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Possibility Theory with an Application to Volatility Estimation

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SUMMARY

In this paper, we obtain the closed form expressions for covariance of different types of fuzzy numbers including triangular, trapezoidal, parabolic and Gaussian. An estimate of the fuzzy parameter is obtained by minimizing the possibilistic mean square error and the method is applied to fuzzy volatility estimation problem. We also study the recursive estimation for some fuzzy volatility models with asymmetric innovations using possibility theory.

Keywords: Fuzzy volatility models, Possibilistic mean square error, Recursive estimates, Triangular fuzzy number, Trapezoidal fuzzy number.

1. INTRODUCTION

Possibility theory is a mathematical theory for dealing with certain types of uncertainty and is an alternative to probability theory. Zadeh (1978) introduced possibility theory as an extension to his fuzzy set theory. The use of fuzzy set theory as a methodology for modeling and analyzing certain time series is of particular interest to a number of researchers due to fuzzy set theory's ability to quantitatively and qualitatively model those problems which involve vagueness and imprecision. Fuzzy time series models provide a new avenue to deal with subjectivity observed in most financial time series models. Most of the fuzzy financial models developed so far have generally, been confined to modeling parameters by fuzzy numbers such as Triangular Fuzzy Number (Δ F.N.) or Trapezoidal Fuzzy Number (T.F.N.). The main reason for using triangular and trapezoidal is to incorporate asymmetry. Recently there has been growing interest in using fuzzy numbers and associated fuzzy inference in finance and economics (see for example Buckley 2004, Thavaneswaran *et al.* 2007, 2009, Thiagarajah *et al.* 2007). Thavaneswaran *et al.* (2009) have recently used the fuzzy estimate and demonstrated the superiority of the fuzzy forecasts over the minimum mean square error forecasts. In this paper, we first provide some illustrative examples for obtaining closed form expressions for covariance of some fuzzy numbers. Then we show that the mean square estimate of the fuzzy parameter can be expressed as a function of possibilistic moments. In analogy with the theorem on normal correlation, we obtain the optimal fuzzy estimate by minimizing the mean square error between the fuzzy parameter and its estimate.

In the last two decades, volatility models have received considerable attention with the emphasis being placed on state space models (see for example Gong *et al.* 2008, Taylor 2005 and Kirby 2006). In the literature, mostly the filtering had been studied for state space models (see for example Abraham and Thavaneswaran 1991 and Granger 1998). In the state space models the conditional mean of the observed process is modeled as a stochastic process. In order to model the changing volatility, stochastic volatility

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models (in which the volatility had been modeled as a stochastic process) had been introduced by Taylor (2005). For fuzzy volatility models with volatility being modeled as a fuzzy sequence, once a new observation is coming in, a new volatility parameter is added, and hence it is almost impossible to estimate every parameter. However, we can construct a recursive estimate which updates the estimated parameter whenever a new observation is included. Recently, Matia *et al.* (2006) and Mottaghi-Kashtiban *et al.* (2008) have studied the filters using possibilistic techniques for fuzzy state space models with asymmetric fuzzy innovations such as trapezoidal and triangular. Here, we study the recursive estimation for some fuzzy volatility models with asymmetric innovations using possibilistic theory.

The remainder of the article is organized as follows. We summarize the preliminaries in the rest of Section 1. In Section 2, we obtain the closed form expression for the possibilistic covariance between some fuzzy numbers. In Section 3, a minimum mean square error fuzzy estimate is defined. In Section 4, recursive estimates are obtained for fuzzy regression with time varying fuzzy parameters and for a fuzzy autoregressive model. We also study the recursive estimation for some fuzzy volatility models.

1.1 Preliminaries and Notations

Following Carlsson and Fuller (2001) and Dubois and Prade (1980), in this section some notations and definitions are given that will be used in the sequel.

Definition 1.1 A fuzzy set A in $X \subset \mathfrak{R}$, where $\mathfrak{R}$ the set of real numbers, is a set of ordered pairs $A = \{(x, \mu(x) : x \in X)\}$, where $\mu(x)$ is the membership function or a grade of membership, or a degree of compatibility or a degree of truth of $x \in X$ which maps $x \in X$ on the real interval [0,1].

