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JOURNAL OF THE INDIAN SOCIETY OF
AGRICULTURAL STATISTICS 65(3) 2011 347-358

Model-Based Direct vs Indirect Estimators for Small Areas

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Received 15 April 2010; Accepted 24 August 2011

SUMMARY

Unbiased direct estimators for small area quantities are usually considered too variable to be of any practical use. This paper describes a class of model-based direct estimators for small area quantities that appears to overcome this objection, in the sense that these estimators are comparable in efficiency to the indirect model-based small area estimators such as empirical best linear unbiased predictor (EBLUP) or Pseudo-EBLUP that are now widely used. There are many practical advantages associated with such model-based direct estimation (MBDE), arising from the fact that they are computed as weighted linear combinations of the actual sample data from the small areas of interest. Note that in this case the weights 'borrow strength' via a model that explicitly allows for small area effects. Empirical results show that the MBDE estimator represents a real alternative to the EBLUP and Pseudo-EBLUP, with the simple MSE estimator associated with the MBDE estimator providing good coverage performance. The results further indicate that the MBDE estimator may be more robust than the EBLUP and Pseudo-EBLUP when the small area model is incorrectly specified.

Keywords: Small area estimation, Model-based direct estimation, Linear mixed model, EBLUP, Pseudo-EBLUP.

1. INTRODUCTION

The dominant paradigm in survey estimation for populations is weighted linear estimation, typically based on linear regression models, while the rapidly expanding field of small area estimation is currently dominated by a model-based predictive approach where the survey weights have little or no relevance. See Rao (2003). Many of the practical advantages of weighted linear estimation are lost when one adopts predictive approach. Perhaps the most important of these are the simplicity of both the estimation process and estimation of mean squared error (MSE). Further, the linear mixed model underpining small area estimation usually assumes that samples are drawn independently across small areas according to a specified sampling design such that the sample design within small areas is ignorable or alternatively selection bias is absent. The estimation based on such models (e.g., empirical best linear unbiased predictors) do not make use of unit level survey weights and the corresponding estimators are not design consistent unless the sampling design is self weighting within small areas. The design-based direct estimators are design consistent but fail to borrow strength from the related areas.

In recent years, some methods proposed in the literature make use of survey weights in model-based small area estimation. Kott (1989) proposed a design consistent estimator, also model unbiased under the simple random effect model with the same assumption of random errors as in linear mixed model defining the EBLUP. He showed that this estimator is robust with respect to model failure under certain conditions and derived an estimator of MSE without including the random effect component. Empirical results show the MSE estimates are quite unstable and even take negative values. Consequently, this approach cannot be used to the compare proposed design-consistent small area estimator and the conventional design-based direct

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estimator. Prasad and Rao (1999) and You and Rao (2002) proposed a model assisted estimator for small area estimation called the pseudo empirical best linear unbiased predictor (pseudo-EBLUP), which depends on the survey weights and remains design consistent as the sample sizes in the small areas increased.

Chandra and Chambers (2009) introduced the calibrated weighting based approach for small area estimation (SAE) and desribed the model-based direct estimation (MBDE) for small areas. This approach uses the calibrated sample weights derived under a population level version of the linear mixed model to define weighted linear small area estimators as well as a simple expression for the MSE. In contrast to design-based direct estimators, MBDE "borrow strength" from other areas via the linear mixed model used in defining the sample weights. There are many practical advantages associated with the MBDE, arising from the fact that the estimators are computed as weighted linear combinations of the actual sample data from the small areas of interest. Perhaps the most important of these are the simplicity of both the estimation process and the estimation of the MSE. Further, the MBDE estimator is easy to interpret and to build into a survey processing system.

This paper studies the performance of EBLUP and two weighting based methods of SAE (i.e., pseudo-EBLUP and MBDE) using real data from the Australian Agricultural and Grazing Industries Survey (AAGIS). It is noteworthy that different SAE methods are evaluated in a realistic situation where underlying model is 'working' model and 'true' model is unknown. The robustness of these estimators are also examined under wrong model specifications. An exploratory data analysis (EDA) is also carried to illustrate how one should proceed for appropriate model specification while doing SAE. The following Section reviews the linear mixed model used in many SAE applications, introduces survey weights on based on this model and descibes the model-based direct estimation (MBDE) for small areas using these weights.The EBLUP and Pseudo-EBLUP are then defined based on same linear mixed model. Section 3 provides illustrative empirical results that compare the EBLUP and Pseudo-EBLUP with MBDE estimators defined under the same model. Finally, Section 4 presents some important issues that arise when a weighting approach is used in SAE and identify related topics that require further attention.

2. SMALL AREA ESTIMATION BASED ON A LINEAR MIXED MODEL

Let U denote a population of size N and let $\mathbf{Y}_U$ denote the N-vector of population values of a characteristic Y of interest, and suppose that our primary aim is estimation of the total $T_y = \sum_U y_j$ of the values in $\mathbf{Y}_U$ (or their mean $m_y = N^{-1} \sum_U y_j$). In order to assist us in this objective, we shall assume that we have 'access' to $\mathbf{X}_U$, an $N \times p$ matrix of values of p auxiliary variables that are related, in some sense, to the values in $\mathbf{Y}_U$. In particular, we assume that the individual sample values in $\mathbf{X}_U$ are known. The non-sample values in $\mathbf{X}_U$ may not be individually known, but are assumed known at some aggregate level. At a minimum, we know the population totals T_x of the columns of $\mathbf{X}_U$. Suppose that the regression of Y on X in the population is linear of form

$$E(\mathbf{Y}_U \mid \mathbf{X}_U) = \mathbf{X}_U\boldsymbol{\beta} \quad \text{and} \quad Var(\mathbf{Y}_U \mid \mathbf{X}_U) = \mathbf{V}_U \qquad (1)$$

where V_U is a positive definite matrix of order N, known up to a multiplicative constant. Without loss of generality, let us arrange the vector $\mathbf{Y}_U$ so that its first n elements correspond to the sample units. Then conformably partition $\mathbf{Y}_U$, $\mathbf{X}_U$ and $\mathbf{V}_U$ according to sample and non-sample units as

$$\mathbf{Y}_U = \begin{bmatrix} \mathbf{Y}_s \\ \mathbf{Y}_r \end{bmatrix}, \mathbf{X}_U = \begin{bmatrix} \mathbf{X}_s \\ \mathbf{X}_r \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} \mathbf{V}_{ss} & \mathbf{V}_{sr} \\ \mathbf{V}_{rs} & \mathbf{V}_{rr} \end{bmatrix}.$$

