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Recent Developments in Fractional Factorial Designs*

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SUMMARY

Fractional factorial designs have received considerable attention in the last twenty five years due to their applicability in a wide variety of situations. These have been successfully used for planning experiments in agriculture, physical and chemical sciences, medicine and industry and have been found very useful in quality improvement work. The literature on this subject is already voluminous and continues to grow. In this article, an overview of some of the recent developments in this area is presented.

Keywords : Resolution, Minimum aberration, Estimation capacity, Blocking, Optimality.

1. INTRODUCTION AND PRELIMINARIES

Factorial experiments are planned for exploring the effects of several controllable variables. The general scenario is one in which there is an output variable which is hypothesized to depend on some controllable or, input variables. The input variables are termed as *factors.* Each factor has at least two settings, these settings being called *levels.* Any combination of the levels of all the factors under consideration is called a *treatment combination.* The aim of the experiment is to explore the effects of individual factors and also their possible interrelationships. Factorial experiments have wide applications in many diverse areas of human investigation, including agriculture, physical and chemical sciences, medicine, manufacturing industry and quality control work.

Consider a factorial experiment involving n (≥ 2) factors $F_1, F_2, ..., F_n$, where for $1 \leq i \leq n$, F_i appears at m_i levels and $m_i \geq 2$. If in particular, $m_1 = m_2 = \cdots = m_n = m$, say, then this set up corresponds to a *symmetric* m^n factorial; otherwise, it corresponds to an *asymmetric* or, *mixed level* factorial. For $1 \leq i \leq n$, let the levels of F_i be coded as 0, 1, ..., $m_i - 1$. A typical treatment combination is represented by an n-tuple $j_1 j_2 \ldots j_n$ and the effect due to this treatment combination is denoted by $\tau(j_1 j_2 \ldots j_n)$ ($0 \leq j_i \leq m_i - 1$; $1 \leq i \leq n$). There are altogether $v = \prod_{i=1}^{n} m_i$ treatment combinations which will, hereafter, be assumed to be lexicographically ordered. The set of v treatment combinations will be denoted by $\mathcal{V}$. Let τ be a $v \times 1$ vector with elements $\tau(j_1 j_2 \ldots j_n)$ arranged in lexicographic order. The treatment effects, that is, the elements of τ, are unknown parameters. In the context of a factorial experiment, interest centres around contrasts belonging to *factorial effects.* A linear parametric function

$$\sum \ldots \sum l(j_1 \ldots j_n)\, \tau(j_1 \ldots j_n), \tag{1.1}$$

where $\{l(j_1 \ldots j_n)\}$ are real numbers, not all zeros, such that $\Sigma \ldots \Sigma l(j_1 \ldots j_n) = 0$ and the summation extends over $j_1 \ldots j_n \in \mathcal{V}$, is called a *treatment contrast*. Following Bose (1947), a treatment contrast of the type

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(1.1) is said to belong to the factorial effect $F_{i_1} \ldots F_{i_g}$ $(1 \le i_1 < \ldots < i_g \le n;\ 1 \le g \le n)$ if

(i) $l(j_1 \ldots j_n)$ depends only on $j_{i_1}, \ldots, j_{i_g}$, and

(ii) writing $l(j_1 \ldots j_n) = \bar{l}(j_{i_1} \ldots j_{i_g})$ in consideration of (i) above, the sum of $\bar{l}(j_{i_1} \ldots j_{i_g})$ separately over each of the arguments $j_{i_1}, \ldots, j_{i_g}$ is zero.

For $1 \le i_1 < \ldots < i_g \le n (1 \le g \le n)$, $F_{i_1} F_{i_2} \ldots F_{i_g}$ denotes a typical factorial effect involving the factors $i_1, \ldots, i_g$. There are $\prod_{u=1}^{g}(m_{i_u} - 1)$ linearly independent contrasts belonging to the factorial effect $F_{i_1} \ldots F_{i_g}$. A factorial effect is called a *main effect* if it involves exactly one factor (*i.e.*, $g = 1$) and an *interaction* if it involves more than one factor (*i.e.*, $g > 1$). Following the standard convention, F_i denotes the ith factor as well as the main effect thereof. Clearly, there are a totality of $\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n - 1$ factorial effects in a factorial experiment involving n factors.