Definition 1.2 A fuzzy set A in $\mathfrak{R}^n$ is said to be a convex fuzzy set if its α-level sets $A(\alpha)$ are (crisp) convex sets for all $\alpha \in [0, 1]$. Alternatively, a fuzzy set A in $\mathfrak{R}^n$ is a convex fuzzy set if and only if for all $x_1, x_2 \in \mathfrak{R}^n$ and $0 \le \alpha \le 1$,

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \ge \text{Min}\,(\mu_A(x_1), \mu_A(x_2)).$$

Definition 1.3 A Triangular Fuzzy Number (Δ F.N.) can be represented completely by a triplet $A = (a_1, a_2, a_3)$, where $a_1 < a_2 < a_3 \in \mathfrak{R}$ with membership function $\mu(x)$ as

$$\mu(x) = \begin{cases} 0 & x \le a_1,\ x \ge a_3 \\ \dfrac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\ \dfrac{x-a_3}{a_2-a_3} & a_2 \le x \le a_3 \end{cases} \tag{1.1}$$

and the corresponding α-cut is given by

$$A(\alpha) = [a_1(\alpha), a_2(\alpha)] = [a_1 + \alpha\,(a_2 - a_1),\ a_3 + \alpha\,(a_2 - a_3)],\ \forall\ \alpha \in [0, 1]. \tag{1.2}$$

Definition 1.4 A Trapezoidal Fuzzy Number (T.F.N.) is represented completely by a quadruplet $A = (a_1, a_2, a_3, a_4)$, where $a_1 < a_2 < a_3 < a_4 \in X$ with membership function $\mu(x)$ given by

$$\mu(x) = \begin{cases} 0 & x \le a_1,\ x \ge a_4 \\ \dfrac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\ \dfrac{x-a_4}{a_3-a_4} & a_3 \le x \le a_4 \end{cases} \tag{1.3}$$

and the corresponding α-cut is given by

$$A(\alpha) = [a_1(\alpha), a_2(\alpha)] = [a_1 + \alpha(a_2 - a_1),\ a_4 + \alpha(a_3 - a_4)],\ \forall\ \alpha \in [0, 1]. \tag{1.4}$$

Definition 1.5 A fuzzy number $A \in \mathfrak{I}$ is said to be parabolic if it assumes membership function of the form

$$\mu_A(x) = 1 - \left(\frac{x-a}{a}\right)^2 \text{ if } 0 \le x \le 2a. \tag{1.5}$$

where a is a positive real number.

The corresponding α-cut is given by

$$A(\alpha) = [(a_1(\alpha), a_2(\alpha)] = \left[\left(1-\sqrt{(1-\alpha)}\right)a, \left(1+\sqrt{(1-\alpha)}\right)a\right] \tag{1.6}$$

Definition 1.6 A fuzzy number $A \in \mathfrak{I}$ is said to be Gaussian (see Saeidifar and Pasha 2009 for details) if it has the membership function as

$$\mu_A(x) = \exp\left(-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2\right) \tag{1.7}$$

where x is the independent variable on the universe, θ is the mean, and $\sigma > 0$ is the standard deviation.

The α-cut is given by

$$A(\alpha) = [a_1(\alpha), a_2(\alpha)]$$

$$= \left[\theta - \sigma\sqrt{(-2\ln\alpha)}, \theta + \sigma\sqrt{(-2\ln\alpha)}\right], \quad 0 \le \alpha \le 1. \tag{1.8}$$

In the next section, following Thavaneswaran *et al.* (2009), we discuss certain properties of the centered weighted possibilistic higher order moments of some fuzzy numbers.