Here $\mathbf{Y}_s$ is the $n \times 1$ vector defined by the sample values in $\mathbf{Y}_U$, $\mathbf{X}_s$ is the corresponding $n \times p$ matrix of sample values of the auxiliary variable and $\mathbf{V}_{ss}$ is the $n \times n$ component of $\mathbf{V}_U$ associated with $\mathbf{Y}_s$. A subscript of r is used to denote corresponding quantities defined by the $N - n$ non-sample units, e.g. $\mathbf{V}_{rs}$ is the $(N - n) \times n$ matrix defined by $Cov(\mathbf{Y}_r, \mathbf{Y}_s) = \sigma^2\mathbf{V}_{rs}$. Given this set-up, and assuming (1) holds, the vector of weights that defines the Best Linear Unbiased Predictor (BLUP) of the population total of Y is given by

$$\mathbf{w}_s^{BLUP} = \left(w_i^{BLUP}; i \in s\right) = \mathbf{1}_n + \mathbf{H}'\left(\mathbf{X}_U'\mathbf{1}_N - \mathbf{X}_s'\mathbf{1}_n\right) + \left(\mathbf{I}_n - \mathbf{H}'\mathbf{X}_s'\right)\mathbf{V}_{ss}^{-1}\mathbf{V}_{sr}\mathbf{1}_{N-n} \qquad (2)$$

where $\mathbf{I}_n$ is the identity matrix of order n, $\mathbf{1}_N$, $\mathbf{1}_n$, $\mathbf{1}_r$ are vectors of one's with dimensions N, n and $N - n$ respectively, and $\mathbf{H} = \left(\mathbf{X}_s'\mathbf{V}_{ss}^{-1}\mathbf{X}_s\right)^{-1}\mathbf{X}_s'\mathbf{V}_{ss}^{-1}$. See Royall (1976).

The most commonly used class of models in small area inference is the class of linear mixed models, described as follows. Let $\mathbf{Y}_j$ be the $N_j \times 1$ vector of values of variable of interest in small area j and let $\mathbf{X}_j$ be the $N_j \times p$ matrix of values of the auxiliary variables associated with. We consider the following specification for the distribution of $\mathbf{Y}_j$ given $\mathbf{X}_j$:

$$\mathbf{Y}_j = \mathbf{X}_{j\beta} + \mathbf{Z}_j u_j + \mathbf{e}_j. \quad (3)$$

Here $\boldsymbol{\beta}$ is a $p \times 1$ vector of fixed effects, $\mathbf{Z}_j$ is a $N_j \times q$ matrix of known covariates characterising differences between the J small areas, $\mathbf{u}_j$ is a random area effect associated with the j^{th} small area and $\mathbf{e}_j$ is a $N_j \times 1$ vector of individual level random errors. The random vectors $\mathbf{u}_j$ and $\mathbf{e}_j$ are assumed to be independently distributed, with zero means and with variances $Var(\mathbf{u}_j) = \boldsymbol{\Sigma}$ and $Var(\mathbf{e}_j) = \sigma_e^2\mathbf{I}_{N_j}$ respectively, so that the covariance matrix of $\mathbf{Y}_j$ is then $Var(\mathbf{Y}_j) = \mathbf{V}_j = \sigma_e^2\mathbf{I}_{N_j} + \mathbf{Z}_j\boldsymbol{\Sigma}\mathbf{Z}'_j$, which depends on a $k \times 1$ vector of parameters $\theta = (\Sigma, \sigma_e^2)$, usually called the variance components of the model. Finally, it is usually assumed that sampling is uninformative given the values of the auxiliary variables, so the sample data also follow the population model (3). By aggregating the area-specific models (3) over the J small areas, we are led to the population level model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e} \quad (4)$$

where $\mathbf{Y} = (\mathbf{Y}'_1, \ldots\ldots, \mathbf{Y}'_J)'$, $\mathbf{X} = (\mathbf{X}'_1, \ldots\ldots, \mathbf{X}'_J)'$, $\mathbf{Z} = diag(\mathbf{Z}_j; 1 \le j \le J)$, $\mathbf{u} = (\mathbf{u}'_1, \ldots\ldots, \mathbf{u}'_J)'$ and $\mathbf{e} = (\mathbf{e}'_1, \ldots\ldots, \mathbf{e}'_J)'$. The variance-covariance matrix of Y is $\mathbf{V} = diag(\mathbf{V}_j;\ 1 \le j \le J)$. It is assumed that X has full column rank p. This is the general linear mixed model, which includes most of the small area models used in practice (Rao 2003, page 107). Again, we consider the decomposition of $\mathbf{Y}$, $\mathbf{X}$, $\mathbf{Z}$ and $\mathbf{V}$ into sample and non-sample components as mentioned after (1). We use similar notation at the small area level by introducing an extra subscript j to denote small area. For example, we denote by s_j the set of n_j sample units in area j, r_j the corresponding $N_j - n_j$ non-sampled units in the area and put $\mathbf{V}_{jss} = \sigma_e^2\mathbf{I}_{n_j} + \mathbf{Z}_{js}\boldsymbol{\Sigma}\mathbf{Z}'_{js}$ and $\mathbf{V}_{jsr} = \mathbf{Z}_{js}\boldsymbol{\Sigma}\mathbf{Z}'_{jr}$. Given the values of the variance components, it is straightforward to see that (4) is just a special case of the model (1) that underpins the BLUP weights (2). In particular, under (4)

$$\begin{aligned}\mathbf{V}_{ss} &= diag\left\{\mathbf{V}_{jss};\ j = 1,\ldots,J\right\} \\ &= diag\left\{\sigma_e^2\mathbf{I}_{n_j} + \mathbf{Z}_{js}\boldsymbol{\Sigma}\mathbf{Z}'_{js};\ j = 1,\ldots,J\right\}\end{aligned}$$

and

$$\begin{aligned}\mathbf{V}_{sr} &= diag\left\{\mathbf{V}_{jsr};\ j = 1,\ldots,J\right\} \\ &= diag\left\{\mathbf{Z}_{js}\boldsymbol{\Sigma}\mathbf{Z}'_{jr};\ j = 1,\ldots,J\right\}.\end{aligned}$$

In practice the variance components $\boldsymbol{\theta} = (\boldsymbol{\Sigma}, \sigma_e^2)$ that define $\mathbf{V}$ are unknown and must be estimated from the sample data using suitable estimation methods such as maximum likelihood (ML), restricted maximum likelihood (REML) or method of moments. We use a 'hat' to denote an estimate. Given estimated values $\hat{\boldsymbol{\theta}} = \left(\hat{\boldsymbol{\Sigma}}, \hat{\sigma}_e^2\right)$ of the variance components we can obtain estimates $\hat{\mathbf{V}}_{ss}$ and $\hat{\mathbf{V}}_{sr}$ of $\mathbf{V}_{ss}$ and $\mathbf{V}_{sr}$ respectively, and therefore compute 'empirical' BLUP weights, or EBLUP weights as

$$\begin{aligned}\mathbf{w}_s^{EBLUP} &= \left(w_{ji}^{EBLUP};\ i \in s_j;\ j = 1,\ldots,J\right) \\ &= \mathbf{1}_n + \hat{\mathbf{H}}'\left(\mathbf{X}'_U\mathbf{1}_N - \mathbf{X}'_s\mathbf{1}_n\right) \\ &\quad + \left(\mathbf{I}_n - \hat{\mathbf{H}}'\mathbf{X}'_s\right)\hat{\mathbf{V}}_{ss}^{-1}\hat{\mathbf{V}}_{sr}\mathbf{1}_{N-n}\end{aligned} \quad (5)$$

where
$$\begin{aligned}\hat{\mathbf{H}} &= \left(\mathbf{X}'_s\hat{\mathbf{V}}_{ss}^{-1}\mathbf{X}_s\right)^{-1}\mathbf{X}'_s\hat{\mathbf{V}}_{ss}^{-1} \\ &= \left(\sum_j \mathbf{X}'_{js}\hat{\mathbf{V}}_{jss}^{-1}\mathbf{X}_{js}\right)^{-1}\left(\sum_j \mathbf{X}'_{js}\,\hat{\mathbf{V}}_{jss}^{-1}\right).\end{aligned}$$

It is easy to see that these 'EBLUP' weights (5) are the empirical version of the BLUP weights (2) under (4).