There is a convenient way to represent a complete set of orthonormal treatment contrasts belonging to the factorial effects. Suppose Ω is the set of all binary n-tuples and let Ω^* be the subset of Ω consisting of its non-null members, *i.e.*, $\Omega^* = \Omega \setminus \{0, 0, \ldots, 0\}$. It is easy to see that there is a 1 – 1 correspondence between Ω^* and the set of all factorial effects in the sense that a typcial factorial effect $F_{i_1} \ldots F_{i_g}$ $(1 \le i_1 < \ldots < i_g \le n;\ 1 \le g \le n)$ corresponds to the element $\boldsymbol{x} = x_1 x_2 \ldots x_n$ of Ω^* such that $x_{i_1} = \ldots = x_{i_g} = 1$ and $x_u = 0$ for $u \ne i_1, \ldots, i_g$. Thus the $2^n - 1$ factorial effects may be represented by $F^{\mathbf{x}}$, $\boldsymbol{x} \in \Omega^*$. For instance, if $n = 3$, then the effect F^{100} represents the main effect of the first factor, F^{110} represents the 2-factor interaction of the first two factors, and so on.

For $1 \le i \le n$, let $\mathbf{1}_{m_i}$, denote an $m_i \times 1$ vector of all ones, $\mathbf{I}_{m_i}$, an identity matrix of order m_i and $\mathbf{P}_i$ be an $(m_i - 1) \times m_i$ matrix such that the $m_i \times m_i$ matrix $\mathbf{A}_i = \begin{pmatrix} m_i^{-1/2} \mathbf{1}'_{m_i} \\ \mathbf{P}_i \end{pmatrix}$ is an orthogonal matrix, *i.e.*, $\mathbf{A}_i \mathbf{A}'_i = \mathbf{I}_{m_i} = \mathbf{A}'_i \mathbf{A}_i$. For a pair of matrices $\mathbf{E} = (e_{ij})$ and $\mathbf{F}$, of orders $s \times t$ and $p \times q$, respectively, let $\mathbf{E} \otimes \mathbf{F}$ denote their tensor (Kronecker) product, *i.e.*, $\mathbf{E} \otimes \mathbf{F} = (e_{ij}\, \mathbf{F})$, which is a matrix of order $sp \times tq$. For $\boldsymbol{x} \in \Omega$, let $\alpha(\boldsymbol{x}) = \prod_{i=1}^{n}(m_i - 1)^{x_i}$. For each $\boldsymbol{x} = (x_1, \ldots, x_n) \in \Omega$, define the $\alpha(\boldsymbol{x}) \times v$ matrix

$$\mathbf{P}^{\boldsymbol{x}} = \mathbf{P}_1^{x_1} \otimes \ldots \otimes \mathbf{P}_n^{x_n} = \bigotimes_{i=1}^{n} \mathbf{P}_i^{x_i}, \tag{1.2}$$

where for $1 \le i \le n$,

$$\mathbf{P}_i^{x_i} = \begin{cases} m_i^{-1/2} \mathbf{1}'_{m_i} & \text{if} \quad x_i = 0 \\ \mathbf{P}_i & \text{if} \quad x_i = 1. \end{cases} \tag{1.3}$$

Then one can show that for each $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $\boldsymbol{x} \ne \boldsymbol{y}$,

(a) $\mathbf{P}^{\boldsymbol{x}}(\mathbf{P}^{\boldsymbol{x}})' = \mathbf{I}_{\alpha(\boldsymbol{x})}$, and

(b) $\mathbf{P}^{\boldsymbol{x}}(\mathbf{P}^{\boldsymbol{y}})' = \mathbf{O}$ (a null matrix).