2. WEIGHTED POSSIBILISTIC MOMENTS

In this section following Carlsson and Fuller (2001), we introduce the following moments. Let $\mathfrak{J}$ be the class of all fuzzy numbers. The first order f-WPM (or weighted possibilistic mean) of $A \in \mathfrak{J}$ is given by

$$M_f(A) = \int_0^1 f(\alpha)\frac{(a_1(\alpha) + a_2(\alpha))}{2}\, d\alpha \tag{2.1}$$

where $f(\alpha)$ is a weight function such that $\int_0^1 f(\alpha)d\alpha = 1$. Similarly, the centered WPV (or weighted possibilistic variance) of $A \in \mathfrak{J}$ is

$$\text{Var}_f(A) = \frac{1}{2}\int_0^1 f(\alpha)\, [(a_1(\alpha)\text{-}M_f(A))^2 + (a_2(\alpha) - M_f(A))^2]d\alpha \tag{2.2}$$

and for any positive integer r, the f-WPM (weighted possibilistic moment) of order r about the weight possibilistic mean value of A is defined as

$$E_r(A) = \frac{1}{2}\int_0^1 f(\alpha)\, [(a_1(\alpha) - M_f(A))^r + (a_2(\alpha) - M_f(A))^r]\, d\alpha \tag{2.3}$$

In analogy with Thavaneswaran *et al.* (2009), the f-Weighted possibilistic skewness of fuzzy number A is defined as $S_f(A) = \dfrac{E_3(A)}{\left(\sqrt{E_2(A)}\right)^3}$ and similarly, the f-Weighted possibilistic kurtosis of A is defined as $K_f(A) = \dfrac{E_4(A)}{(E_2(A))^2}$. The f-Weighed possibilistic covariance between two fuzzy numbers A and B is given by

$$\text{Cov}_f(A, B) = \frac{1}{2}\int_0^1 f(\alpha)[(a_1(\alpha) - M_f(A))(b_1(\alpha) - M_f(B)) + (a_2(\alpha) - M_f(A))(b_2(\alpha) - M_f(B))]\, d\alpha \tag{2.4}$$

A weighted fuzzy possibilistic correlation is an index of the linear strength of the relationship between two fuzzy numbers and is defined as follows.

$$\rho_f = \frac{\text{Cov}_f(A, B)}{\sqrt{\text{Var}_f(A)}\sqrt{\text{Var}_f(B)}}$$

Thavaneswaran *et al.* (2009) has shown that the weighted possibilistic correlation ρ_f satisfy the inequality that $-1 \le \rho_f \le 1$.

Definition 2.1 A time series of fuzzy numbers $\{A_t\}$ is said to be weakly stationary if

(i) possibilistic mean of $\{A_t\}$ is a constant,

(ii) possibilistic variance of $\{A_t\}$ is a constant, and

(iii) possibilistic covariance $\text{Cov}_f(A_t, A_s)$, depends only on lag time $|t - s|$.

Lemma 2.1 Let Δ, T, P and G be triangular, trapezoidal, parabolic, and Gaussian fuzzy numbers having membership function given by (1.1), (1.3), (1.5), and (1.7) with corresponding α-cuts $[a_1(\alpha), a_2(\alpha)]$, $[b_1(\alpha), b_2(\alpha)]$, $[c_1(\alpha), c_2(\alpha)]$ and $[d_1(\alpha), d_2(\alpha)]$ respectively.

Let the weight function is $f(\alpha) = 2\alpha$, $\forall\ 0 \le \alpha \le 1$. Then

(a) $\text{Cov}\,(\Delta, \text{G}) = \sigma\sqrt{\pi}\left(\dfrac{9 - 2\sqrt{6}}{36}\right)(a_3 - a_1)$

(b) $\text{Cov}(T, P) = -\dfrac{4}{105}\, a(3a_1 + 4a_2 - 4a_3 - 3a_4)$

(c) $\text{Cov}(T, G) = -\dfrac{\sqrt{\pi}}{4}\,\sigma(a_4 - a_1) + \dfrac{\sqrt{2\pi}}{6\sqrt{3}}\,\sigma(a_1 - a_2 + a_3 - a_4)$

(d) for two Gaussian fuzzy numbers G_1 *and* G_2, $\text{Cov}(G_1, G_2) = \sigma_1\sigma_2$

Proof :

(a) The covariance between a triangular Δ and a Gaussian G fuzzy number is given by

