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Hierarchical Bayes Small Area Estimation Approach for Spatial Data*

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SUMMARY

In this paper, spatial model for small area estimation described in Chandra *et al.* (2007) has been studied under Hierarchical Bayes (HB) framework and small area HB estimates for means have been obtained using Gibbs sampling. In this framework, small area mean is estimated by its posterior mean and posterior variance of the small area mean is used as a measure of precision of the estimate. Note that the proposed estimators automatically takes into account the extra uncertainty associated with the hyper-parameters in the model. The performance of the proposed estimators was evaluated via simulation studies. Our empirical results show that proposed method is efficient than the existing methods. Further, we examined three types of spatial weighting function to incorporate the spatial effects. In addition, we also did sensitivity analysis of the small area means estimators by the choice of different priors. Sensitivity analysis indicates that prior choice has impact on variance estimates.

Keywords: SAR model, Prior sensitivity, Hierarchical Bayes, Gibbs sampling.

1. INTRODUCTION

Small area estimation received a lot of attention in recent past years due to growing demand for reliable small area statistics. Mixed effect models or mixed models are particularly suitable for small area estimation because of its flexibility in effectively combining different sources of information and explaining different sources of errors. Mixed models typically incorporate area-specific random effects that explain additional variation i.e. between area variations in the data not explained by the fixed effects part of the model and called "small area models". Small area models consider the random area effects as independent.

Consider, basic unit level model given by Battese *et al*. (1988). Let y_{ij} denote the value of the variable of interest for the j^{th} (j = 1, 2, ..., N_i) unit in small area i (i = 1, 2, ..., m) and let $\mathbf{x}_{ij}$ denote the vector of values of the p unit level auxiliary variables associated with j^{th} unit of i^{th} small area. Now consider a nested error regression model of form

$$y_{ij} = \mathbf{x}_{ij}^T \boldsymbol{\beta} + v_i + e_{ij},\ j = 1, 2, ..., N_i; i = 1, 2, ..., m \quad (1.1)$$

where, $\boldsymbol{\beta}$ is a vector of p unknown fixed effects, v_i is the random area effect associated with small area i, assumed to have mean zero and variance σ_v^2 and e_{ij} is an individual unit level random error with mean zero and variance σ_e^2. The two error terms are assumed to be mutually independent, both across individuals as well as across areas. In addition, it is often assumed that they are normally distributed. In matrix notation, (1.1) is expressed as

$$\mathbf{Y}_i = \mathbf{X}_i\beta + v_i\mathbf{1}_{N_i} + \mathbf{e}_i,\ j = 1, 2, ..., N_i; \quad i = 1, 2, ..., m \quad (1.2)$$

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where $\mathbf{Y}_i = (y_{i1}, y_{i2}, ..., y_{iN_i})^T$ is $N_i \times 1$ vector of observations, $\mathbf{X}_i = (\mathbf{x}_{i1}, \mathbf{x}_{i2}, ..., \mathbf{x}_{iN_i})^T$ is a $N_i \times p$ matrix and $\mathbf{e}_i = (e_{i1}, e_{i2}, ..., e_{iN_i})^T$ is $N_i \times 1$ vector of error terms. Here, N_i is the number of population units in small area i.

It is assumed that, samples are drawn independently across small areas according to a specified sampling design. The sample data $\{y_{ij}, \mathbf{x}_{ij}, j = 1, 2, ..., n_i; i = 1, 2, ..., m\}$ are assumed to obey the population model, *i.e.*,

$$y_{ij} = \mathbf{x}_{ij}^T \beta + v_i + e_{ij},\ j = 1, 2, ..., n_i;\ i = 1, 2, ..., m \quad (1.3)$$

where n_i is the sample size in the i^{th} area. It implies that the sampling design is ignorable or selection bias is absent, which is true in the case of simple random sampling design within small areas. For more general designs, the sample indicator variable, should be unrelated to the y_{ij} conditional on $\mathbf{x}_{ij}$

In matrix notation, (1.3) is expressed as:

$$\mathbf{Y}_i = \mathbf{X}_i \beta + v_i \mathbf{1}_{n_i} + \mathbf{e}_i,\ j = 1, 2, ..., n_i;\ i = 1, 2, ..., m \quad (1.4)$$

The covariance matrix of $\mathbf{Y}_i$ is $\text{Var}(\mathbf{Y}_i) = \mathbf{V}_i = \sigma_e^2 \mathbf{I}_{n_i} + \sigma_v^2 \mathbf{1}_{n_i} \mathbf{1}_{n_i}^T$, which depends on the vector $\delta = (\sigma_v^2, \sigma_e^2)^T$ of variance components of the model. Here $\mathbf{1}_{n_i}$ is the unit vector of dimension n_i and $\mathbf{I}_{n_i}$ is the identity matrix of order $n_{i.}$ Assuming (1.4) holds, the population mean of Y in area i is, $\bar{Y}_i = \bar{\mathbf{X}}_i \beta + v_i + \bar{e}_{i'}$, where $\bar{\mathbf{X}}_i = N_i^{-1} \sum_{j=1}^{N_i} \mathbf{x}_{ij}$ is assumed to be known.

