeneous blocks. For such situations, Hinkelmann (1965) developed partial triallel crosses (PTC) involving only a sample of all possible crosses by establishing a correspondence between PTC and generalized partially balanced incomplete block designs (GPBIBD). Ponnuswamy and Srinivasan (1991) and Subbarayan (1992) obtained PTC using a class of balanced incomplete block (BIB) designs. Dharmlingum (2002) also constructed PTC using the Trojan squares. Actually Trojan squares are MOLS. Our method of construction of PTC is based on Rao (1956).

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Other research workers who contributed in this area are Arora and Aggarwal (1984, 1989), Ceranka *et al.* (1990). More details on triallel cross experiments can be found in Hinkelmann (1975) and Narain (1990).

Following Gupta and Kageyama (1994) and Dey and Midha (1996), Das and Gupta (1997) constructed block designs for triallel crosses by using the nested balanced block design with parameters $v = p$, b_1, b_2, k_1, $k_2 = 3$. Their method yields designs which are universally optimal in $\boldsymbol{D}(p, b, k)$, the class of connected block designs for triallel crosses in p lines with b blocks each of size k such that the total number of experimental units are $< 3\ {}^pC_3$. For optimality of two lines and four lines crosses see Parsad *et al.* (2005).

In this paper, we are proposing a method of construction of block designs for triallel cross experiments by using the mutually orthogonal Latin squares. These designs are found to be optimal in the sense of Das and Gupta (1997). The paper is structured as: in section 2 we gave some definitions and in section 3 we discussed a method of construction of these designs. In section 4 we discussed the optimality of these designs.

2. SOME DEFINITIONS

1. **Definition:** The triallel cross (T.C.) has been defined by Rawlings and Cockerham (1962) as a set of all possible three-way hybrids among a group of (inbred) lines. Given three lines i, j and k, there are distinct triallel crosses, namely $(i\,j)\,k$, $(j\,k)\,i$ and $(i\,k)\,j$ involving these three lines.

   Thus given a set of p lines, the triallel cross will consist of a set of $[p\,(p-1)\,(p-2)/2]$ three way crosses.

2. Hinkelmann (1965) proposed the definition of PTC as given below:

   Suppose we have p lines which are denoted by $i = 1, 2, \ldots, p$. A three way cross is then represented by a triplet $(i\,j)\,k$, where $(i\,j)$ stands for an offspring of the single cross $i \times j$. We shall call i and j half-parents and k full-parent. The crosses $(i\,j)\,k$, $(j\,i)\,k$, $k\,(i\,j)$, and $k\,(j\,i)$ are considered to be identical in three way crosses. Then PTC can be defined as follows:

   A set of matings is said to be a PTC if it satisfies the following conditions:

   (i) Each line occurs exactly r_H times as half-parent and r_F times as full parent.

   (ii) Each cross $(i\,j)\,k$ occurs either once or not at all.

   The total number of crosses is $p\,r_F$ and $r_H = 2\,r_F$. Let $r_F = r$, whence $r_H = 2r$.

3. CONSTRUCTION OF PARTIAL TRIALLEL CROSSES

It is known that when p is prime, it is possible to construct $(p-1)$ orthogonal Latin squares in such a way that they differ only in a cyclical interchange of the rows from 2nd to p^{th}. Such squares are taken for the construction of incomplete block designs for partial triallel crosses. For $p = 6$, such squares cannot be constructed.

Assume that there are p inbred lines and it is desired to find an incomplete block design for a mating design for partial triallel crosses. Out of $(p-1)$ MOLS, consider any two MOLS and superimpose one square over the other square. We obtain a composite Latin square in which each cell contains ordered pair of integers (i, j) taking values from 0 to $p-1$. These ordered pairs of integers occur once in the composite square. Border the columns of the composite Latin square with integers from 0 to $p-1$ in the same order as they occur in the first row. Omitting the first row we now have, including the bordering elements, $p\,(p-1)$ ordered pair of integers corresponding to the $p\,(p-1)$ cells. To the ordered pair (i, j) corresponding to each cell, we attach border elements corresponding to their columns. This provides a mating design for partial triallel cross. If we consider the rows of these mating designs as blocks, we get two types of block designs for triallel cross with parameters $v = p\,(p-1)$, $b = p-1$, $k = p$, $r = 1$ and $v = p\,(p-1)/2$, $b = p-1$, $k = p$, $r = 2$.