$\text{Cov}_f(\Delta, G)$

$$= \frac{1}{2}\int_0^1 f(\alpha)[(a_1(\alpha) - M_f(\Delta))\,(d_1(\alpha) - M_f(G)) + (a_2(\alpha) - M_f(\Delta))\,(d_2(\alpha) - M_f(G))]\,d\alpha$$

$$= \frac{1}{2}\int_0^1 f(\alpha)[(a_1 + \alpha(a_2 - a_1) - M_f(\Delta))(\theta - \sigma\sqrt{(-2\ln\alpha)} - M_f(G)) + (a_3 + \alpha(a_2 - a_3) - M_f(\Delta))(\theta + \sigma\sqrt{(-2\ln\alpha)} - M_f(G))]$$

$$= \sigma\sqrt{\pi}\left(\frac{9 - 2\sqrt{6}}{36}\right)(a_3 - a_1),$$

(b) The covariance between a trapezoidal T and a parabolic P fuzzy number is given by

$\text{Cov}_f(T, P)$

$$= \frac{1}{2}\int_0^1 f(\alpha)[(a_1 + \alpha(a_2 - a_1) - M_f(T))(a - a\sqrt{(1-\alpha)} - M_f(P)) + (a_4 + \alpha(a_3 - a_4) - M_f(T))(a + a\sqrt{(1-\alpha)} - M_f(P))]d\alpha$$

$$= -\frac{4}{105}\; a(3a_1 + 4a_2 - 4a_3 - 3a_4),$$

(c) The covariance between a trapezoidal T and a Gaussian G fuzzy number is given by

$\text{Cov}_f(T, G)$

$$= \frac{1}{2}\int_0^1 f(\alpha)[(a_1 + \alpha(a_2 - a_1) - M_f(T))(\theta - \sigma\sqrt{(-2\ln\alpha)} - M_f(G)) + (a_4 + \alpha(a_3 - a_4) - M_f(T))(\theta + \sigma\sqrt{(-2\ln\alpha)} - M_f(G))]d\alpha$$

$$= -\frac{\sqrt{\pi}}{4}\,\sigma(a_4 - a_1) + \frac{\sqrt{2\pi}}{6\sqrt{3}}\;\sigma(a_1 - a_2 + a_3 - a_4),$$

(d) The covariance between two Gaussian fuzzy numbers G_1 and G_2 is given by

$$\text{Cov}_f(G_1, G_2) = \frac{1}{2}\int_0^1 f(\alpha)\Big[\left(\theta_1 - \sigma_1\sqrt{(-2\ln\alpha)} - M_f(G_1)\right)\left(\theta_2 - \sigma_2\sqrt{(-2\ln\alpha)} - M_f(G_2)\right) + \left(\theta_1 + \sigma_1\sqrt{(-2\ln\alpha)} - M_f(G_1)\right)\left(\theta_2 + \sigma_2\sqrt{(-2\ln\alpha)} - M_f(G_2)\right)\Big]\,d\alpha$$

$$= \sigma_1\sigma_2.$$

3. MINIMUM POSSIBILISTIC MEAN SQUARE ERROR FUZZY ESTIMATE

In state space filtering, theorem on normal correlation plays an important role (see for example Gong *et al.* 2008). In this section, by minimizing the possibilistic mean square error on observed fuzzy number Y and an unobserved fuzzy number Θ, we obtained an estimate of Θ based on observed Y. Let Y and Θ be two centered fuzzy numbers having possibilistic correlation ρ_f (Thavaneswaran *et al.* 2009 for details). Then we have the following theorem.

Theorem 3.1 The minimum mean square linear estimate $\hat{\Theta}$ of Θ is given by

(a) $\hat{\Theta} = \dfrac{\text{Cov}_f(\Theta, Y)}{\text{Var}_f(Y)}$ and

(b) $\text{PMSE}_f^2(\hat{\Theta}, \lambda Y) = (1 - \rho_f^2)\text{Var}_f(\Theta)$

Proof: Let $\hat{\Theta} = \lambda Y$ where λ is obtained by minimizing the possibilistic mean square error between Θ and $\hat{\Theta}$. The possibilistic mean square error between Θ and λY is given by

$$\text{PMSE}_f(\hat{\Theta}, \lambda Y) = \text{Var}_f(\Theta - \lambda Y) = \text{Var}_f(\Theta) + \lambda^2\text{Var}_f(Y) - 2\lambda\text{Cov}_f(\Theta, Y) \quad (3.1)$$