By virtue of (a) above, Rank $(\mathbf{P}^{\boldsymbol{x}}) = \alpha(\boldsymbol{x})$ and this equals the number of linearly independent treatment contrasts belonging to the effect $\mathbf{F}^{\boldsymbol{x}}$. Thus for each $\boldsymbol{x} \in \Omega^*$, the elements of $\mathbf{P}^{\boldsymbol{x}}\tau$ represent a *complete set* of orthonormal treatment contrasts belonging to the effect $\mathbf{F}^{\boldsymbol{x}}$. Also, constrasts belonging to different factorial effects are mutually orthogonal.

It is also possible to provide an interpretation for $\mathbf{P}^{000\ldots0}\tau$. Since $\mathbf{P}^{00\ldots0} = v^{-1/2}\, \mathbf{1}'_v$, we have $\mathbf{P}^{00\ldots0}\, \tau = v^{1/2}\, \bar{\tau}$, where $\bar{\tau}$, the general mean, is the arithmetic mean of the quantities $\{\tau(j_1 \ldots j_n)\}$.

2. FRACTIONAL FACTORIAL PLANS

2.1 Basic Ideas

The number of treatment combinations in a factorial experiment increases rapidly with the increase in the number of levels and/or factors. Even with 2 levels of each factor, if the number of factors is 10, one has to experiment with 1024 treatment combinations in a single replicate. Such a large experiment, apart from being too expensive and impractical, may not at all be necessary if the interest is in estimating lower order factorial effects (say, the main effects and possibly all or some 2-factor interactions). Economy of space and material in such situations can be achieved by considering a suitable subset of the set of all treatment combinations. The underlying experimental strategy then, is called a *fractional factorial plan.* Such a plan aims at drawing, under appropriate assumptions, valid statistical inference about the relevant factorial effects through an optimal utilization of the available resources.

Consider an N–run fractional factorial plan d for an $m_1 \times ... \times m_n$ factorial, where $0 < N < v$ $(= \Pi\, m_i)$. According to the plan d, suppose $r_d(j_1 \ldots j_n)$ observations are to be made with treatment combination $j_1 \ldots j_n$ for each $j_1 \ldots j_n \in \mathcal{V}$, where $\{r_d(j_1 \ldots j_n)\}$ are non-negative integers satisfying $\sum\limits_{j_1 \ldots j_n \in v} \ldots \sum r_d(j_1 \ldots j_n) = N.$ For $1 \le u \le N$, let Y_u be the u-th observation according to d. If Y_u corresponds to the treatment combination $j_1 \ldots j_n$, then we shall assume that $\mathbb{E}(Y_u)$ equals $\tau(j_1 \ldots j_n)$, where $\mathbb{E}(.)$ stands for expectation. Furthermore, $Y_1, ..., Y_N$ are assumed to be uncorrelated and to have a common variance $\sigma^2 (0 < \sigma^2 < \infty)$.

It will be convenient to express the above model in matrix notation. To that end, for $1 \le u \le N, j_1 \ldots j_n \in \mathcal{V}$, define the indicator $\mathcal{X}_d(u; j_1 \ldots j_n)$, which assumes the value 1 if the u-th observation according to d corresponds to $j_1 \ldots j_n$ and the value zero otherwise. Let $\mathbf{X}_d$ be the $N \times v$ design matrix, with rows indexed by u and columns indexed by $j_1 \ldots j_n$, such that

$$\mathbf{X}_d = (\mathcal{X}_d(u; j_1 \ldots j_n))_{\substack{u=1,\ldots,N. \\ j_1 \ldots j_n \in \mathcal{V}}} \qquad (2.1)$$

The columns of $\mathbf{X}_d$ are assumed to be lexicographically ordered. Then, with $\mathbf{Y} = (Y_1, ..., Y_n)'$, the linear model can be expressed as

$$\mathbb{E}(\mathbf{Y}) = \mathbf{X}_d \tau,\ \mathbb{D}(\mathbf{Y}) = \sigma^2 \mathbf{I}_N, \qquad (2.2)$$

where $\mathbb{E}(\cdot)$ as before, stands for the expectation and $\mathbb{D}(\cdot)$, for the dispersion (variance-covariance) matrix.