Since we have $(p-1)$ MOLS and taking any two at a time, we obtain $(p-1)\,(p-2)/2$ composite Latin squares. Using the procedure described above we get $(p-1)\,(p-2)/2$ mating designs for partial triallel cross which can be classified in two types as (i) $(p-1)$ $(p-3)/2$ mating designs as d_1 in which all triallel crosses are repeated once. (ii) $(p-1)/2$ mating designs as d_2 in which all triallel crosses repeated two times. The degree of fractionation for both d_1 and d_2 is $2/(p-2)$ and $1/(p-2)$, respectively.

The method of construction is illustrated below.

Example: Let us consider the mating design for partial triallel cross experiment for $p = 5$ parents. For $p = 5$, we have four mutually orthogonal Latin square of order 5 as given below:

L_1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

L_2

0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2

L_3

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

L_4

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

Consider any two mutually orthogonal Latin squares and superimpose any one over the other. Remove first row in each composite Latin square, and then attach border elements in each cell corresponding to their column. We get following six mating designs for partial triallel cross. If we consider the rows of these mating designs as blocks, we get two types of block designs **D**(v, b, k) for triallel cross with parameters $v = 20, b = 4, k = 5, r = 1$ and $v = 10, b = 4, k = 5, r = 2$.

I

1. (1 × 2) 0 (2 × 3) 1 (3 × 4) 2 (4 × 0) 3 (0 × 1) 4
2. (2 × 4) 0 (3 × 0) 1 (4 × 1) 2 (0 × 2) 3 (1 × 3) 4
3. (3 × 1) 0 (4 × 2) 1 (0 × 3) 2 (1 × 4) 3 (2 × 0) 4
4. (4 × 3) 0 (0 × 4) 1 (1 × 0) 2 (2 × 1) 3 (3 × 2) 4

II

1. (2 × 3) 0 (3 × 4) 1 (4 × 0) 2 (0 × 1) 3 (1 × 2) 4
2. (4 × 1) 0 (0 × 2) 1 (1 × 3) 2 (2 × 4) 3 (3 × 0) 4
3. (1 × 4) 0 (2 × 0) 1 (3 × 1) 2 (4 × 2) 3 (0 × 3) 4
4. (3 × 2) 0 (4 × 3) 1 (0 × 4) 2 (1 × 0) 3 (2 × 1) 4

III

1. (1 × 3) 0 (2 × 4) 1 (3 × 0) 2 (4 × 1) 3 (0 × 2) 4
2. (2 × 1) 0 (3 × 2) 1 (4 × 3) 2 (0 × 4) 3 (1 × 0) 4
3. (3 × 4) 0 (4 × 0) 1 (0 × 1) 2 (1 × 2) 3 (2 × 3) 4
4. (4 × 2) 0 (0 × 3) 1 (1 × 4) 2 (2 × 0) 3 (3 × 1) 4

IV

1. (1 × 4) 0 (2 × 0) 1 (3 × 1) 2 (4 × 2) 3 (0 × 3) 4
2. (2 × 3) 0 (3 × 4) 1 (4 × 0) 2 (0 × 1) 3 (1 × 2) 4
3. (3 × 2) 0 (4 × 3) 1 (0 × 4) 2 (1 × 0) 3 (2 × 1) 4
4. (4 × 1) 0 (0 × 2) 1 (1 × 3) 2 (2 × 4) 3 (3 × 0) 4

V

1. (2 × 4) 0 (3 × 0) 1 (4 × 1) 2 (0 × 2) 3 (1 × 3) 4
2. (4 × 3) 0 (0 × 4) 1 (1 × 0) 2 (2 × 1) 3 (3 × 2) 4
3. (1 × 2) 0 (2 × 3) 1 (3 × 4) 2 (4 × 0) 3 (0 × 1) 4
4. (3 × 1) 0 (4 × 2) 1 (0 × 3) 2 (1 × 4) 3 (2 × 0) 4

VI

1. (3 × 4) 0 (4 × 0) 1 (0 × 1) 2 (1 × 2) 3 (2 × 3) 4
2. (1 × 3) 0 (2 × 4) 1 (3 × 0) 2 (4 × 1) 3 (0 × 2) 4
3. (4 × 2) 0 (0 × 3) 1 (1 × 4) 2 (2 × 0) 3 (3 × 1) 4
4. (2 × 1) 0 (3 × 2) 1 (4 × 3) 2 (0 × 4) 3 (1 × 0) 4

By inspecting above mating designs we find that the mating designs II and IV are replica of each other and each triple cross is repeated twice in each mating design. The rest four mating designs are also replica of each other but each triple cross occurs only once in each mating design. Thus we get two types mating designs, designs I, III, V and VI belong to class d_1 and designs II and IV belong to class d_2. When $(p - 1)$ is even we get above two types of mating designs otherwise we will get only one type of design. The choice between d_1 and d_2 depends on the degree of fractionation of the triallel crosses.