By differentiating $\text{PMSE}_f(\hat{\Theta}, \lambda Y)$ with respect to λ and equating the first derivative function to zero,

$$\frac{\partial \text{PMSE}_f(\hat{\Theta}, \lambda Y)}{\partial\lambda} = 2\lambda\text{Var}_f(Y) - 2\text{Cov}_f(\Theta, Y) = 0 \quad (3.2)$$

We have $\lambda = \dfrac{\text{Cov}_f(\Theta, Y)}{\text{Var}_f(Y)}$. This is the minimum value of λ as the second derivative of $\text{PMSE}_f(\hat{\Theta}, \lambda Y)$ with respect to λ is positive. Let $\hat{\Theta} = \dfrac{\text{Cov}_f(\Theta, Y)}{\text{Var}_f(Y)} Y$, then

$$\text{PMSE}_f^2(\hat{\Theta}, \Theta) = (1-\rho_f^2)\,\text{Var}_f(\Theta)$$
$$= \text{Var}_f(\Theta) - \frac{\text{Cov}_f^2(\Theta, Y)}{\text{Var}_f(Y)}$$
$$= (1-\rho_f^2)\,\text{Var}_f(\Theta) \quad (3.3)$$

Corollary 3.1 Linear prediction of Θ

Let Θ, A be two uncorrelated fuzzy numbers having possibilistic mean μ_Θ, μ_A and possibilistic variance $\sigma_\Theta^2, \sigma_A^2$ respectively. If $Y = \Theta + A$ is observed, then the minimum mean square fuzzy linear predictor of Θ based on observed Y is given by

$$\hat{\Theta} = \frac{\sigma_\Theta^2}{\left(\sigma_\Theta^2 + \sigma_A^2\right)(Y - \mu_\Theta - \mu_A)} \quad (3.4)$$

Moreover if Θ and A are Gaussian fuzzy numbers, then the corresponding α-cut of the fuzzy predictor $\hat{\Theta}$ is given by

$$\hat{\Theta} = \left[\hat{\theta}_1(\alpha), \hat{\theta}_2(\alpha)\right]$$
$$= \left[-\frac{\sigma_\Theta^2}{\sqrt{\left(\sigma_\Theta^2 + \sigma_A^2\right)}}\sqrt{(-2\ln\alpha)}, \quad (3.5)\right.$$
$$\left.\frac{\sigma_\Theta^2}{\sqrt{\left(\sigma_\Theta^2 + \sigma_A^2\right)}}\sqrt{(-2\ln\alpha)}\right]$$

Corollary 3.2 For any two symmetric fuzzy numbers Θ, A having the corresponding α-cuts $\Theta = [\mu_\Theta - \sigma_\Theta S_\Theta(\alpha),\ \mu_\Theta + \sigma_\Theta S_\Theta(\alpha)]$ and $A=[\mu_A - \sigma_A S_A(\alpha),\ \mu_A + \sigma_A S_A(\alpha)]$ then the minimum mean square linear predictor of Θ based on observed Y is

$$\hat{\Theta} = \left[\frac{\sigma_\Theta^2\int_0^1 f(\alpha)S_\Theta^2(\alpha)d\alpha}{\sigma_\Theta^2\int_0^1 f(\alpha)S_\Theta^2(\alpha)d\alpha + \sigma_A^2\int_0^1 f(\alpha)S_A^2(\alpha)d\alpha}\right](Y - \mu_\Theta - \mu_A) \quad (3.6)$$

4. RECURSIVE ESTIMATION

The Kalman filtering for state space model is typically obtained under the assumption that the random innovations are normally distributed. However, these models fail to capture the skewness and the leptokurtosis in financial data. In state space modeling, process noise and observation noise are *not always symmetric* and hence the *Gaussian distribution* assumption is not appropriate. Replacing Gaussian membership function by a non-symmetrical triangular or a trapezoidal membership function allows the introduction of asymmetries in a natural manner. A fuzzy representation of state space model can be given as

$$Y_{t+1} = c\Theta_t + dA_{t+1} \quad (4.1)$$
$$\Theta_{t+1} = a\Theta_t + bB_{t+1}$$

where a, b, c and d are crisp numbers and $\{A_t\}$, $\{B_t\}$ are two uncorrelated sequences of uncorrelated centered fuzzy numbers having the membership functions $\mu_A(x)$ and $\mu_B(x)$ respectively. $E_r(A)$ and $E_r(B)$ denote the r^{th} possibilistic moments of the sequences $\{A_t\}$ and $\{B_t\}$ respectively. We propose an optimal (minimum possibilistic mean square error) filtered estimate of Θ_t based on observed $Y_1, \ldots, Y_t$.