A linear parametric function for the form $\boldsymbol{l}'\tau$ is said to be estimable in d if it has a linear unbiased estimator. It is well known that $\boldsymbol{l}'\tau$ is estimable under d if and only if $\boldsymbol{l}' \in \mathcal{R}(\mathbf{X}_d)$, where $\mathcal{R}(\cdot)$ denotes the row space of a matrix. Hence, with $\mathbf{P}^x$ as in (1.2), for any $\boldsymbol{x} \in \Omega$, $\mathbf{P}^x\tau$ is estimable in d (*i.e.*, each element of $\mathbf{P}^x\tau$ is estimable in d) if and only if

$$\mathcal{R}(\mathbf{P}^x) \subset \mathcal{R}(\mathbf{X}_d). \qquad (2.3)$$

Now if $r_d(j_1 \ldots j_n) = 0$ for some $j_1 \ldots j_n \in \mathcal{V}$, then the corresponding column of $\mathbf{X}_d$ equals the null vector. However, from the definition of $\mathbf{P}^x$, it is easily seen that $\mathbf{P}^x$ cannot have a zero column. Hence, the relation (2.3) cannot hold for any $\boldsymbol{x} \in \Omega$. In other words, for any $\boldsymbol{x} \in \Omega$, in order to ensure the estimability of $\mathbf{P}^x\tau$ in d, it is necessary that $r_d(j_1 \ldots j_n) \ge 1$ for each $j_1 \ldots j_n \in \mathcal{V}$, *i.e.*, $N \ge v$. This means in particular that, unless further assumptions are made, a fractional factorial plan (for which N is strictly smaller than v) is incapable even of ensuring the estimability of complete sets of main effect contrasts which are invariably the parametric functions of interest in any factorial set up.

It is possible to overcome the above stated difficulty if from a knowledge of the physical process underlying the experimental set up or from past experience, one can assume the absence or negligibility of certain factorial effects, typically the higher order factorial effects. Fortunately, such an assumption is reasonable in many practical situations. In situations where the absence of higher order effects can be validly assumed, fractional factorials provide useful information on lower order effects at a considerable saving. Even the highly fractionated designs, like the designs of Resolution III, can be used in the initial stages of an exploratory programme for screening among a large number of potential factors, a few factors with large effects, quickly and at a reasonable cost. Detailed experiment can then be performed with the reduced set of factors. Thus, fractional factorials are a useful class of designs for situations where many factors have to be studied simultaneously and where, at least in the initial stages of investigation, an economical assessment of the effects of many factors, with possibly some ambiguous conclusions, is to be preferred over an experiment involving a detailed examination of only a few factors. In view of these considerations, fractional factorial plans have been used extensively in many areas in the recent years, notably in manufacturing and high-tech industries and in quality improvement work. Simultaneously, numerous theoretical results have also been found. In this communication, we present an overview of some of the major developments in the area of fractional factorials.

2.2 Notion of Resolution

Under the absence of factorial effects involving $t + 1$ or more factors, suppose interest lies in the general mean and contrasts belonging to the lower order factorial effects, say those involving at most f factors, where $1 \le f \le t$. A fractional factorial plan ensuring the estimability of all factorial effects involving f factors or less, under the assumption that all factorial effects involving $t + 1$ factors or more, $1 \le f \le t \le n - 1$, is called a Resolution (f, t) plan. This definition, due to Dey and Mukerjee (1999b), of resolution of a fractional factorial plan is somewhat different from and more general than the standard definition, according to which a Resolution (f, t) plan, as defined above, would have simply been called a Resolution-($f + t + 1$) plan. The modification in the definition is necessary, because the resolution of a plan is *not always* dependent on the integers f, t through their sum, $f + t$. Examples can be found where a plan is of Resolution (f, t) but not of Resolution (f', t') even though $f + t = f' + t'$.

Below we give two examples of Resolution (f, t) plans for 2-level symmetric experiments. The plan in (i) below has $f = 1 = t$, $n = 3$, $N = 4$ while the plan in (ii) has $f = 1$, $t = 2$, $n = 4$, $N = 8$.