The model-based direct estimator (MBDE) of the j^{th} small area mean of Y (i.e., $m_{yj} = N_j^{-1}\sum_{U_j} y_j$) is the direct estimator of this quantity based on the EBLUP weights (5). That is, it is defined as

$$\hat{m}_{yj}^{MBDE} = \left\{\sum_{s_j} w_i^{EBLUP}\right\}^{-1}\sum_{s_j} w_i^{EBLUP}y_i \quad (6)$$

where the weights used in (6) are those associated with the sample units in small area j in (5). We refer to (6) as a direct estimator because it is a weighted mean of the sample data from the small area of interest. However, this does not mean that it can be calculated just using these data. Note that the EBLUP sample weights (5) will be a function of the data from the entire sample. That is, they 'borrow strength' from other areas through the model (4).

An important consideration in small area estimation is estimation of the mean squared error (MSE) of the small area estimator. We can easily adapt straightforward methods of MSE estimation for population level estimators to estimation of the MSE of (6). Well known results indicate that robust model-based methods as well as appropriately conditioned design-based methods lead to MSE estimators $v(\hat{m}_y) = \sum_s w_i^2 (y_i - \hat{y}_i)^2$ + lower order terms, where $\hat{y}_i$ denotes the fitted value for y_i under the linear model implied by the calibration constraints. In order to estimate the mean squared error of (6), we note that the implied population level model (4) includes random area effects and so one needs to consider whether it is appropriate to condition on these effects when estimating this MSE. For example, the rather complicated MSE estimator (12) of the EBLUP (11) does involve this conditioning. On the other hand, estimation of the MSE of (6) is straightforward if we do not condition on random area effects, treat the EBLUP weights (5) as fixed and use standard methods for estimating the MSE of a weighted linear estimator of a domain mean under the population model (1). See Royall and Cumberland (1978). The choice between these two approaches is largely philosophical and depends on how much one 'believes' the linear mixed model (4). We write down a first order approximation to prediction variance for the area j weighted mean (6) as

$$Var(\hat{m}_{yj}^{MBDE} - m_{yj}) = Var\left\{\left(\sum_{s_j} w_i\right)^{-1}\left(\sum_{s_j} w_i y_i\right) - N_j^{-1}\left(\sum_{s_j} y_i + \sum_{r_j} y_i\right)\right\}$$

$$\approx N_j^{-2}\left(\sum_{s_j} a_i^2 Var(y_i) + \sum_{r_j} Var(y_i)\right) \quad (7)$$

where $a_i = \left(\sum_{s_j} w_k\right)^{-1}\left(N_j w_i - \sum_{s_j} w_k\right)$. A robust model-based estimate of (7) is obtained by substituting the squared residual $(y_i - x_i'\hat{\boldsymbol{\beta}})^2$ for $Var(y_i)$ in the first (leading) term on the right hand side of (7). If these squared sample residuals are also used to estimate the second term, the resulting estimator of (7) is

$$v(\hat{m}_{yj}^{MBDE}) = \sum_{s_j} \lambda_i (y_i - x_i'\hat{\boldsymbol{\beta}})^2 \quad (8)$$

where $\lambda_i = N_j^{-2}\left(a_i^2 + (N_j - n_j)/(n_j - 1)\right)$. Using (8) to estimate the prediction mean squared error of $\hat{m}_{yj}^{MBDE}$ implicitly assumes that this weighted mean is unbiased for m_{yj}. However, this is not generally the case, since $E(\hat{m}_{yj}^{MBDE} - m_{yj}) \approx (\hat{m}_{xj}^{MBDE} - m_{xj})'\boldsymbol{\beta}$ under (4), where $\hat{m}_{xj}^{MBDE}$ denotes the weighted average of the sample values of the auxiliary variables in area j. Calibration on X ensures that this term vanishes at population level, but not necessarily at small area level. A simple estimate of this bias is

$$b(\hat{m}_{yj}^{MBDE}) = (\hat{m}_{xj}^{MBDE} - m_{xj})'\hat{\boldsymbol{\beta}}. \quad (9)$$

Our suggested estimator of the mean squared error of (6) is therefore

$$m\hat{s}e(\hat{m}_{yj}^{MBDE}) = v(\hat{m}_{yj}^{MBDE}) + \left(b(\hat{m}_{yj}^{MBDE})\right)^2 \quad (10)$$

Note that one could alternatively 'bias correct' $\hat{m}_{yj}^{MBDE}$ directly using $b(\hat{m}_{yj}^{MBDE})$. However, this is not recommended since this correction increases the variability of our estimator much more than it reduces its bias. Using it in (10) is a more conservative, and safer, approach.

Assuming model (3) holds, the EBLUP for the j^{th} small area mean m_{yj} (Prasad and Rao 1990) is

$$\hat{m}_{yj}^{EBLUP} = f_j \bar{Y}_{js} + (1 - f_j)[\bar{\mathbf{X}}'_{jr}\hat{\boldsymbol{\beta}} + \bar{\mathbf{Z}}'_{jr}\hat{\boldsymbol{\Sigma}}\,\mathbf{Z}'_{js}\hat{\mathbf{V}}_{jss}^{-1}(\mathbf{Y}_{js} - \mathbf{X}_{js}\hat{\boldsymbol{\beta}})] \quad (11)$$

where $f_j = n_j / N_j$ and $\bar{\mathbf{X}}_{jr}$ and $\bar{\mathbf{Z}}_{jr}$ are vectors of means for the $N_j - n_j$ non-sampled units in small area j. Note that the MBDE (6) is not the same as EBLUP (11), even though both sum to the same population level EBLUP. This is because there is no unique representation of (11) as a weighted mean of the sample data values from small area j.