Theorem 4.1 In the case of all estimate of the form

$$\hat{\Theta}_{t+1} = a\hat{\Theta}_t + G_t(Y_{t+1} - c\hat{\Theta}_t), \quad (4.2)$$

(i) the G_t which minimizes the possibilistic mean square error

$$PMSE_f(\Theta_{t+1}, \hat{\Theta}_{t+1}) = M_f(\Theta_{t+1}, \hat{\Theta}_{t+1})^2 = K_{t+1}$$

is given by $\hat{G}_t = \dfrac{acK_t}{c^2K_t + d^2E_2(A_{t+1})}$ and

(ii) the recursive estimates are given by

$$\hat{\Theta}_{t+1} = a\hat{\Theta}_t + \frac{acK_t}{c^2K_t + d^2E_2(A_{t+1})}\left(Y_{t+1} - c\hat{\Theta}_t\right) \quad (4.3)$$

$$K_{t+1} = \frac{a^2d^2K_tE_2(A_{t+1})}{c^2K_t + d^2E_2(A_{t+1})} + b^2E_2\left(B_{t+1}\right). \quad (4.4)$$

(iii) for $a = 1$ and $c = 1$, the limiting MSE of filtering (see Shiryavev 1995 for details), $K = \lim_{t\to\infty} K_t$ is given by the positive root of the equation

$$K^2 - b^2K - b^2d^2 = 0. \quad (4.5)$$

Proof :

(i) The difference of $\Theta_{t+1} - \hat{\Theta}_{t+1} = a(\Theta_t - \hat{\Theta}_t) - G_t(Y_{t+1} - c\hat{\Theta}_t) + bB_{t+1}$. Then

$$K_{t+1} = E[\Theta_{t+1} - \hat{\Theta}_{t+1}]^2 = (a - cG_t)^2 K_t + d^2 G_t^2 E_2(A_{t+1}) + b^2 E_2(B_{t+1}) \tag{4.6}$$

By differentiating K_{t+1} and equating the first derivative function to zero,

$$\frac{\partial K_{t+1}}{\partial G_t} = -2c(a - cG_t)K_t + 2d^2 E_2(A_{t+1})G_t = 0,$$

we obtain $\hat{G}_t = \dfrac{acK_t}{c^2 K_t + d^2 E_2(A_{t+1})}$. The second derivative of K_t with respect to G_t is positive, hence K_t attains its minimum value of $\hat{G}_t$.

(ii) By substitution of the value of G_t in (4.2) and (4.6) we get the expression for recursive estimates.

(iii) Proof follows by taking the limit in (4.4) and is omitted.

The algorithm in (4.3) gives the new linear estimate at time $t + 1$ as the old estimate at time t plus an adjustment. This adjustment is based on the prediction error $(Y_{t+1} - c\hat{\Theta}_t)$. Given starting values Θ_0 and K_0, we can compute the estimator recursively using (4.3) and (4.4). The recursive estimate Θ_{t+1} in (4.3) is usually referred to as an *on-line* estimate and it is very appealing computationally, especially when data are gathered sequentially. Θ_0 and K_0 can usually be obtained from an initial stretch of data. It is of interest to note that the recursive estimate obtained here is derived from the possibilistic techniques. This algorithm may be interpreted in the Bayesian framework by considering the following state space form, which is obtained by taking $a = 1$, $b = 0$ and $d = 1$ in (4.1).