In this model, it is assumed that random area effects are independent. But in practice, it should be more reasonable to assume that the random effects between the neighbouring areas are correlated and the correlation decays to zero as distance increases. In this context, Petrucci *et al.* (2006) extended the Fay-Herriot model by incorporating spatial correlation between the random small area effects modelled through the Simultaneously Autoregressive (SAR) process. The best linear unbiased predictor under this model is called spatial best linear unbiased predictor (BLUP). Its empirical version (EBLUP) has been obtained and an estimator of its MSE has been proposed. Pratesi and Salvati (2008) explored the performance of this estimator with a Monte Carlo simulation study on lattice data. They showed that the introduction of spatially correlated random area effects reduce both the variance and the bias of the EBLUP estimator. Subsequently, Chandra *et al.* (2007) investigated small area estimation based on basic unit level models with spatially correlated small area effects where the neighbourhood structure is described by a contiguity matrix. They considered the linear regression model with spatial dependence in the error structure. In particular, they have considered a Simultaneously Autoregressive (SAR) error process, where the vector of random area effects $\mathbf{v} = (v_i)$ satisfies

$$\mathbf{v} = \rho \mathbf{W} \mathbf{v} + \mathbf{u} \quad (1.5)$$

Here, ρ is the spatial autoregressive coefficient, $\mathbf{W}$ is the spatial weight matrix of order m and $\mathbf{u} \sim N(\mathbf{0}, \sigma_u^2 \mathbf{I})$.

Since $\mathbf{v} = (\mathbf{I} - \rho \mathbf{W})^{-1} \mathbf{u}$ with $E(\mathbf{u}) = \mathbf{0}$ and $\text{Var}(\mathbf{u}) = \sigma_u^2 \mathbf{I}_m$. Thus $E(\mathbf{v}) = \mathbf{0}$ and $\text{Var}(\mathbf{v}) = \sigma_u^2 [\mathbf{I} - \rho \mathbf{W})\ (\mathbf{I} - \rho \mathbf{W})^{\mathrm{T}}]^{-1} = \mathbf{G}.$

The $\mathbf{W}$ matrix describes how random effects from neighbouring areas are related, where ρ defines the strength of this spatial relationship. The simplest way to define $\mathbf{W}$ matrix is as a contiguity matrix. That is, the elements of $\mathbf{W}$ take non-zero values only for those pairs of areas that are adjacent. Generally, for ease of interpretation, this matrix is defined in row standardized form, in which case ρ is called the spatial autocorrelation parameter. Formally, the element γ_{ik} of a contiguity matrix takes the value 1 if area i shares an edge with area k and 0 otherwise. In row- standardized form this becomes

$$\gamma_{ik} = r_i^{-1} \text{ if } i \text{ and } k \text{ are contiguous}$$
$$0 \text{ otherwise.}$$

where, r_i is the total number of areas that share an edge with area i (including area i itself). Under this situation, the linear mixed model can be written as:

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{Z}\mathbf{v} + \mathbf{e} \quad (1.6)$$

where $\mathbf{v}$ is an m-vector of spatially correlated area-effects that satisfy the SAR model (1.5), with $\text{Var}(\mathbf{e}) = \sigma_e^2 \mathbf{I}_N$ and $\text{Var}(\mathbf{v}) = \mathbf{G}$. This model can be written as

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{Z}(\mathbf{I} - \rho \mathbf{W})^{-1} \mathbf{u} + \mathbf{e} \quad (1.7)$$

It follows that the covariance matrix of $\mathbf{Y}$ is $\text{Var}(\mathbf{Y}) = \mathbf{V} = \sigma_e^2 \mathbf{I}_N + \mathbf{Z}\mathbf{G}\mathbf{Z}^T$. In practice, the vector of parameters $\theta = (\sigma_u^2, \sigma_e^2, \rho)^T$ is unknown. Replacing it with an asymptotically consistent estimator $\hat{\theta} = (\hat{\sigma}_u^2, \hat{\sigma}_e^2, \hat{\rho})^T$ and assuming that (1.7) holds, the spatial-EBLUP (SEBLUP) for the i^{th} small area mean $\bar{\mathbf{Y}}_i$ is

$$\hat{\bar{\mathbf{Y}}}_{i,\,\text{SEBLUP}} = f_i\, \bar{\mathbf{Y}}_{is} + (1 - f_i)(\bar{\mathbf{X}}_{ir}^T \hat{\beta}^s + \mathbf{t}_i^T \hat{\mathbf{v}}) \qquad (1.8)$$

where,

$\hat{\beta}^s = (\mathbf{X}_s^T \hat{\mathbf{V}}_{ss}^{-1} \mathbf{X}_s)^{-1} \mathbf{X}_s^T \hat{\mathbf{V}}_{ss}^{-1} \mathbf{Y}_s$ is the empirical BLU estimator of the β under (1.7), $\mathbf{t}_i$ is the vector (0, 0, 0, ..., 1, 0, 0)^T with the 1 in the i^{th} position, s denote the sample values, r denote non-sample values and

$$\hat{\mathbf{v}} = \hat{\mathbf{G}} \mathbf{Z}_s^T \hat{\mathbf{V}}_{ss}^{-1} (\mathbf{Y}_s - \mathbf{X}_s \hat{\beta}^s)$$

and

$$\hat{\mathbf{G}} = \hat{\sigma}_u^2 [(\mathbf{I} - \hat{\rho}\, \mathbf{W})\, (\mathbf{I} - \hat{\rho}\, \mathbf{W})^T]^{-1}$$

and

$$\hat{\mathbf{V}}_{ss} = \hat{\sigma}_e^2 \mathbf{I}_N + \mathbf{Z}_s \hat{\sigma}_u^2 [(\mathbf{I}_m - \hat{\rho}\, \mathbf{W})\, (\mathbf{I}_m - \hat{\rho}\, \mathbf{W})^T]^{-1} \mathbf{Z}_s^T$$

where $\mathbf{X}_s$ is the $n \times p$ matrix of sample values of auxiliary variables, $\mathbf{Z}_s$ is the corresponding $n \times m$ matrix of sample components of $\mathbf{Z}$ and $\mathbf{V}_{ss}$ is the $n \times n$ covariance matrix associated with the n sample units that make up the $n \times 1$ sample vector $\mathbf{Y}_s$. A subscript r is used to denote corresponding quantities defined by the N-n non-sample units of the population, with $\mathbf{V}_{rs}$ denoting the $(N - n) \times n$ matrix defined by Cov($\mathbf{Y}_r$, $\mathbf{Y}_s$) and $\mathbf{V}_{iss}$ is the variances and covariances for the i^{th} area.