MSE estimation for (11) is usually carried out using the theory described in Prasad and Rao (1990). Although this MSE estimator is somewhat complicated, it works well under (3). However, when (3) fails it can be misleading. Following Prasad and Rao (1990) an approximately unbiased estimator of the MSE of (11) is

$$v(\hat{m}_{yj}^{EBLUP}) = (1-f_j)^2\left[g_{1j}(\hat{\boldsymbol\theta}) + g_{2j}(\hat{\boldsymbol\theta}) + 2g_{3j}(\hat{\boldsymbol\theta})\right] + N_j^{-1}(1-f_j)\hat{\sigma}_e^2 \quad (12)$$

where

$$g_{1j}(\hat{\boldsymbol\theta}) = \bar{\mathbf{Z}}'_{jr}\left(\hat{\boldsymbol\Sigma} - \hat{\boldsymbol\Sigma}\mathbf{Z}'_{js}\hat{\mathbf{V}}_{jss}^{-1}\mathbf{Z}_{js}\hat{\boldsymbol\Sigma}\right)\bar{\mathbf{Z}}_{jr},$$

$$g_{2j}(\hat{\boldsymbol\theta}) = \left(\bar{\mathbf{X}}'_{jr} - b'_j\mathbf{X}_{js}\right)\left(\sum\nolimits_j \mathbf{X}'_{js}\hat{\mathbf{V}}_{jss}^{-1}\mathbf{X}_{js}\right)^{-1}\left(\bar{\mathbf{X}}'_{jr} - \mathbf{b}'_j\mathbf{X}_{js}\right)'$$

$$g_{3j}(\hat{\boldsymbol\theta}) = tr\left\{\left(\nabla\mathbf{b}'_j\right)\hat{\mathbf{V}}_{jss}\left(\nabla\mathbf{b}_j\right)v(\hat{\boldsymbol\theta})\right\}$$

with $\mathbf{b}'_j = \bar{\mathbf{Z}}'_{jr}\hat{\boldsymbol\Sigma}\mathbf{Z}'_{js}\hat{\mathbf{V}}_{jss}^{-1}$, $\nabla\mathbf{b}'_j = \partial\mathbf{b}'_j/\partial\boldsymbol\theta$ and where $v(\hat{\boldsymbol\theta})$ is the estimate of the asymptotic covariance matrix of $\hat{\boldsymbol\theta}$ defined by the inverse of the relevant observed information matrix. See Prasad and Rao (1990) and Rao (2003, pp. 107-110).

Let π_{ji} denote the sample inclusion probability of population unit *i* in small area *j*. A design-based direct estimator for area *j* mean m_{yj} is

$$\hat{m}_{yj}^{\pi} = \left(\sum\nolimits_{i\in s_j}\pi_{ji}^{-1}\right)^{-1}\sum\nolimits_{i\in s_j}\pi_{iji}^{-1}y_{ji} = \sum\nolimits_{i\in s_j}\tilde{w}_{ji}y_{ji}$$

with $\sum\nolimits_{i\in s_j}\tilde{w}_{ji} = 1. \quad (13)$

The estimator (13) uses sampling weights and also design consistent but fails to borrow strength. One alternative approach in the literature is the pseudo-EBLUP (You and Rao 2002, Rao 2003, section 7.2.7), which is a model assisted method of small area estimation. Recollect from (11) that the EBLUP is defined by replacing the unknown area *j* mean m_{yj} by an estimate of its expected value given the observed sample values of *Y* in area *j* and the area *j* values of *X*. The pseudo-EBLUP is then defined by replacing m_{yj} by an estimate of its expected value given the value of its design-consistent estimate $\hat{m}_{yj}^{\pi}$ and the area *j* values of *X*. That is, under (3) the pseudo-EBLUP of m_{iy} (You and Rao 2002, Rao 2003, section 7.2.7) is

$$\hat{m}_{yj}^{psuedoEBLUP} = f_j\bar{Y}_{js} + (1-f_j)\left\{\bar{\mathbf{X}}'_{jr}\hat{\boldsymbol\beta}_{\tilde{w}} + \bar{\mathbf{Z}}'_{jr}\hat{\boldsymbol\Sigma}_{\tilde{w}}\bar{\mathbf{Z}}_{js\tilde{w}}\hat{\mathbf{V}}_{jss\tilde{w}}^{-1}(\hat{m}_{yj}^{\pi} - \hat{m}_{xj}'^{\pi}\hat{\boldsymbol\beta}_{\tilde{w}})\right\} \quad (14)$$

with $\hat{\mathbf{V}}_{jss\tilde{w}} = \bar{\mathbf{Z}}'_{js\tilde{w}}\hat{\boldsymbol\Sigma}_{\tilde{w}}\bar{\mathbf{Z}}_{js\tilde{w}} + \hat{\sigma}_{e\tilde{w}}^2\sum\nolimits_{i\in s_j}\tilde{w}_{ji}^2$. Here $\hat{\boldsymbol\beta}_{\tilde{w}}$, $\hat{\boldsymbol\Sigma}_{\tilde{w}}$ and $\hat{\sigma}_{e\tilde{w}}^2$ are pseudo-maximum likelihood estimates based on the weights $\tilde{w}_{ij}$ and $\mathbf{Z}_{js\tilde{w}}$ and $\hat{m}_{xj}^{\pi}$ are design-consistent estimates of $\bar{\mathbf{Z}}_j$ and m_{xj} that are defined in exactly the same way as $\hat{m}_{iy}^{\pi}$ above.

An approximately model-unbiased estimator of the MSE of pseudo-EBLUP is

$$v(\hat{m}_{yj}^{psuedoEBLUP}) = (1-f_j)^2\left\{g_{1j\tilde{w}}(\hat{\boldsymbol\theta}_{\tilde{w}}) + g_{2j\tilde{w}}(\hat{\boldsymbol\theta}_{\tilde{w}}) + 2g_{3j\tilde{w}}(\hat{\boldsymbol\theta}_{\tilde{w}})\right\} + N_j^{-1}(1-f_j)\hat{\sigma}_{e\tilde{w}}^2 \quad (15)$$

where

$$\mathbf{g}_{1j\tilde{w}}(\hat{\boldsymbol\theta}_{\tilde{w}}) = \bar{\mathbf{Z}}'_{jr}\left(\hat{\boldsymbol\Sigma}_{\tilde{w}} - \hat{\boldsymbol\Sigma}_{\tilde{w}}\bar{\mathbf{Z}}_{js\tilde{w}}\hat{\mathbf{V}}_{jss\tilde{w}}^{-1}\bar{\mathbf{Z}}'_{js\tilde{w}}\hat{\boldsymbol\Sigma}_{\tilde{w}}\right)\bar{\mathbf{Z}}_{jr},$$

$$g_{2j\tilde{w}}(\hat{\boldsymbol\theta}_{\tilde{w}}) = \left(\bar{\mathbf{X}}'_{jr} - \mathbf{b}'_{j\tilde{w}}\hat{\mathbf{m}}_{xj}'^{\pi}\right)v(\hat{\boldsymbol\beta}_{\tilde{w}})\left(\bar{\mathbf{X}}'_{jr} - \mathbf{b}'_{j\tilde{w}}\hat{m}_{xj}'^{\pi}\right)',$$

and

$$g_{3j\tilde{w}}(\hat{\boldsymbol\theta}_{\tilde{w}}) = tr\left\{\left(\nabla\mathbf{b}'_{j\tilde{w}}\right)\hat{\mathbf{V}}_{jss\tilde{w}}\left(\nabla\mathbf{b}_{j\tilde{w}}\right)v(\hat{\boldsymbol\theta}_{\tilde{w}})\right\}$$

with

$$\mathbf{b}'_{j\tilde{w}} = \bar{\mathbf{Z}}'_{jr}\hat{\boldsymbol\Sigma}_{\tilde{w}}\bar{\mathbf{Z}}_{js\tilde{w}}\hat{\mathbf{V}}_{jss\tilde{w}}^{-1},$$

$$\nabla\mathbf{b}'_{j\tilde{w}} = \partial\nabla\mathbf{b}'_{j\tilde{w}}/\partial\boldsymbol\theta = \left[\partial\mathbf{b}'_{j\tilde{w}}/\partial\sigma_e^2, \partial\mathbf{b}'_{j\tilde{w}}/\partial\Sigma\right]$$

and where $v(\hat{\boldsymbol\theta}_{\tilde{w}})$ is the estimate of the asymptotic covariance matrix of $\hat{\boldsymbol\theta}_{\tilde{w}}$.