$$Y_{t+1} = c\Theta_t + A_{t+1} \tag{4.7}$$
$$\Theta_{t+1} = \Theta_t$$

Then the recursive estimates are given by

$$\hat{\Theta}_{t+1} = \hat{\Theta}_t + \frac{cK_t}{c^2 K_t + d^2 E_2(A_{t+1})}\left(Y_{t+1} - c\hat{\Theta}_t\right)$$

$$K_{t+1} = \frac{K_t d^2 E_2(A_{t+1})}{c^2 K_t + d^2 E_2(A_{t+1})} \tag{4.8}$$

$$\frac{1}{K_{t+1}} = \frac{c^2}{d^2 E_2(A_{t+1})} + \frac{1}{K_t}$$

Example 5. Fuzzy regression with time varying fuzzy parameters $\{\Theta_t\}$ and crisp valued explanatory variable $\{X_t\}$

$$Y_{t+1} = \Theta_t X_{t+1} + A_{t+1} \tag{4.9}$$
$$\Theta_{t+1} = \Theta_t$$

where $\Theta_t = \Theta$ and $\{A_{t+1}\}$ is an uncorrelated sequence of centered fuzzy numbers having membership function $\mu_A(x)$ and the second moment $E_2(A)$. Then from Theorem (4.1), the recursive estimate of Θ and its possibilistic MSE are given by

$$\hat{\Theta}_{t+1} = \hat{\Theta}_t + \frac{X_{t+1}K_t}{E_2(A) + X_{t+1}^2 K_t}\left(Y_{t+1} - X_{t+1}\hat{\Theta}_t\right) \tag{4.10}$$

$$K_{t+1} = \frac{E_2(A_{t+1})K_t}{E_2(A) + X_{t+1}^2 K_t}.$$

Example 6 For a fuzzy autoregressive model of the form

$$Y_{t+1} = \Theta_t Y_t + A_{t+1} \tag{4.11}$$
$$\Theta_{t+1} = \Theta_t = \Theta$$

Then the recursive estimate of Θ and its possibilistic MSE are given by

$$\hat{\Theta}_{t+1} = \hat{\Theta}_t + \frac{Y_t K_t}{E_2(A) + Y_t^2 K_t}\left(Y_{t+1} - Y_t\hat{\Theta}_t\right) \tag{4.12}$$

$$K_{t+1} = \frac{E_2(A)K_t}{E_2(A) + Y_t^2 K_t}$$

Hence, the updating formula for possibilistic MSE is given by

$$\frac{1}{K_t} - \frac{1}{K_{t-1}} = \frac{Y_{t-1}^2}{E_2(A)} \tag{4.13}$$

If we solve the recursive relations (4.12), using initial values Θ_0 and K_0, we obtain an expression for $\hat{\Theta}_t$, the *off-line* version (see Thavaneswaran and Abraham (1988) for details) as

$$\hat{\Theta}_t = \frac{\sum_{s=2}^{t} Y_s Y_{s-1}}{\sum_{s=2}^{t} Y_{s-1}^2} \tag{4.14}$$

This version will sometimes be better for certain theoretical investigations. In the remainder of this section, we illustrate this optimal possibilistic MSE approach for some fuzzy volatility models.

4.1 Some Illustrations

4.1.1 Fuzzy Volatility Models

We consider the fuzzy analog of Kirby's (2006) volatility model of the form

$$Y_{t+1} = \Theta_t A_{t+1}$$
$$\Theta_{t+1} = k + a\Theta_t + b\Theta_t B_{t+1}$$

where $\{A_t\}$ and $\{B_t\}$ are two correlated sequences of uncorrelated centered fuzzy numbers having the membership function $\mu_A(x)$, $\mu_B(x)$ respectively, the possibilistic corr$(A_t, B_t) = \rho$ and the second moment $E_2(A)$. Now we obtain the fuzzy optimal linear recursive estimate. Then it can be easily shown that the optimal recursive estimate of Θ_{t+1} and corresponding possibilistic MSE K_{t+1}= E$[\Theta_{t+1} - \hat{\Theta}_{t+1}]^2$ satisfy the following:

$$\hat{\Theta}_{t+1} = k + a\hat{\Theta}_t - \frac{b\rho\sqrt{E_2(B)}}{\sqrt{E_2(A)}} Y_{t+1}$$

$$K_{t+1} = a^2 K_t + [K_t^2 E_2(B) - b^2\rho^2 E_2(B)]$$

$$\frac{k^2 + ak^2}{(1-a^2-b^2E_2(B))(1-a)}$$

4.1.2 Generalized Fuzzy Volatility Model

We propose a fuzzy volatility model of the form

$$Y_{t+1} = \sigma e^{\Theta_t/2} A_{t+1}$$
$$\Theta_{t+1} = \phi\Theta_t + (1 + \Theta_t)\, B_{t+1}$$