If all random effects are normally distributed, the parameter vector $\theta = (\sigma_u^2, \sigma_e^2, \rho)^T$ can be estimated via maximum likelihood (ML) as well as restricted maximum likelihood (REML) (Pratesi and Salvati 2005, Singh *et al.* 2005, Petrucci and Salvati 2006).

Chandra *et al.* (2007) showed that such models allow efficient use of spatial auxiliary information in small area estimation. They developed MSE for this SEBLUP estimator for small area means. Due to incorporating spatial effects, number of parameters became more. So, MSE estimation of spatial EBLUP estimator is more complicated than EBLUP estimator. In this situation, Hierarchical Bayes estimation is one of the remedy because this approach is straightforward and inferences for parameters are "exact" unlike the EBLUP approach. It automatically takes into account the extra uncertainties associated with unknown parameters in the model. This motivates us to use HB approach to obtain efficient spatial EBLUP estimators of small area means.

They also mentioned that the spatial models considered, have been based on neighbourhoods defined by contiguous areas *i.e.*, contiguity matrix. It is easy to see that this is just one way of introducing spatial dependence between area effects and there may be several other options to introduce spatial dependence. Keeping these points in view, HB estimates of SEBLUP have been obtained using three types of different weight matrices. Apart from the contiguity matrix other two weight matrices used, are described below:

Gaussian distance-decay based weighting is given as

$$\gamma_{ik} = \exp\left(-\frac{d_{ik}^2}{2r_0^2} \right), \; r_0 = \max(d_{ik}) \qquad (1.9)$$

where d_{ik} is the distance between i^{th} and k^{th} area.

Here, the value of the weight would decay gradually with distance, to the extent that when $d_{ik} = r_0$ the weighting would be 0.5.

The simplest way of denoting the spatial dependence of these areas is the use of authorized variogram function. In this approach, elements of W_i *i.e.*, γ_{ik} are obtained by using following equations.

$$\gamma_{ik} = 1 - r(d_{ik}) \qquad (1.10)$$

This also preserves the symmetric nature of weight matrix. Main authorized variogram functions are exponential, spherical and Gaussian variograms. Spherical variogram function is defined as

$$r(d_{ik}) = \left\{ \left(\frac{3d_{ik}}{2r_m} - \frac{1}{2} \left(\frac{d_{ik}}{r_m} \right)^3 \right) \right\}, \; \text{if } d_{ik} < r_0 \qquad (1.11)$$

What follows next, in Section 2, we now introduce HB approach for spatial data. In particular, we consider the spatial model described in Chandra *et al.* (2007). In this case we use Gibbs sampling method. Section 3 represents the simulation studies and sensitivity analysis and finally in section 4, we give some concluding remarks.

2. HIERARCHICAL BAYES APPROACH FOR SPATIAL MODELS

In this section, the spatial model (1.6) described by Chandra *et al.* (2007) has been studied in Hierarchical Bayes framework.

Consider the spatial model (1.6) in hierarchical bayes framework such that:

$$y_{ij} \mid \beta, \mathrm{v}_i, \sigma_e^2 \sim \mathrm{N}(\mathbf{x}_{ij}^T\beta + \mathrm{v}_i, \sigma_e^2),\ j = 1, 2, ..., n_i;\quad i = 1, 2, ..., m$$

$$(\mathbf{v}/\sigma_u^2) \sim \mathrm{N}_m(\mathbf{0}, \mathbf{G})$$

where $\mathbf{G} = \sigma_u^2[(\mathbf{I}-\rho\mathbf{W})\,(\mathbf{I}-\rho\mathbf{W})^T]^{-1}$

Assuming prior distributions for parameters are as

$$\beta \sim \mathrm{N_p}(\mathbf{0}, \mathbf{C}),$$

$$\sigma_e^2 \sim \mathrm{IG}(a_0, b_0) \text{ and } \sigma_u^2 \sim \mathrm{IG}(a_1, b_1)$$

In this approach, the posterior mean $\mathrm{E}(\bar{\mathrm{Y}}_i \mid \{y_{ij}\})$ is used as a point estimate for mean and the posterior variance $\mathrm{V}(\bar{\mathrm{Y}}_i \mid \{y_{ij}\})$ as a measure of variability. In order to estimate $\mathrm{E}(\bar{\mathrm{Y}}_i \mid \{y_{ij}\})$ and $\mathrm{V}(\bar{\mathrm{Y}}_i \mid \{y_{ij}\})$, the Gibbs sampling method given by Gelfand and Smith (1990) has been used. Full conditional distributions of the parameters for the Gibbs sampler are as follows

$$(\beta \mid \mathbf{v}, \sigma_u^2, \sigma_e^2, \mathbf{Y}) \sim \mathrm{N}_p((\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X} + \mathbf{C}^{-1})^{-1}\,\mathbf{X}^T\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{Zv}), (\mathbf{X}^T\mathbf{V}^{-1}\mathbf{X} + \mathbf{C}^{-1})^{-1})$$