Note: For given $(p - 1)$ MOLS, if we superimpose three MOLS at a time and omitting the first row, we obtain a mating design for partial triallel cross. So in this way we get $[(p - 1)(p - 2)(p - 3)/6]$ mating designs. Now exchanging the position of superimposing squares, for example (L_1, L_2: L_3) we may get another two mating designs (L_1, L_3: L_2) and (L_2, L_3: L_1). With the result we get two types of mating designs containing $3p(p-1)$ and $p(p-1)/2$ distinct crosses, respectively, satisfying the property of triallel crosses. Thus in total we get $[(p-1)(p-2)(p-3)/3]$ mating designs. The degree of fractionation for these designs are of order $6/(p-2)$ and $1/(p-2)$, respectively. Dharmlingam (2002) used this technique in the construction of PTC by the name of Trojan square design.

4. OPTIMALITY

For the analysis of data obtained from design d we will follow Das and Gupta (1997). Let r_t and s_i denote the number of replication of the t^{th} cross and the number of replications of the i^{th} line in different crosses, respectively, in d [$t = 1, 2, ..., p(p-1)(p-3)/3$; $i = 1, 2, ..., p$]. Evidently, $\Sigma r_t = bk$, $\Sigma s_i = 3bk$ and $n = bk$, the total number of observations. In a triallel cross experiment, the genotypic effect of the hybrid consists of single line effects, two line effects and three line specific effects. However, if we assume that for a partial triallel cross experiment (in which every line appears as half parent an equal number of times, say r_H, and every line appears as full parent an equal number of times, say r_F, and each of the crosses $(ij)k$ appears at most once) the two line specific effects and three line specific effects are not of importance, still the line effects are of two types viz effects as half parent and effect as full parent *i.e.*, the ordering of lines in a triallel cross is important. Some plant breeders argue that these ordering effects can also be averaged over line effects. Das and Gupta (1997) considered the situations where ordering of lines in a triallel cross is not of importance. We will take the following additive model for the observations obtained from design *d*.

$$\mathbf{y} = \mu \mathbf{1}_n + \Delta_1' \mathbf{g} + \Delta_2' \beta + \mathbf{e} \quad (3.1)$$

where $\mathbf{y}$ be $n \times 1$ vector of observations, $\mathbf{1}$ is the $n \times 1$ vector of ones, Δ_1' is the $n \times p$ design matrix for lines and Δ_2' is a $n \times b$ design matrix for blocks, that is, the $(h, l)^{th}$ element of Δ_1' (respectively, of Δ_2') is 1 if the h^{th} observation pertains to the l^{th} line (respectively, of block) and is zero otherwise. μ is a general mean, $\mathbf{g}$ is a $p \times 1$ vector of line parameters, β is a $b \times 1$ vector of block parameters and $\mathbf{e}$ is a $n \times 1$ vector of residuals. It is assumed that vector β is fixed and $\mathbf{e}$ is normally distributed with $E(\mathbf{e}) = \mathbf{0}$, $V(\mathbf{e}) = \sigma^2 \mathbf{I}$ and $Cov(\beta, \mathbf{e}) = \mathbf{0}$, where $\mathbf{I}$ is the identity matrix of conformable order.

The method of least squares for the analysis of proposed design d leads to the following reduced normal equations for estimating the linear functions of the general combining effects of lines under model (3.1).

$$\mathbf{C}_d = \mathbf{G}_d - \mathbf{N}_d \mathbf{K}_d^{-1} \mathbf{N}_d' = (c_{ij}) \quad (i, j = 1, 2, ..., p) \quad (3.2)$$

where $\mathbf{G}_d = \Delta_1 \Delta_1' = (g_{dii'})$, $g_{dii} = s_{di} = 3(p-1)$ and for $i \neq i'$, $g_{dii'}$ is the number of crosses in d in which the lines i and i' appear together. $\mathbf{N}_d = \Delta_1 \Delta_2' = (n_{dij})$, n_{dij} is the number of times the line i occurs in block j of d and $\mathbf{K}_d = \Delta_2 \Delta_2'$ is the diagonal matrix of block sizes.

Following Dey and Midha (1996) we have the following identities which are useful for obtaining the information matrix $\mathbf{C}_d$.