Example 2.1

$$(\text{i}) : \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}, \qquad (\text{ii}) : \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

2.3 Role of Orthogonal Arrays

A natural question that arises in the context of fractional factorials is how to choose the fraction so that the parameters of interest, under suitable assumptions, are estimated most efficiently. To that end, orthogonal arrays provide a solution. A definition of orthogonal arrays follows.

Definition 2.1 : *An orthogonal array $OA(N, n, m_1 \times \ldots \times m_n, g)$, having N rows, $n(\geq 2)$ columns, $m_1, \ldots, m_n$ symbols and strength g $(2 \leq g < n)$, is an $N \times n$ array, with elements in the ith column from a set of $m_i \geq 2$ distinct symbols $(1 \leq i \leq n)$, in which all possible g-tuples of symbols appear equally often as rows in every $N \times g$ subarray.*

Without loss of generality, the symbols appearing in the ith column of an $OA(N, n, m_1 \times \ldots \times m_n, g)$ may be supposed to be $0, 1, \ldots, m_i - 1$. Note that an orthogonal array of strength g is also of strength g' $(1 \leq g' < g)$. If in particular, $m_1 = \ldots = m_n = m$, say, then we get a symmetric orthogonal array, which is denoted simply by $OA(N, n, m, g)$, otherwise, the array is an asymmetric or mixed orthogonal array.

The number of rows in an orthogonal array is bounded below by an integer that depends on n and the m_i's. For an $OA(N, n, m_1 \times \ldots \times m_n, 2)$,

$$N \geq 1 + \sum_{i=1}^{n} (m_i - 1),$$

and for an $OA(N, n, m_1 \times \ldots \times m_n, 3)$,

$$N \geq 1 + \sum_{i=1}^{n} (m_i - 1) + (m^* - 1)\left\{\sum_{i=1}^{n} (m_i - 1) - (m^* - 1)\right\},$$

where $m^* = \max_{1 \leq i \leq n} m_i$. Such lower bounds on the number of rows of an orthogonal array of arbitrary strength $g \geq 2$ are known; see *e.g.*, Hedayat *et al.* (1999). These lower bounds are often referred to as *Rao's bounds* as, Rao (1947) first obtained such bounds in the context of symmetric orthogonal arrays. Arrays with the number of rows attaining these bounds are called *tight*. For a comprehensive treatment of orthogonal arrays including applications and tables, see Hedayat *et al.* (1999). Tables of fractional factorial plans, including symmetric and asymmetric orthogonal arrays also appear in Dey (1985) and Wu and Hamada (2000). For more recent results on the construction of orthogonal arrays, see *e.g.*, Suen *et al.* (2001) and Suen and Dey (2003).

Example 2.2. In Example 2.1, the plan (i) is an $OA(4, 3, 2, 2)$ and plan (ii), an $OA(8, 4, 2, 3)$. In (a), (b) below, we show an asymmetric $OA(8, 5, 4 \times 2^4, 2)$ and a symmetric $OA(9, 4, 3, 2)$.

$$(\text{a})\ OA(8, 5, 4 \times 2^4, 2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{(b) } OA(9, 4, 3, 2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

The rows of an $OA(N, n, m_1 \times ... \times m_n, g)$ can be identified with the treatment combinations of an $m_1 \times ... \times m_n$ factorial set up. Thus the array itself represents an N–run plan for such a factorial. For example, the array $OA(9, 4, 3, 2)$ shown in Example 2.2(b) represents a 9–run plan for a 3^4 factorial. This is given by {0000, 0111, 0222, 1012, 1120, 1201, 2021, 2102, 2210}, as obtained by listing the rows of the array. Similarly, the array in Example 2.2(a) represents an 8–run plan for an asymmetric 4×2^4 experiment.

Let $\mathcal{D}_N$ be the class of all N–run plans for an $m_1 \times ... \times m_n$ factorial such that each member of $\mathcal{D}_N$ allows the estimability of all factorial effects with f factors or less under the absence of all factorial effects involving $t + 1$ factors or more, where $1 \le f \le t \le n - 1$. The following result shows the role of orthogonal arrays in the choice of a 'good' fractional factorial plan.