$$(\mathbf{v} \mid \beta, \sigma_u^2, \sigma_e^2, \mathbf{Y}) \sim \mathrm{N}_m(\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z} + \mathbf{G}^{-1})^{-1}\,\mathbf{Z}^T\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\beta), (\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z} + \mathbf{G}^{-1})^{-1})$$

$$(\sigma_e^2 \mid \beta, \mathbf{v}, \sigma_u^2, \mathbf{Y}) \sim \mathrm{IG}\left[a_0 + \frac{n}{2},\ b_0 + \frac{1}{2}\,[\mathbf{Y} - (\mathbf{X}\beta + \mathbf{Zv})]^T\,[\mathbf{Y} - (\mathbf{X}\beta + \mathbf{Zv})]\right]$$

$$(\sigma_u^2 \mid \beta, \mathbf{v}, \sigma_e^2, \mathbf{Y}) \sim \mathrm{IG}\left(\left(a_1 + \frac{m}{2}\right), b_1 + \frac{1}{2}\mathbf{v}^T\mathbf{v}\right)$$

Here also proper priors on all the parameters have been used to ensure the proper posterior distributions. Values for the parameters of the inverted gamma priors were chosen very small to reflect the lack of knowledge about the parameters.

It is very straight forward to draw samples from these full conditional distributions as they have closed-form. After a "burn-in" period of B = 5000 iterations, the next 5000 samples were considered to obtain $\beta^{(k)}, \mathbf{v}^{(k)}, \sigma_e^{2(k)}$ and $\sigma_u^{2(k)}$, $k = 1, 2, ..., d$.

Now conditional on $\{y_{ij}\}$, β, σ_e^2 and σ_u^2, the posterior distribution of $\mathbf{v}$ can be obtained as

$$(\mathbf{v} \mid \beta, \sigma_u^2, \sigma_e^2, \mathbf{Y}) \sim \mathrm{N}_m((\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z} + \mathbf{G}^{-1})^{-1}\mathbf{Z}^T\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\beta), (\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z} + \mathbf{G}^{-1})^{-1})$$

Thus,

$$\mathrm{E}(\mathrm{v}_i \mid \beta, \sigma_u^2, \sigma_e^2, \mathbf{Y}_i) = \hat{\mathrm{v}}_i(\beta, \sigma_u^2, \sigma_e^2, \mathbf{Y}_i) = \mathbf{t}^T(\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z} + \mathbf{G}^{-1})^{-1}\mathbf{Z}^T\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\beta)$$

Thus, the conditional posterior mean of $\mathbf{Y}_i$ can be obtained as

$$\mathrm{E}(\bar{Y}_i \mid \beta, \sigma_u^2, \sigma_e^2, \mathbf{Y}_i) = \hat{\bar{Y}}_i(\beta, \sigma_u^2, \sigma_e^2) = \bar{\mathbf{X}}_i^T\,\hat{\beta} + \hat{\mathrm{v}}_i(\beta, \sigma_u^2, \sigma_e^2, \mathbf{Y}_i)$$

Similarly, the conditional posterior variance of $\bar{Y}_i$ is given by

$$\mathrm{V}(\bar{Y}_i \mid \beta, \sigma_u^2, \sigma_e^2, \mathbf{Y}_i) = \hat{\mathrm{V}}_i(\beta, \sigma_u^2, \sigma_e^2) = \mathbf{t}^T(\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z} + \mathbf{G}^{-1})^{-1}\,\mathbf{t}$$

Now, the simulated samples $\{\beta^{(k)}, \mathbf{v}^{(k)}, \sigma_e^{2(k)}$ and $\sigma_u^{2(k)}, k = 1, 2, ..., d\}$ were used to estimate the unconditional posterior mean and posterior variance of $\bar{\mathrm{Y}}_i$. Denote the values of β, σ_e^2 and σ_u^2 for the k^{th} simulated sample as $\beta^{(k)}$, $\sigma_e^{2(k)}$ and $\sigma_u^{2(k)}$ respectively. Then posterior mean $\mathrm{E}(\bar{Y}_i \mid \{y_{ij}\})$ can be estimated as

$$\hat{\bar{Y}}_i^{\mathrm{HB}} = \frac{1}{d}\sum_{k=1}^{d}\hat{\bar{Y}}_i(\beta^{(k)}, \mathbf{v}^{(k)}, \sigma_u^{2(k)}, \sigma_e^{2(k)})$$

Similarly, the posterior variance $\mathrm{V}(\bar{Y}_i\{y_{ij}\})$ is estimated as

$$\hat{V}^{HB}(\bar{Y}_i) = \frac{1}{d}\sum_{k=1}^{d}\hat{\mathrm{V}}_i\left(\beta^{(k)}, \sigma_u^{2(k)}, \sigma_e^{2(k)}\right) + \frac{1}{d}\sum_{k=1}^{d}\left[\hat{\bar{Y}}_i\left(\beta^{(k)}, \mathbf{v}^{(k)}, \sigma_u^{2(k)}, \sigma_e^{2(k)}\right)\right]^2 - \left[\frac{1}{d}\sum_{k=1}^{d}\hat{\bar{Y}}_i\left(\beta^{(k)}, \mathbf{v}^{(k)}, \sigma_u^{2(k)}, \sigma_e^{2(k)}\right)\right]^2$$

In following section, a comparison between spatial (SAR) small area estimates through Hierarchical Bayes and with EBLUP approach is presented.