where Y_t is the observed fuzzy sequence. $\{A_t\}$ and $\{B_t\}$ are two uncorrelated sequences of uncorrelated centered fuzzy numbers and $\sigma = \sqrt{E_2(A)}$. Then the recursive estimate of Θ_{t+1} and corresponding $K_{t+1} = E[\Theta_{t+1} - \hat{\Theta}_{t+1}]^2$ can be given as

$$\hat{\Theta}_{t+1} = \phi\hat{\Theta}_t + \frac{2\phi K_t}{K_t + 8E_2(A)}\left(\log Y_{t+1} - \log\sigma - \frac{\hat{\Theta}_t}{2}\right)$$

$$K_{t+1} = \frac{64E_4(A)\phi^2 K_t + 4\phi^2 E_2(A) K_t^2}{(K_t + 8E_2(A))^2}$$

$$+E_2(B)\left(1 + \frac{2E_2(A)}{1-\phi-E_2(A)} + \frac{2E_4(A) + E_2(A)(1-\phi-E_2(A))}{(1-\phi^2-E_2(A))(1-\phi+E_2(A))}\right)$$

4.1.3 GARCH Model with Fuzzy Volatility

In analogy to the GARCH model, the following model has the second fuzzy noise term B_t beside the first one A_t.

$$Y_{t+1} = \Theta_t + \Theta_t \mathrm{A}_{t+1}$$
$$\Theta_{t+1} = a\Theta_t + B_{t+1}$$

where $\{A_t\}$ and $\{B_t\}$ are two uncorrelated sequences of uncorrelated centered fuzzy numbers having the membership functions $\mu_A(x)$, $\mu_B(x)$ respectively. Then the recursive estimate of Θ_{t+1} and corresponding K_{t+1}= $E[\Theta_{t+1} - \hat{\Theta}_{t+1}]^2$ can be constructed as

$$\hat{\Theta}_{t+1} = a\hat{\Theta}_t + \frac{a(1-a^2)K_t}{(1-a^2)K_t + E_2(A)E_2(B)}\left(Y_{t+1} - \hat{\Theta}_t\right)$$

$$K_{t+1} = \frac{a^2 E_2(A)E_2(B)K_t}{(1-a^2)K_t + E_2(A)E_2(B)} + E_2(B).$$

4.1.4 Fuzzy Coefficient GARCH model

When the observed process is correlated with its volatility process, we use the following model

$$Y_{t+1} = \Theta_t A_{t+1}$$
$$\Theta_{t+1} = k + a\Theta_t + b\,\Theta_t A_{t+1}$$

where $\{A_t\}$ is a uncorrelated sequences of centered fuzzy numbers having the membership functions $\mu_A(x)$ and the second moment $E_2(A)$. Then the recursive estimate of Θ_{t+1} and corresponding K_{t+1}= $E[\Theta_{t+1} - \hat{\Theta}_{t+1}]^2$ are given by

$$\hat{\Theta}_{t+1} = k + a\hat{\Theta}_t + \frac{b(1-a)(1-a^2-b^2E_2(A))}{(1-a)k^2 + 2ak^2} Y_{t+1}$$

$$K_{t+1} = a^2 K_t + b^2 E_2(A)\frac{(1-a)k^2 + 2ak^2}{(1-a)(1-a^2-b^2E_2(A))}$$

$$-b^2 E_2(A).$$

5. CONCLUSIONS

Abraham and Thavaneswaran (1991) had studied filtering for state space models, which was cited by Granger (1998), a Nobel Prize winner (2003). Matia *et al.* (2006) and Mottaghi Kashtiban *et al.* (2008) have studied the fuzzy filters using possibilistic techniques for state space models. In Thavaneswaran *et al.* (2009), we have demonstrated the superiority of the fuzzy forecast over the minimum mean square error forecasts.

In this paper, we extended the results to fuzzy volatility models. We also showed that the mean square between fuzzy parameter and its estimate can be expressed as a function of possibilistic moments and obtained the optimal fuzzy estimate by minimizing the mean square error.

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