Theorem 2.1 : *Let d_0 be a fractional factorial plan belonging to $\mathcal{D}_N$, represented by an $OA(N, n, m_1 \times ... \times m_n, g)$, where $2 \le g < n$. Then d_0 is a universally optimal Resolution (f, t) plan in $\mathcal{D}_N$ for every choice of integers f, t such that $f + t = g$ and $1 \le f \le t \le n - 1$.*

In the above theorem, the term 'universal optimality' is used in the sense of Kiefer (1975). Note that a universally optimal plan is, in particular, also optimal according to the more commonly used criteria, like A–, D– and E– criteria. Theorem 2.1 shows that fractional factorial plans represented by orthogonal arrays are strongly optimal. For instance, a fractional factorial plan represented by an orthogonal array of strength *two* is universally optimal for the estimation of the mean and contrasts belonging to the main effects under the assumption that all 2-factor and higher order interactions have negligible magnitudes. Similarly, a fractional factorial plan represented by an orthogonal array of strength four ensures the optimal estimation of the mean, all main effects and all 2-factor interactions, when 3-factor and higher order factorial effects are assumed negligible. For fuller details and more results on optimality aspects of fractional factorial plans, see Dey and Mukerjee (1999b, Chapters 2 and 6).

Most of the available optimality results on fractional factorials relate to situations where all factorial effects involving the same number of factors are considered equally important and thus, the underlying model involves the general mean and all factorial effects involving up to a specified number of factors. In practice however, the presumption of equality in the importance of all factorial effects involving the same number of factors may not always be an appropriate one. For instance, there may be a situation where it is known *a priori* that only one of the factors can possibly interact with each of the other factors, all other 2-factor and higher order factorial effects being absent. The model then includes the mean, all main effects and a specified set of 2-factor interactions. Work on the issue of estimability and optimality in situations of this kind in the context of 2-level factorials has been addressed by Hedayat and Pesotan (1992, 1997), Wu and Chen (1992) and Chiu and John (1998). Further work in this area was done by Dey and Mukerjee (1999a) who considered arbitrary factorials, including the asymmetric ones and gave a combinatorial characterization for a fractional factorial plan to be universally optimal under a hierarchical model. We give an example of a universally optimal plan based on the results of Dey and Mukerjee (1999a).

Example 2.3 : Suppose it is desired to find an optimal fraction for a 5×2^3 experiment in $N = 20$ runs under a model that includes the mean, all the main effects and the three 2-factor interactions F_1F_j, $2 \le j \le 4$, where the factor F_1 appears at 5 levels and the factors F_j, $2 \le j \le 4$, are each at two levels. The following plan with columns representing the runs is universally optimal under the above stated model. The plan is also saturated.

F_1	0000	1111	2222	3333	4444
F_2	0011	0011	0011	0011	0011
F_3	0101	0101	0101	0101	0101
F_4	0110	0110	0110	0110	0110

Continuing with this line of work, Dey and Suen (2002) used tools from a finite projective geometry to obtain a large number of fractional factorial plans for symmetric m^n factorials, where $m \geq 2$ is a prime or a prime power, which are universally optimal under a model that includes the mean, all main effects and a specified set of 2-factor interactions. This work was extended by Dey *et al.* (2005) to cover asymmetric factorials, again using ideas from a finite projective geometry. See also Chatterjee *et al.* (2002) for some additional results.

3. MORE ON THE CHOICE OF FRACTIONAL FACTORIALS

A regular fraction of symmetric factorials is characterized by a set of defining contrasts. For example, in a $1/2^k$ fraction of a 2^n factorial, the set of defining contrasts consists $2^k - 1$ factorial effects. The resolution of a regular fraction is the number of factors involved in the smallest interaction in the set of defining contrasts. Equivalently, an m^{n-k} plan (*i.e.*, a $1/m^k$ fraction of an m^n factorial) of Resolution R (≥ 3) keeps all treatment contrasts belonging to factorial effects involving at most f factors estimable under the absence of all factorial effects involving R – f or more factors, whenever f satisfies $1 \leq f \leq \frac{1}{2}(R-1)$ (In the literature on applied experimental designs, the value of resolution is generally indicated by a Roman numeral such as III, IV, etc.)