3. EMPIRICAL EVALUATIONS

3.1 Simulation Study

In this study, spatial population structure has been generated assuming mean value of the dependent variable y fixed for a given area which is located at the centre of the population in the map. Since total 15 areas has been considered in the population, therefore, mean value of the other 14 areas for the dependent variable y has been generated assuming spatial pattern. In this pattern, the values of y will depend on distance from the central area. Further, the mean values of auxiliary variable x have been generated using mean values of y and bivariate normal distribution keeping the value of correlation coefficient between mean values of y and mean values x fixed, *i.e.*, $\rho = 0.7$. In order to generate the unit level data for each area, the bivariate normal population has been assumed and coefficient of variation has been fixed at 15% for each y and x. It may be noted that number of units in each area ranging from 335 to 430 with total population of 5945 units.

Overall sample size n is 50 from all 15 areas and the sample size n_i ($i = 1, ..., 15$) within each area ranges from 2 to 5 has been taken for the study. Sampling units were selected with simple random sampling without replacement. The spatial unit level model is given by:

$$y_{ij} = \beta_0 + x_{1ij}\beta_1 + v_i + e_{ij}, j = 1, 2, ..., n_i;$$

$$i = 1, 2, ..., m \quad (3.1.1)$$

In order to implement the Gibbs sampler based on above model (3.1.1), the following priors were assumed for different parameters $\beta_0 \sim N(0, 10^4)$, $\beta_1 \sim N(0, 10^4)$,

$$\mathbf{v} \sim N_{15}(\mathbf{0}, \sigma_u^2[(\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W})^T]^{-1},$$

$\sigma_u^2 \sim IG(0.001, 0.001)$, $\sigma_e^2 \sim IG(0.001, 0.001)$. Here, $\mathbf{W}$ is the weight matrix obtained by three different approaches *i.e.*, Neighbourhood Criteria, Gaussian Decay and Spherical method and ρ was set equal to 0.7. After a burn-in period of 5000 iterations, next 5000 samples were kept for $\{\beta_0^{(k)}, \beta_1^{(k)}, \mathbf{v}^{(k)}, \sigma_e^{2(k)}$ and $\sigma_u^{2(k)}, k = 1, 2, ..., d\}$. Further, $v_i^{(k)}$ was obtained by multiplying $\mathbf{v}^{(k)}$ with vector (0, 0, 0, ..., 1, 0, 0, ..., 0), 1 in the i^{th} position.

Gibbs sampler for the model (3.1.1) was implemented using WinBugs software. The WinBugs program constructs the necessary full conditional distributions and carries out the Gibbs sampling. Prior and initial values were generated using this software. Gibbs sampler was first run for a burn-in period of 5000 iterations and then 5000 more iteration were run and kept for analysis and estimation. This was done for all different weighting approaches and posterior mean and posterior variances were obtained. Following table shows the mean and variances of Hierarchical Bayesian estimates of spatial (SAR) model for three different weighting methods.

Table 1 shows that posterior means obtained through different weighting approaches are of same order however posterior variances for all three weighting approaches are not same. Fig. 1 shows that

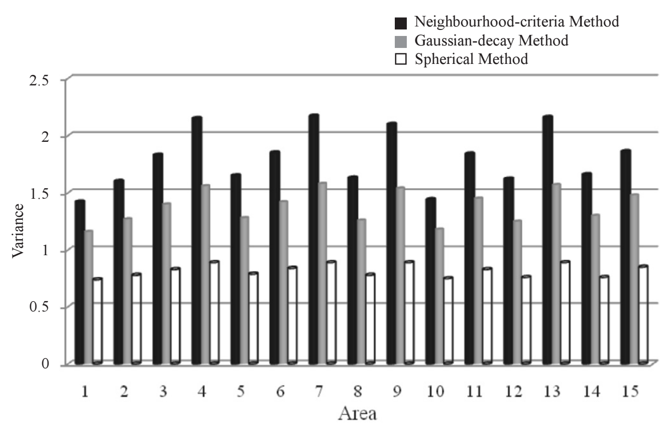

Fig. 1. Posterior variances of Hierarchical Bayes estimates of spatial model for three different weighting approaches.

variances are least in case of spherical variogram approach followed by Gaussian-decay and neighbourhood criteria. Further the percentage bias and percentage gain in efficiency of spatial model under HB framework has been obtained with respect to SEBLUP technique for each of three different weighting approaches (Table 2).

Table 1. Posterior means and Posterior variances of Hierarchical Bayes estimates of spatial model for three different weighting approaches.

Area	Neighbourhood criteria method		Gaussian method		Spherical method	
	mean	variance	mean	variance	mean	variance
1.	28.07	1.42	28.71	1.16	28.77	0.74
2.	29.15	1.60	29.72	1.27	29.30	0.78
3.	29.38	1.83	30.13	1.40	29.52	0.83
4.	28.67	2.15	29.56	1.56	29.30	0.89
5.	31.03	1.65	31.28	1.28	29.87	0.79
6.	29.41	1.85	30.19	1.42	30.10	0.84
7.	25.76	2.17	27.42	1.58	28.77	0.89
8.	27.95	1.63	28.86	1.26	29.30	0.78
9.	28.62	2.10	29.63	1.54	29.52	0.89
10.	25.92	1.44	27.21	1.18	29.30	0.75
11.	28.33	1.84	29.41	1.45	29.87	0.83
12.	29.74	1.62	30.39	1.25	30.10	0.76
13.	28.12	2.16	29.29	1.57	29.52	0.89
14.	30.86	1.66	31.28	1.30	30.10	0.76
15.	31.38	1.86	32.02	1.48	31.42	0.85

Table 2. Percentage bias and percentage gain in efficiency of spatial model in HB approach with respect to the SEBLUP approach for three weighting approaches.