Note that the pseudo-EBLUP (14) is essentially motivated by the idea of estimating the area *j* mean by its conditional expectation under (3) given the value of the usual design-consistent estimator (13) for this quantity. As such, this is indirect estimators like the EBLUP. By construction, (6) is a *direct* estimator of m_{yj}, because it is a weighted mean of the area *j* sample values of *Y*. In contrast, (11) and (14) are the *indirect* estimators because they cannot be expressed in this form, being a weighted mean of *all* the sample values of Y. Clearly, under (3), both MBDE (6) and pseudo-EBLUP (14) will not be as efficient as the EBLUP. The pseudo-EBLUP estimator relies on the design consistency of $\hat{m}_{jy}^{\pi}$ for robustness. Indeed relying on a large sample property of a small sample statistic seems rather optimistic. However, (6) has the advantage of being a simple weighted mean of the area i sample data, and therefore should be more robust to misspecification of (3) than the more model-dependent estimator (11) as well as model-assisted estimator (14). Direct estimators like (6), i.e. estimators that are defined as weighted averages of the sample data from the small areas of interest, have a number of practical advantages, including simplicity of construction and aggregation consistency.

3. EMPIRICAL RESULTS

In this section we illustrate the performance of various small area estimation methods via design-based simulation. Our basic data come from the same sample of 1652 Australian broadacre farms data from the Australian Agricultural and Grazing Industries Survey (AAGIS) that were used in the simulation study reported in Chambers (1996). Here however we used these sample farms to generate a target population of 81982 farms by sampling with replacement from them with probabilities proportional to their sample weights. We then drew 1000 independent stratified random samples from this (fixed) population, with total sample size in each simulation equal to the original sample size (1652) and with strata defined by the 29 different Australian broadacre agricultural regions. Sample sizes within these strata were fixed to be the same as in the original sample. Note that these varied from a low of 6 to a high of 117, allowing an evaluation of the performance of different small area estimation methods across a range of realistic small area sample sizes. Table 1 shows the stratum population and sample sizes for this population.

We considered the 29 regions as small areas and total cash costs (A$) of the farm business over the surveyed year (i.e., TCC) as variable of interest. Our aim was to estimate the average total cash costs in each of the 29 different regions. In doing so, we used the fact that these regions can be grouped into three zones (Pastoral, Mixed Farming, and Coastal, see Fig. 1), with farm area (hectares) known for each farm in the population. This auxiliary variable is referred to as Size in what follows.

Table 1. Regional population and sample sizes

Region	N	n	Region	N	n
1	79	6	16	2683	60
2	115	10	17	2689	60
3	189	30	18	2847	34
4	330	25	19	3056	74
5	388	36	20	3139	51
6	465	19	21	3910	73
7	604	36	22	4486	117
8	729	40	23	4550	80
9	737	30	24	4587	95
10	964	30	25	5368	83
11	1586	51	26	5528	103
12	1778	62	27	6489	108
13	1984	55	28	6980	81
14	2182	47	29	10933	77
15	2607	79			

The linear relationship between the target variable (TCC) and Size was rather weak in the original sample data with R^2 and Root MSE value of 0.05 and 970358

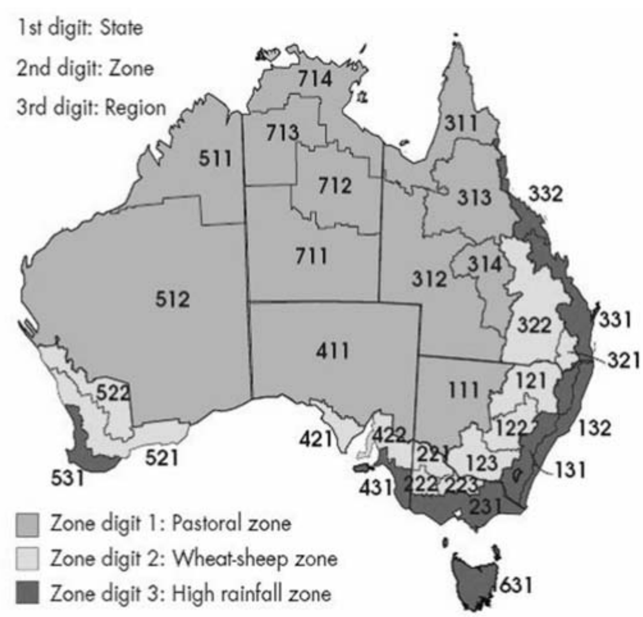

Fig. 1. Map of Australian broadacre zones and farming regions

respectively. Further, in the original sample data there were two massive outlier points. When model was fitted by excluding these two outliers, linear relationship between TCC and Size improved with comparatively larger value of R^2 (=0.24) and smaller Root MSE (=410044). However, these outlier data points were retained while generating the population as well as in small area estimation to examine the performance to this effect. This relationship between TCC and Size further improved when separate linear models were fitted within six post strata. These post-strata were defined by splitting each zone into small farms (farm area less than zone median) and large farms (farm area greater than or equal to zone median). The statistical details for model fitted with these post-strata given in Table 2 clearly reflect this.

Table 2. Statistical details for model fitted with post-strata

Source	DF	Sum of Squares	F Ratio	Prob > F
Size	1	6.262e+11	5.3970	0.0203
Stratum	5	5.625e+13	96.9683	0.0001
Size*Stratum	5	6.866e+13	118.3618	0.0001
R^2 = 0.48 and Root MSE = 340620				

The model fitting was also done to test whether linear mixed models are suitable for AAGIS data. The results in Table 3 show that region specific intercepts are significantly different for different regions. The model further improved when we used region specific slope for Size along with region specific intercepts. Consequently, the matrix X of auxiliary variable values in linear mixed model was defined so as to include an effect for Size, effects for the post-strata and effects for interactions between Size and the post strata. Two different specification for X (corresponding to whether an intercept was included or not) and two different specifications for Z (corresponding to whether a random slope on Size was included or not) were used to specify linear mixed model and hence the different small area estimation methods based on this model. These four specifications are set out in Table 5.