In view of the above definition of resolution, in a Resolution III design (*i.e.*, $R = 3$), no main effect is aliased with another main effect but a main effect is aliased with one or more 2-factor interaction(s). In a fractional factorial plan, a factorial effect involving $f \geq 1$ factors is said to be *clear*, if it is not aliased with another factorial effect involving the same number ($= f$) of factors. Hence, in a Resolution III plan, the main effects are *clear.* Similarly, in a Resolution IV plan, the main effects are clear, but the 2-factor interactions are not clear as, a 2-factor interaction is aliased with another 2-factor interaction.

From the definition of resolution, it is clear that the choice of a fractional factorial plan can be based on the resolution of the plan, fractions with higher resolution being preferred over the ones with smaller resolution. It was later realized that this criterion is not discriminating enough in the sense that fractions with the same resolution can have entirely different properties when judged by other considerations. For example, consider two 2^{7-2} plans, d_1 and d_2, involving factors $F_1, ..., F_7$, with the following sets of defining contrasts:

$$d_1 : I = F_4F_5F_6F_7 = F_1F_2F_3F_4F_6 = F_1F_2F_3F_5F_7,$$
$$d_2 : I = F_1F_2F_3F_6 = F_1F_4F_5F_7 = F_2F_3F_4F_5F_6F_7.$$

Clearly, both the plans are of Resolution IV. However, d_1 has three pairs of 2-factor interactions that are aliased with each other, viz., (F_4F_5, F_6F_7), (F_4F_6, F_5F_7) and (F_4F_7, F_5F_6) while d_2 has six such pairs. Thus, in d_1, the number of clear 2-factor interactions is more than that in d_2 and based on this criterion, d_1 is preferable to d_2, even though both the plans have the same resolution.

Fries and Hunter (1980) proposed a more discriminating criterion than resolution, called *minimum aberration* (MA) for selecting optimal fractions. Unfortunately, barring the work of Franklin (1984, 1985), the MA criterion went unnoticed for nearly a decade. Only in the early nineties, Wu and his collaborators recognized the crucial role of the MA criterion in the selection of optimal fractions. For references to the work of Wu and others on this topic, see Mukerjee and Wu (2006).

For explaining the concept of minimum aberration, we consider regular fractions of the type m^{n-k}, where m is a prime or a prime power. As stated earlier, a regular fraction m^{n-k} is specified by a set of defining contrasts or, a set of defining *words*, where a word consists of letters which are the names of the factors. The number of letters in a word so defined is its *wordlength.* For an m^{n-k} design, let A_i denote the number of words of length i in its set of defining contrasts. The vector

$$W = (A_3, A_4, \ldots, A_n)$$

is called the *wordlength pattern* of the design. Note that, in W, one starts with A_3, because a design with positive A_1 or A_2 is useless. For any two m^{n-k} designs d_1 and d_2, let r be the smallest integer such that $A_r(d_1) \neq A_r(d_2)$. Then d_1 is said to have *less aberration* than d_2 if $A_r(d_1) < A_r(d_2)$. If there is no design with less aberration than d_1, then d_1 has *minimum aberration.*

Example 3.1 : Consider two 3^{5-2} designs d_1 and d_2 involving the factors $F_1, ..., F_5$, with the following sets of defining contrasts:

$$d_1 : I = F_1F_2F_4^2 = F_1F_2^2F_3F_5^2 = F_1F_3^2F_4F_5 = F_2F_3F_4F_5^2$$
$$d_2 : I = F_1F_2F_4^2 = F_1F_3F_5^2 = F_1F_2^2F_3^2F_4F_5 = F_2F_3^2F_4^2F_5.$$

Clearly, both d_1 and d_2 have Resolution III. One can check that d_1 has less aberration than d_2. The implication of this is that in d_1, there are three 2-factor interactions, viz., F_1F_2, $F_1F_4^2$ and $F_2F_4^2$, that get aliased with a main effect whereas, in d_2, there are six 2-factor interactions that get aliased with a main effect. Therefore, if one is not confident about the absence of all 2-factor interactions, then d_1 is preferable to d_2 because d_1 requires less stringent assumptions, even though both d_1 and d_2 have the same resolution.