Area	Sample size	Neighbourhood criteria method		Gaussian method		Spherical method	
		%bias	%gain in efficiency	%bias	%gain in efficiency	%bias	%gain in efficiency
1.	5	2.26	14.53	0.42	30.82	–1.23	56.24
2.	4	4.53	22.24	2.65	38.30	2.49	61.80
3.	3	3.35	36.06	4.46	48.96	4.70	70.20
4.	2	5.15	40.59	1.89	56.84	1.46	76.98
5.	4	7.11	26.01	5.36	37.02	9.45	61.50
6.	3	4.84	28.98	1.56	45.11	1.18	68.57
7.	2	–7.74	38.87	–14.25	57.57	–20.25	77.13
8.	4	–0.17	20.76	–3.17	38.97	–3.32	61.92
9.	2	1.84	42.97	–0.75	59.14	3.33	77.38
10.	5	–3.41	13.52	–9.16	28.38	–18.32	55.31
11.	3	1.27	37.19	–3.33	44.07	–3.91	68.59
12.	4	3.41	20.68	0.83	38.67	3.45	63.39
13.	2	1.56	40.47	–2.40	56.15	–3.97	77.45
14.	4	4.95	22.95	4.10	36.78	8.42	64.07
15.	3	4.78	28.24	2.97	42.74	5.79	68.75

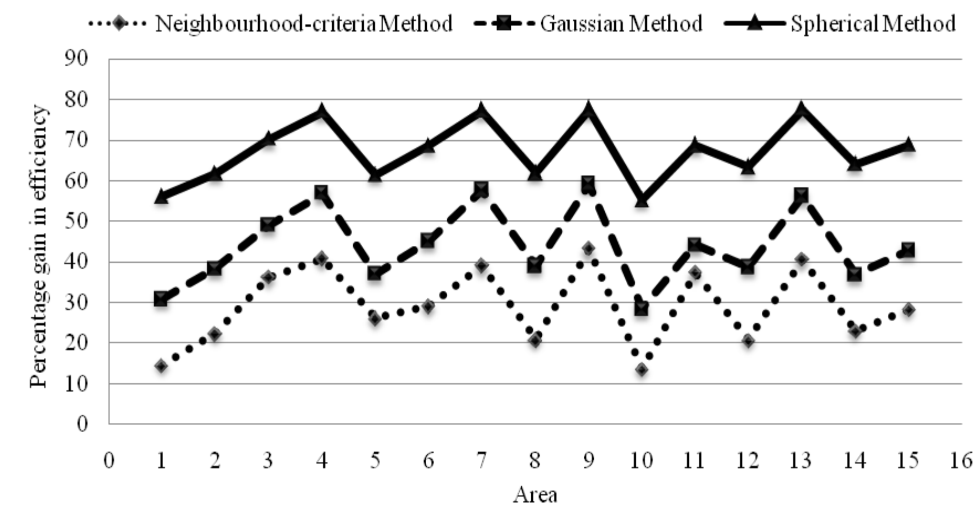

Fig. 2. Percentage gain in efficiency of spatial model in HB approach with respect to the SEBLUP approach for three weighting approaches.

Table 2 shows that proposed estimator is almost unbiased for each of the 15 areas for all three weighting methods with few exceptions. However, the %gain in efficiency is maximum in case of spherical method (between 77.45 to 55.31) followed by Gaussian method (between 59.14 to 28.38) and neighbourhood criteria method (between 42.97 to 13.52) (Fig. 2).

3.2 Sensitivity Analysis

In practice it is always difficult to obtain accurate information about the distribution of the variances. Here, inverted gamma distribution was assumed on variance components. The interest is to know the effects caused by the choice of different priors. Basically, aim is to evaluate the sensitivity of posterior means to the choice of different priors on the variance components. In order to test the sensitivity of the posterior estimates to the choice of a_i and b_i, different values of these parameters *i.e.*, 0.001, 0.01, 0.1, 1, 10 have been set. Table 3 to 5 show the mean and variance for different gamma values for three different weighting approaches.

Above tables show that posterior estimates of small area means and variances are almost same when a_i and b_i are small (< = 0.1). This indicates small area mean estimates and variances are very much stable for

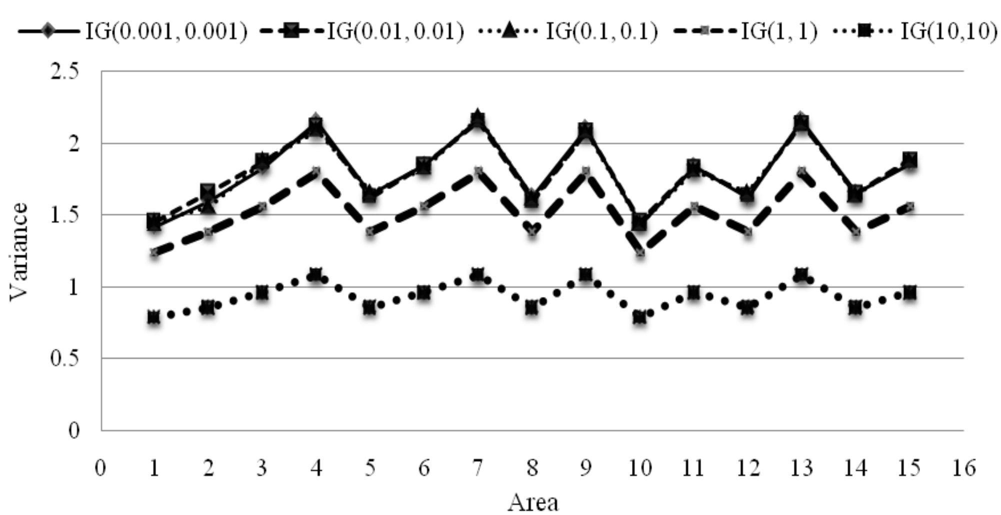

Fig. 3. Posterior variances of Hierarchical Bayes estimates of spatial model for Neighbourhood criteria method.