Table 3. Statistical details for model fitted with Region-specific intercepts

Source	DF	Sum of Squares	Mean Square	F Ratio	Prob > F
Region	28	7.006e+12	2.5e+11	2.2159	0.0003
Error	1621	1.83e+14	1.13e+11		
C. Total	1649	1.9e+14			
R^2=0.037, Root MSE=336031					

Table 4. Statistical details for model fitted with Region-specific intercepts and Region-specific slopes with Size variable

Source	DF	Sum of Squares	F Ratio	Prob > F
Region	28	7.28e+13	42.3776	0.0001
Size	1	5.144e+11	8.3842	0.0038
Region*Size	28	8.536e+13	49.6857	0.0001
R^2=0.486, Root MSE=247702				

Table 5. Different mixed model specifications considered in the simulations

Model	Model Type	X	Z
I	Random Intercepts	Intercept included	Intercept only
II	Random Slopes	Intercept included	Intercept + Size
III	Random Slopes with fixed intercept	Intercept included	Size only
IV	Random Slopes with zero intercept	Intercept excluded	Size only

We use the Akaike Information Criterion (AIC) evaluated as $AIC = -2logLik + 2k$, where k is the number of parameters in the model and *logLik* is the log-likelihood of the model. The smaller the value of *AIC* is better. We also use likelihood ratio (LR) test as criteria to find the best model. The values of test criterions obtained from ANOVA function in R for models I and II are set out in Table 6. The *p*-value for the test statistics comparing models I and II is about 4%, indicates that model II is significantly better fit than model I. The AIC criterion is nearly same for both models but marginally smaller for model II.

For the AAGIS farm data considered in this analysis, models I and II are appropriate (with II fitting

Table 6. Analysis of variance results for comparing two linear mixed models

Model	DF	AIC	logLik	LR	p-value
I	14	49992	–24982		
II	16	49989	–24979	6.43	0.04

marginally better) while models III and IV are badly specified. We use REML estimates of random effects parameters throughout, obtained via the *lme* function in R (Bates and Pinheiro 1998). For each model, three different estimators of the 29 regional means were computed, along with corresponding estimators of their MSE. These were the EBLUP (11) with MSE estimator (12), referred to as EBLUP below; the MBDE estimator (6) based on EBLUP weights (5) and with MSE estimator (10), referred to as MBDE below; the Pseudo-EBLUP (14) with MSE estimator (15), referred to as Pseudo-EBLUP below.

Three measures of estimation performance were computed using the estimates generated in the simulation study. These were the relative bias (RB) and the relative root mean squared error (RMSE), both expressed as percentages, of regional mean estimates and the coverage rate of nominal 95 per cent confidence intervals for regional means. Table 7 presents the distribution of values of these measures (all computed over the 29 regions) generated by EBLUP, Pseudo-EBLUP and MBDE under models I – IV.

In Table 7 we note that the average relative biases under MBDE are smaller than those under EBLUP for all models except model IV. However, the average relative bias of Pseudo-EBLUP is marginally smaller than both MBDE and EBLUP under model II and III. The average relative RMSEs for MBDE are marginally higher than those for EBLUP under models I and II and smaller for models III and IV. Indeed, under model I the average relative RMSEs of Pseudo-EBLUP is marginally lower than both MBDE and EBLUP. Average coverage rates for MBDE are relatively higher than those for EBLUP and Pseudo-EBLUP under all models. It is evident that for correct specifications of the working models (i.e. model I and II), between EBLUP and Pseudo-EBLUP, the EBLUP is marginal better. However, for wrong model specifications Pseudo-EBLUP is worse (e.g., models IV) and still dominated by the EBLUP. In contrast, MBDE is performing well and generates robust sets of small area estimates. Although neither approach dominates, it seems clear that, MBDE is more robust to model misspecification than the indirect estimators (i.e., EBLUP and Pseudo-EBLUP).

Figs. 2 – 4 show the region-specific performances generated by EBLUP, Pseudo-EBLUP and MBDE (ordered by increasing population size). Fig. 2 shows the better relative bias performances of all methods under models I and II and their worse relative bias performance under model IV. Fig. 3 shows that the relative RMSEs of regional estimates generated by MBDE are comparable with those generated under EBLUP, with neither approach dominating. The Pseudo-EBLUP is more volatile under model IV. Overall, with the exception of two regions (3 and 21), it seems that MBDE under model II performs marginally better overall.

In the two regions (3 and 21) where MBDE fails, inspection of the population and sample data indicated that this is because of a few outlying estimates. In fact, the outlying values of MBDE for region 21 are all caused by the presence of a single massive outlier (TCC > A$30,000,000) in the original sample. This outlier was included in the simulation population (twice) and then selected (in one case, twice) in 37 of the 1000 simulation samples. Recall that in explanatory data analysis described above these outlier data points were very well identified but were retained in small area estimation to explore the performance of different methods in such cases. If we discard the outlier driven estimates in regions 3 and 21 then the MBDE approach seems the method of choice for regional estimation in our simulation study. This is confirmed when we return to Table 7 and now consider the columns containing the median values of relative mean error and relative RMSE.

Fig. 4 summarizes region-specific variation in the nominal 95 per cent confidence interval coverage rates generated by EBLUP, Pseudo-EBLUP and MBDE. If we ignore the outlier driven results for regions 3 and 21, the results displayed in Fig. 4 show that MBDE approach gives marginally better coverage rates under Models I and II. A close look at these results also indicates that in the event of model misspecification (e.g. under Models III and IV) the MBDE coverage rate is more robust.

4. CONCLUSIONS AND FURTHER RESEARCH

The empirical results reported in the previous section are evidence that the MBDE estimator perform well and represents a real alternative to the indirect estimators (i.e., model-dependent EBLUP estimator and model-assisted Pseudo-EBLUP estimator), with the

Table 7. Distribution of performance measures for EBLUP, MBDE and Pseudo EBLUP methods of small area estimation