(i) $\sum_{l=1}^{b} n_{dil} = 3(p-1)$

(ii) $\sum_{l=1}^{b} n_{dil}^2 = 9(p-1)$

(iii) $\sum_{l=1, i \neq i'}^{b} n_{dil} n_{di'l} = 6$ (3.3)

where n_{dil} is the number of times ith line occurs in lth block in d and n_{dil} is the number of times ith and i'th lines occur in lth block in d. The proofs are all identies are easy. Therefore we are not producing here.

Using above identities (3.3), the information matrix $\mathbf{C}_d$ is given by

$$\mathbf{C}_d = \frac{1}{p} \begin{pmatrix} 3(p-1)(p-3) & 3(3-p) & \cdots & 3(3-p) \\ 3(3-p) & 3(p-1)(p-3) & \cdots & 3(3-p) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 3(3-p) & 3(3-p) & \cdots & 3(p-1)(p-3) \end{pmatrix} \quad (3.4)$$

A design d is said to be connected if and only if rank($\mathbf{C}_d$) = (p – 1) or equivalently, if all elementary contrast among *gca* effects are estimable using design d. A connected design d is variance balanced if and only if the information matrix $\mathbf{C}_d$ is completely symmetric *i.e.*, the matrix $\mathbf{C}_d$ has all the diagonal elements equal and all the off diagonal elements equal. Now we will prove that rank($\mathbf{C}_d$) = p – 1.

Theorem 1. For proposed design d, the rank of information matrix ($\mathbf{C}_d$) is equal to p – 1.

Proof: The information matrix $\mathbf{C}_d$ can be expressed as

$$\mathbf{C}_d = \theta\,(\mathbf{I}_p - p^{-1}\,\mathbf{1}_p\,\mathbf{1}'_p) = \theta\,\mathbf{A} \qquad (3.5)$$

Here $\theta = 3\,(p-3)$ and $\mathbf{A} = (\mathbf{I}_p - p^{-1}\,\mathbf{1}_p\,\mathbf{1}'_p)$, where $\mathbf{I}_p$ is the identity matrix of order p and $\mathbf{1}_p$ is a p-component vector of all 1's. Since $\mathbf{A}$ is an idempotent symmetric matrix of order p and has a rank equal to its trace. The trace of $\mathbf{A}$ is (p – 1). Hence the rank ($\mathbf{C}_d$) is also p – 1. This completes the proof.

Since rank ($\mathbf{C}_d$) = p – 1. Therefore the design d is connected and also variance balanced because the matrix $\mathbf{C}_d$ has all the diagonal elements equal and all the off diagonal elements equal. Now the trace of $\mathbf{C}_d$ is

$$\text{Trace}(\mathbf{C}_d) = 3\,(p-3)\,(p-1) \qquad (3.6)$$

The criterion for the optimality is the constancy of the variances for all pair wise comparisons of the lines together with the minimization of this variance. To show that the designs obtained by using the method of section 3 are universally optimal. We will use the following theorem given by Das and Gupta (1997). For given positive integers p, b, k, $\mathbf{D}(p, b, k)$ denotes the class of all connected block designs with p lines, b blocks and common block size.

Theorem 2. Let $d \in \mathbf{D}(p, b, k)$, be a block design for triallel crosses satisfying

(i) Trace ($\mathbf{C}_d$) $\le k^{-1} b\,\{3k\,(k-1-2x) + p\,x\,(x+1)\}$, and

(ii) $\mathbf{C}_d$ is completely symmetric

Then d is universally optimal in the relevant class of competing design in $\mathbf{D}(p, b, k)$ and in particular is A-optimal.

Now consider $d \in \mathbf{D}(p, b, k)$ constructed by using the mutually orthogonal Latin squares with parameters $v = p\,(p-1)$, $b = p-1$, $k = p$ and applying Theorem 2, we see that

$$\begin{aligned}\text{Trace}(\mathbf{C}_d) &= k^{-1} b\,\{3k\,(k-1-2x) + p\,x\,(x+1)\}\\ &= 3\,(p-1)\,(p-3),\end{aligned}$$

which is equal to the value given at (3.6)

Hence we state the following theorem.

Theorem 3. Let $d \in \mathbf{D}(p, b, k)$, be a block design for triallel crosses constructed by using the mutually orthogonal Latin square satisfying

(i) Trace($\mathbf{C}_d$) = 3 (p – 1) (p – 3)

(ii) $\mathbf{C}_d$ is completely symmetric.

(iii) $3\,k/p$ is an integer.

Then d is universally optimal in the relevant class of competing design in $\mathbf{D}(p, b, k)$ and particularly is A-optimal.

5. CONCLUSION

According to Das and Gupta (1997), the proposed designs belong to a class where $3\,k/p$ is an integer and the optimality result of d is relevant to this class.

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