Table 3. Posterior means and posterior variances of Hierarchical Bayes estimates of spatial model for Neighbourhood Criteria method.

Area	G(0.001, 0.001)		G(0.01, 0.01)		G(0.1, 0.1)		G(1, 1)		G(10, 10)	
	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
1.	28.07	1.42	28.03	1.46	28.11	1.46	28.08	1.24	28.04	0.79
2.	29.15	1.60	29.12	1.66	29.19	1.56	29.12	1.38	29.00	0.86
3.	29.38	1.83	29.35	1.87	29.44	1.88	29.36	1.56	29.23	0.96
4.	28.67	2.15	28.71	2.12	28.74	2.09	28.69	1.80	28.62	1.08
5.	31.03	1.65	30.88	1.63	31.05	1.66	30.88	1.38	30.59	0.86
6.	29.41	1.85	29.43	1.85	29.47	1.83	29.42	1.56	29.36	0.96
7.	25.76	2.17	26.09	2.15	25.89	2.18	26.06	1.80	26.37	1.08
8.	27.95	1.63	28.02	1.60	28.01	1.62	28.02	1.38	28.06	0.86
9.	28.62	2.10	28.67	2.08	28.70	2.08	28.67	1.80	28.64	1.08
10.	25.92	1.44	26.15	1.46	26.00	1.44	26.14	1.24	26.42	0.79
11.	28.33	1.84	28.42	1.83	28.41	1.81	28.44	1.56	28.50	0.96
12.	29.74	1.62	29.62	1.64	29.78	1.66	29.72	1.38	29.63	0.86
13.	28.12	2.16	28.23	2.13	28.21	2.14	28.23	1.80	28.28	1.08
14.	30.86	1.66	30.85	1.65	30.89	1.64	30.75	1.38	30.51	0.86
15.	31.38	1.86	31.35	1.88	31.43	1.88	31.33	1.56	31.17	0.96

Table 4. Posterior means and posterior variances of Hierarchical Bayes estimates of spatial model for Gaussian-decay method.

Area	G(0.001, 0.001)		G(0.01, 0.01)		G(0.1, 0.1)		G(1, 1)		G(10, 10)	
	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
1.	28.71	1.16	28.71	1.18	28.71	1.20	28.71	1.11	28.71	0.75
2.	29.72	1.27	29.74	1.26	29.75	1.32	29.73	1.22	29.70	0.82
3.	30.13	1.40	30.14	1.42	30.17	1.46	30.15	1.35	30.09	0.90
4.	29.56	1.56	29.59	1.63	29.58	1.64	29.57	1.50	29.54	1.00
5.	31.28	1.28	31.34	1.25	31.35	1.32	31.31	1.22	31.21	0.82
6.	30.19	1.42	30.18	1.49	30.20	1.46	30.20	1.35	30.18	0.90
7.	27.42	1.58	27.44	1.64	27.34	1.64	27.38	1.50	27.49	1.00
8.	28.86	1.26	28.85	1.34	28.84	1.32	28.85	1.22	28.88	0.82
9.	29.63	1.54	29.66	1.54	29.64	1.64	29.63	1.50	29.62	1.00
10.	27.21	1.18	27.15	1.15	27.12	1.20	27.17	1.11	27.30	0.75
11.	29.41	1.45	29.38	1.50	29.38	1.46	29.39	1.35	29.42	0.90
12.	30.39	1.25	30.45	1.24	30.41	1.32	30.40	1.22	30.37	0.82
13.	29.29	1.57	29.29	1.59	29.28	1.64	29.29	1.50	29.30	1.00
14.	31.28	1.30	31.35	1.33	31.34	1.32	31.31	1.22	31.22	0.82
15.	32.02	1.48	32.06	2.49	32.06	1.46	32.04	1.35	31.98	0.90

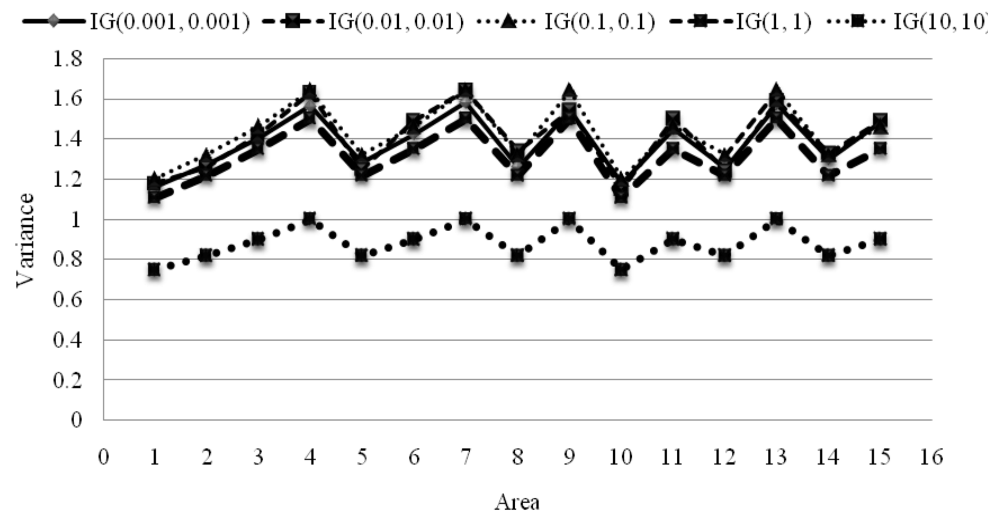

Fig. 4. Posterior variances of Hierarchical Bayes estimates of spatial model for Gaussian-decay method.