Model	Method	Min	Q1	Mean	Median	Q3	Max
Relative Bias,%							
I	EBLUP	–10.75	–4.70	4.24	1.55	6.77	54.49
	MBDE	–18.49	–2.33	–2.49	–0.82	–0.11	03.91
	Pseudo EBLUP	–9.96	–5.05	5.12	2.46	7.79	61.83
II	EBLUP	–7.92	–3.02	2.98	0.61	6.12	32.20
	MBDE	–17.21	–2.20	–2.13	–0.47	0.60	04.07
	Pseudo EBLUP	–13.28	–5.91	1.96	–1.19	5.53	52.53
III	EBLUP	–13.24	–6.42	4.52	1.95	11.31	63.89
	MBDE	–25.44	–7.54	–3.84	0.13	2.11	07.50
	Pseudo EBLUP	–15.77	–6.47	3.49	0.10	6.57	65.45
IV	EBLUP	–24.86	–11.12	1.17	–2.63	7.26	64.06
	MBDE	–25.44	–3.25	2.20	2.06	9.39	27.39
	Pseudo EBLUP	–34.43	–20.50	25.45	–4.56	54.81	267.99
Relative RMSE,%							
I	EBLUP	7.41	11.26	19.92	15.74	21.32	103.99
	MBDE	7.83	11.36	20.56	14.45	21.61	110.95
	Pseudo EBLUP	7.54	11.86	19.86	16.40	20.48	95.61
II	EBLUP	6.90	11.27	19.87	16.40	22.38	53.84
	MBDE	7.51	10.28	20.15	13.16	17.22	110.91
	Pseudo EBLUP	7.96	11.78	23.04	17.43	28.34	102.41
III	EBLUP	6.17	14.79	23.89	19.94	28.39	64.94
	MBDE	7.69	11.06	21.14	14.44	21.24	110.96
	Pseudo EBLUP	5.61	12.86	24.02	18.23	26.41	101.82
IV	EBLUP	6.15	13.01	23.38	19.73	28.22	65.09
	MBDE	7.47	13.39	22.35	20.61	25.43	68.89
	Pseudo EBLUP	5.70	15.55	54.18	29.36	69.66	345.44
Coverage rate							
I	EBLUP	0.22	0.90	0.90	0.98	1.00	1.00
	MBDE	0.48	0.92	0.92	0.94	0.98	1.00
	Pseudo EBLUP	0.24	0.89	0.89	0.97	1.00	1.00
II	EBLUP	0.11	0.86	0.85	0.92	0.98	1.00
	MBDE	0.40	0.93	0.93	0.95	0.99	1.00
	Pseudo EBLUP	0.26	0.89	0.89	0.95	0.99	1.00
III	EBLUP	0.08	0.56	0.69	0.72	0.89	1.00
	MBDE	0.44	0.94	0.94	0.96	0.99	1.00
	Pseudo EBLUP	0.24	0.79	0.83	0.87	0.95	1.00
IV	EBLUP	0.08	0.50	0.65	0.71	0.92	1.00
	MBDE	0.88	0.95	0.97	0.98	1.00	1.00
	Pseudo EBLUP	0.07	0.48	0.69	0.80	0.95	1.00

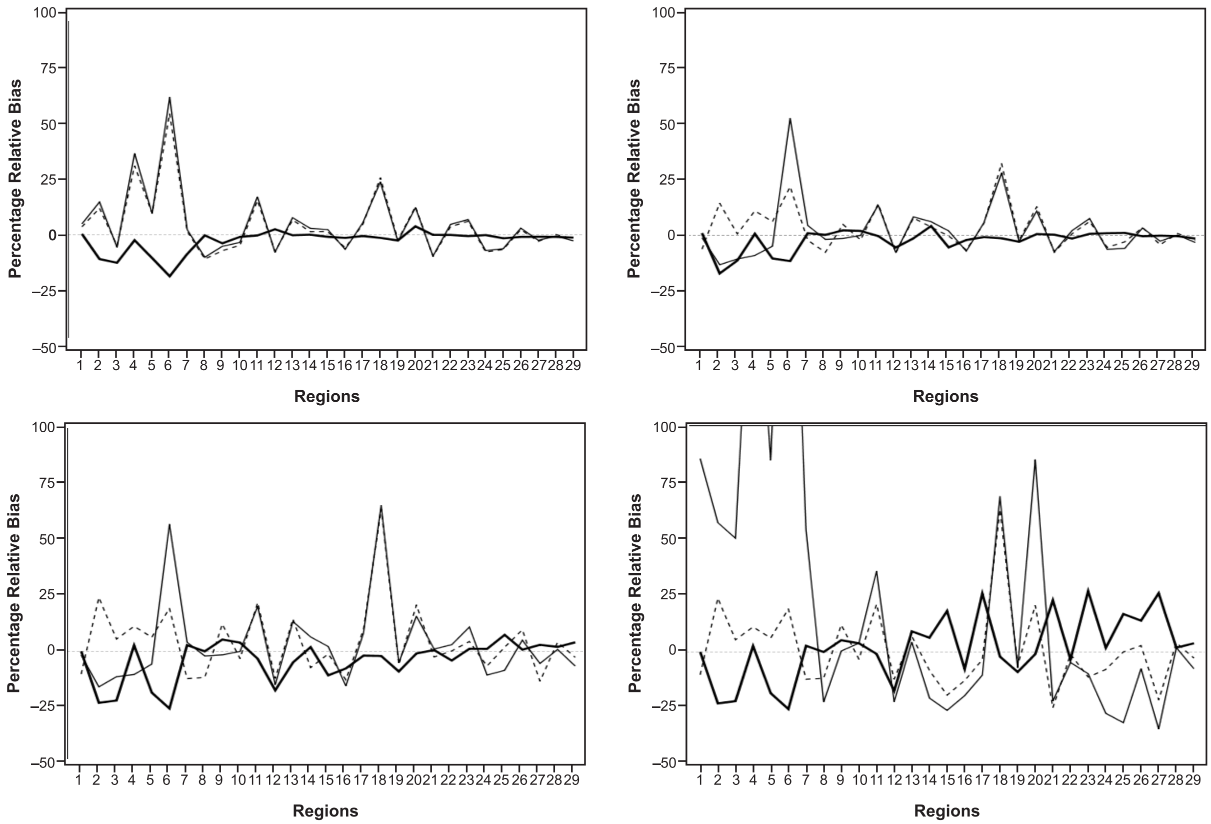

Fig. 2. Region-specific relative mean errors for EBLUP (dashed line), Pseudo EBLUP (thin line) and MBDE (solid line) under models I (top left), II (top right), III (bottom left) and IV (bottom right). Regions are ordered in terms of increasing population size.