Deep and interesting theoretical work on minimum aberration designs was carried out by several authors including Cheng and Mukerjee (1998) and Mukerjee and Wu (1999, 2001). A detailed exposition of such results is available in the book by Mukerjee and Wu (2006). The notion of minimum aberration designs was originally studied in the context of regular fractions of symmetric factorials of the type m^{n-k}, where m is a prime or a prime power. Some progress on the asymmetric or mixed factorials has also been made. See Mukerjee and Fang (2000) and Mukerjee and Wu (2001) in this context.

Another important notion in making an optimal choice of a fractional factorial plan is that of *estimation capacity,* introduced by Sun (1993). It was studied in fuller detail by Cheng *et al.* (1999) and Cheng and Mukerjee (1998). For simplicity, consider a 2^n factorial. In such an experiment, there are $\delta = \binom{n}{2}$ two-factor interactions. For $1 \le r \le \delta$, there are $\binom{r}{\delta}$ possible models that include all the main effects and r 2-factor interactions, the remaining $(\delta - r)$ 2-factor interactions and other higher order factorial effects being assumed negligible. For a fixed r, let $E_r(T)$ be the number of models of this kind that can be estimated by a design T. The estimation capacity aims at maximizing $E_r(T)$ for every r $(1 \le r \le \delta)$. A design which achieves this is said to have *maximum estimation capacity.* This criterion therefore aims at selecting a design that retains full information on the main effects, and as much information as possible on the 2-factor interactions in the sense of entertaining maximum possible model diversity, under the assumption of absence of interactions involving three or more factors. Results on designs with maximum estimation capacity for m^{n-k} fractional factorials, where m is a prime or a prime power, can be found in the above stated references and in Chapter 5 of Mukerjee and Wu (2006).

4. BLOCKING

Blocking is often an effective way to reduce the experimental error when the experimental units are not homogeneous. Early work on blocking of fractional factorials centred around orthogonal blocking. In the context of a main effect plan, for instance, with n factors $F_1, \ldots, F_n$, a common technique is to start with an orthogonal array of strength two, having $n + 1$ columns, and then identify one of these columns with the blocking factor and the remaining columns with $F_1, \ldots, F_n$. In every block, this method allocates all the levels of each F_i equally often and as such, this method is successful when the block size is an integral multiple of the number of levels of each F_i. Hence, the method will not work when for example, the block size is less than the number of levels of any F_i. Mukerjee *et al.* (2002) obtained sufficient conditions for a main effect plan to be universally optimal under possibly non-orthogonal blocking and gave a method of construction of such optimal block designs. An example based on their method follows.

Example 4.1 : Consider a $3^4 \times 2$ factorial experiment and suppose a block design for this experiment is desired in blocks of size 2 each. The parameters of interest are all the main effects, all other factorial effects being assumed to be negligible. Clearly, one cannot have an orthogonally blocked design in this situation. The following design for the problem is universally optimal in $\mathcal{D}(9, 2)$, where in the context of an $m_1 \times \ldots \times m_n$ factorial, $\mathcal{D}(b, k)$ denotes the class of all fractions laid out in $b \ge 2$ blocks each of size $k \ge 2$.

Block 1	Block 2	Block 3	Block 4	Block 5
00000	01110	02220	10120	11200
11111	12221	10001	21201	22011

Block 6	Block 7	Block 8	Block 9
12010	20210	21020	22100
20121	01021	02101	00211

For some additional results in this area, see Das and Dey (2004) and Bose and Bagchi (2007).

Another direction of work in the area of blocked fractional factorials is based on the MA and maximum estimation capacity criteria. Optimal block designs for regular symmetric factorials based on the MA criterion was studied in detail by Mukerjee and Wu (1999), making extensive use of finite projective geometries. More results on block designs of regular fractions having maximum estimation capacity were obtained by Cheng and Mukerjee (2001). Much of this work is elegantly described in Chapter 7 of the book by Mukerjee and Wu (2006), where details of the available results, examples and tables of designs can be found.

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