Table 5. Posterior means and posterior variances of Hierarchical Bayes estimates of spatial model for Spherical method.

Area	G(0.001, 0.001)		G(0.01, 0.01)		G(0.1, 0.1)		G(1, 1)		G(10, 10)	
	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
1.	28.77	0.74	28.77	0.79	28.78	0.69	28.81	0.49	28.83	0.30
2.	29.30	0.78	29.33	0.75	29.31	0.75	29.34	0.51	29.36	0.31
3.	29.52	0.83	29.54	0.82	29.53	0.81	29.56	0.53	29.58	0.31
4.	29.30	0.89	29.34	0.86	29.31	0.89	29.34	0.55	29.36	0.32
5.	29.87	0.79	29.88	0.76	29.89	0.75	29.92	0.51	29.94	0.31
6.	30.10	0.84	30.13	0.88	30.11	0.81	30.14	0.53	30.16	0.31
7.	28.77	0.89	28.77	0.88	28.78	0.89	28.81	0.55	28.83	0.32
8.	29.30	0.78	29.34	0.76	29.31	0.75	29.34	0.51	29.36	0.31
9.	29.52	0.89	29.54	0.92	29.53	0.89	29.56	0.55	29.58	0.32
10.	29.30	0.75	29.33	0.68	29.31	0.69	29.34	0.49	29.36	0.30
11.	29.87	0.83	29.88	0.85	29.89	0.81	29.92	0.53	29.94	0.31
12.	30.10	0.76	30.15	0.79	30.11	0.75	30.14	0.51	30.16	0.31
13.	29.52	0.89	29.53	0.84	29.53	0.89	29.56	0.55	29.58	0.32
14.	30.10	0.76	30.11	0.72	30.11	0.75	30.14	0.51	30.16	0.31
15.	31.42	0.85	31.44	0.83	31.44	0.81	31.47	0.53	31.49	0.31

small values of a_i and b_i, there is almost no difference among the estimates at all. But as a_i and b_i increase, small area means are stable but posterior variances are decreasing rapidly. This indicates that posterior variances decrease as the priors become more informative.

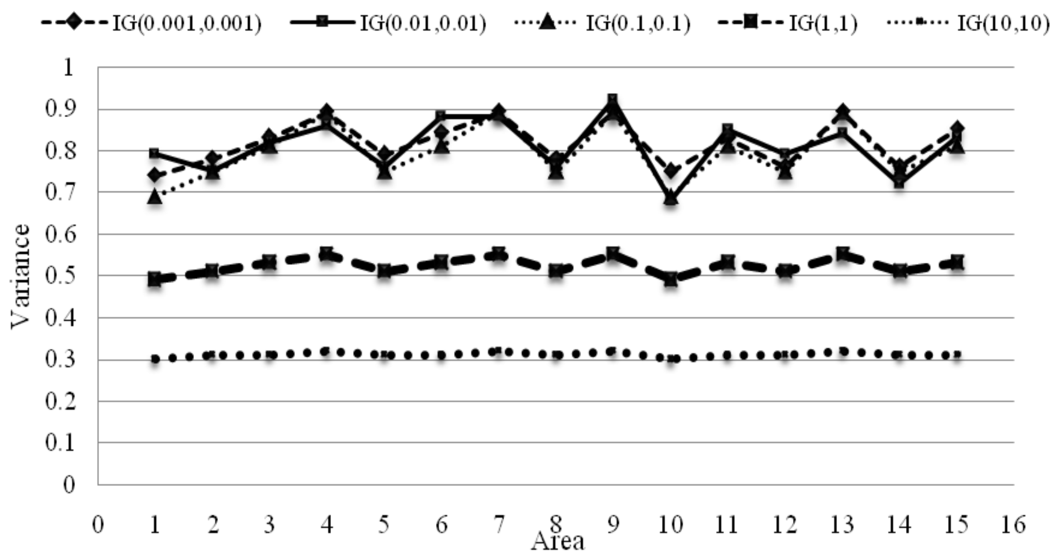

Fig. 5. Posterior variances of Hierarchical Bayes estimates of spatial model for Spherical method.

4. CONCLUSION

In this study, attempt has been made to obtain small area estimates for spatial small area model described in Chandra *et al.* (2007) under Hierarchical Bayes (HB) framework. Three different weighting approaches were used to incorporate the spatial effects in the model. Analysis shows that proposed estimator is almost unbiased for each of the 15 areas for all three weighting methods with few exceptions. However, the percentage gain in efficiency is maximum in case of spherical method followed by Gaussian method and neighbourhood criteria method.

Sensitivity analysis shows that posterior estimates of small area means and variances are almost similar when a_i and b_i are small ($<= 0.1$). This indicates small area mean estimates and variances are very much stable for small values of parameters, there is almost no difference among the estimates at all for these values. Though as a_i and b_i increase, small area means are stable but posterior variances are decreasing rapidly. This indicates that as the priors become more informative, posterior variances decrease. So, taking more informative priors, one can improve the results.

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