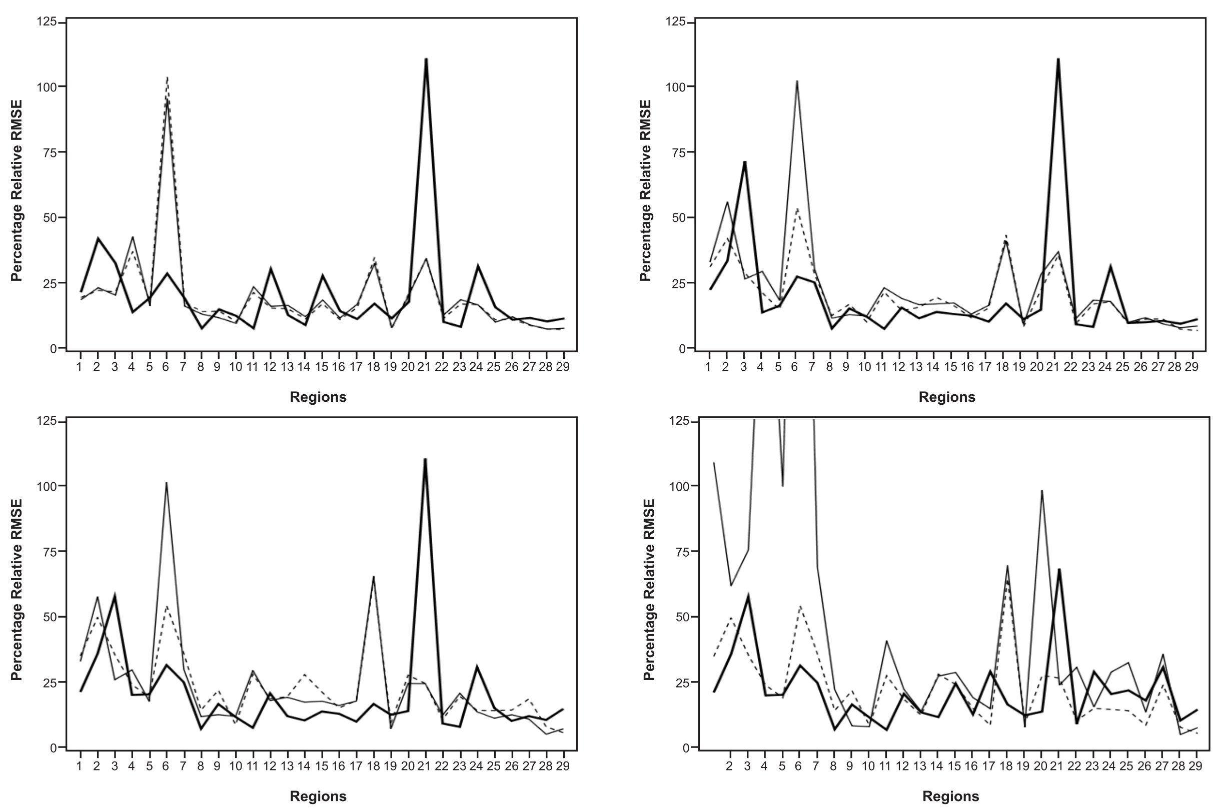

Fig. 3. Region-specific relative RMSEs for EBLUP (dashed line), Pseudo EBLUP (thin line) and MBDE (solid line) under models I (top left), II (top right), III (bottom left) and IV (bottom right). Regions are ordered in terms of increasing population size.

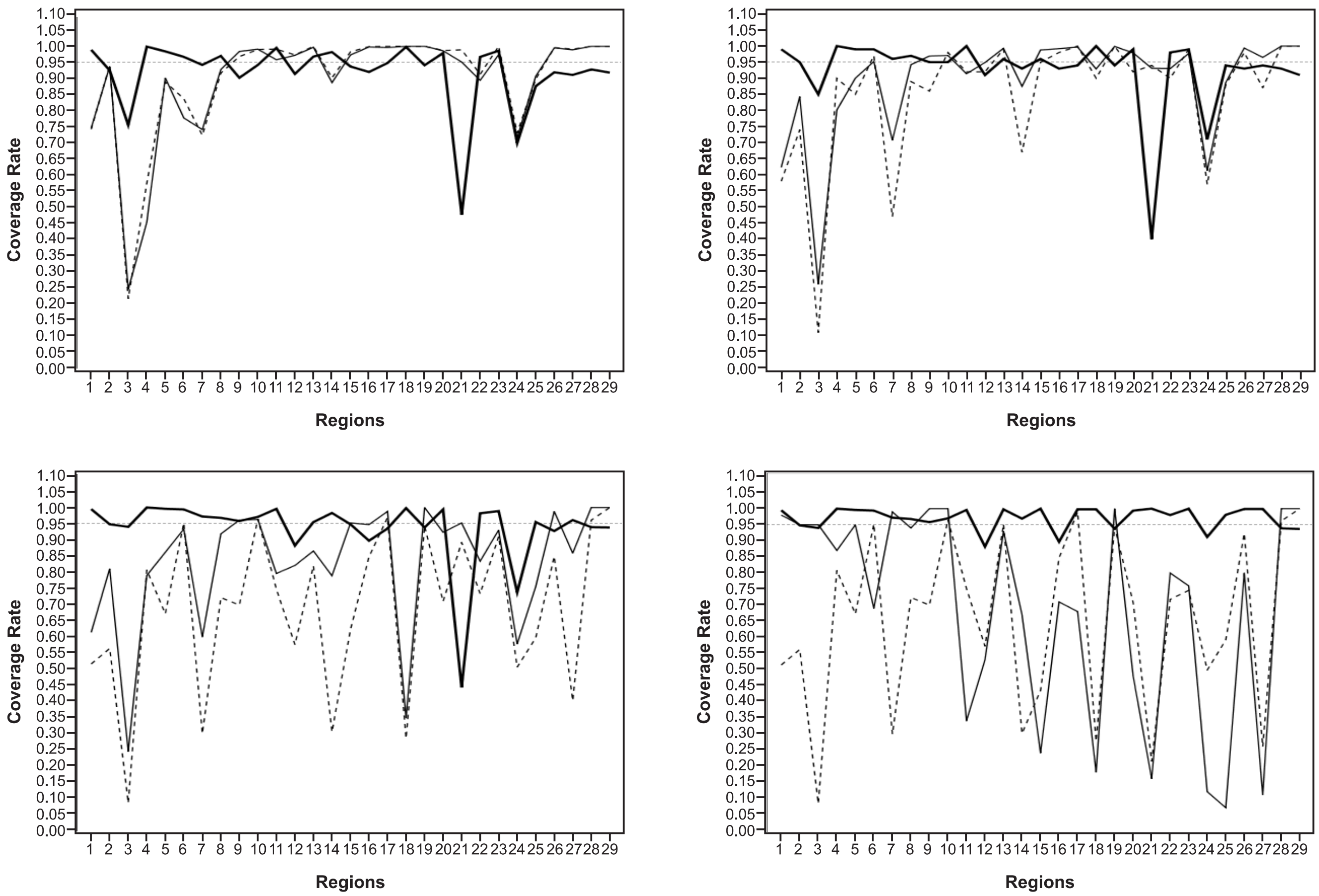

Fig. 4. Region-specific coverage rates for EBLUP (dashed line), Pseudo EBLUP (thin line) and MBDE (solid line) under models I (top left), II (top right), III (bottom left) and IV (bottom right). Regions are ordered in terms of increasing population size.

associated easy to calculate MSE estimator providing good coverage performance. Furthermore, they indicate that the MBDE approach may be more robust than the EBLUP and Pseudo-EBLUP in the realistic situation where linear mixed model (3) is a working model, rather than the (unknown) true model. These results should not be taken as a blanket recommendation for MBDE over EBLUP or Pseudo-EBLUP, however. These results are indication of what happened in practical situation.

There are issues that impact on the utility of the MBDE estimator that remain unresolved. For example, negative weights, can lead to impossible (i.e. negative) estimates. Since such values are easily identified, they should not cause problems in real life. However, the problem remains of how to modify the weights to ensure they are strictly positive. Methods for dealing with negative weights under 'standard' regression models have been discussed in the literature (Huang and Fuller 1978, Deville and Sarndal 1992) but their application in the context of mixed models remains to be explored. Further, the MBDE estimator being a linear combination of just the small area data values is more susceptible to outliers than the EBLUP or Pseudo-EBLUP estimator.

ACKNOWLEDGEMENT

The author gratefully acknowledges to Professor Ray Chambers for his constructive comments and suggestions during analysis of this work.

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