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# On the Construction of Two-level Fractional Factorial Designs when Some Combinations are Debarred

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**SUMMARY**

In choosing a fractional factorial plan, one often decides on an orthogonal plan. Fractional factorial plans represented by orthogonal arrays provide orthogonal plans which also have strong optimality properties. While using such fractional factorial plans, it is possible in some situations that certain treatment combinations are infeasible or, even if these are feasible, no observations can be made on these. In such situations, it is desirable to have an orthogonal fractional factorial plan that does not include the infeasible treatment combinations. In this paper, we provide a method of obtaining such plans represented by two-symbol orthogonal arrays of strength two.

*Keywords* : Debarred combination, Projectivity, Hadamard property.

## 1. INTRODUCTION

Fractional factorial plans have been an active area of research in recent years due to their wide applicability in many diverse fields, including agriculture, physical and chemical sciences, industrial experimentation and quality improvement work. Extensive discussion of fractional factorials together with useful catalogues can be found in Dey (1985) and Wu and Hamada (2000). For a more mathematical treatment of fractional factorials with emphasis on optimality aspects, one may refer to Dey and Mukerjee (1999). Among the fractional factorials, the ones which have each factor at two levels are the most important ones and in this communication, we concern ourselves only to fractions of symmetric 2-level factorials.

In some experimental situations, it may happen that certain factor-level combinations are infeasible in the sense that observations are not available under such level combinations. For example, in an agronomic experiment, amounts of the nutrients, nitrogen and phosphorus and amount of irrigation may be three of the factors and high doses of nitrogen and phosphorus and a low dose of irrigation may lead to burning of the crop. In such a case, clearly no observation can be made on treatment combinations involving the high levels of the factors nitrogen and phosphorus and low dose of irrigation. Furthermore, in some other experimental situations, it is possible that the experiment can be carried out but measurements cannot be made. A real life example reported by Cheng and Li (1993) illustrates this scenario and relates to a study on thermal history control in bar-code printers. Among the several ways to print a bar-code, the thermal transfer is regarded as the best because of its printing quality and suitability for multiple-product-small-quantity production. In one experiment with thermal transfer, no measurements could be made for a treatment combination in which all the factors were at the lowest levels. Additionally, in some experimental situations, certain combinations may be ruled out because of their being prohibitively expensive, even though it might be operationally

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feasible to conduct an experiment with such treatment combinations.

Orthogonal fractional factorial plans provide uncorrelated estimates of all the relevant parameters under a chosen model and these are economical, efficient and easy to analyze and interpret. If an orthogonal plan contains one or more such infeasible treatment combinations as described above, there will be missing data and it may not be possible to augment the plan without losing the attractive property of orthogonality. It is therefore desirable to start with an orthogonal fraction (before the experiment is conducted) which does not include any infeasible treatment combination. Cheng and Li (1993) first considered this problem in the context of regular fractions (for a definition of regular fractions, see e.g., Dey and Mukerjee (1999)). In this paper, we provide results on the same problem for 2-level fractional factorial plans represented by orthogonal arrays of strength two. A brief remark about the applicability of the proposed method to regular fractions is also made.

## 2. RESULTS

Consider a 2-level symmetric factorial experiment involving $n$ factors, $F_1, \ldots, F_n$. Suppose that a certain combination of levels of $k \leq n$ factors, say $F_1, \ldots, F_k$, is infeasible. We call such a combination of the $k$ factors as a *debarred combination*. Note that if $k < n$, then a single debarred combination may lead to several of the treatment combinations to be infeasible. In general, if there is a single debarred combination involving $k$ ($\leq n$) factors, then the number of infeasible treatment combinations is $2^{n-k}$.

An orthogonal array, $OA(N, n, 2, 2)$ involving $n$ columns, $N$ rows, 2 symbols and strength 2 is an $N \times n$ matrix $A$ with two distinct entries in each column such that each of the 4 combinations of the symbols appears equally often as a row in every $N \times 2$ submatrix of $A$. If one identifies the columns of an $OA(N, n, 2, 2)$ with the factors of a factorial experiment and the rows as treatment combinations, then an $OA(N, n, 2, 2)$ represents a fractional factorial plan for a $2^n$ experiment in $N$ runs. Such a fractional factorial plan is orthogonal and universally optimal (in particular, $A$-, $D$- and $E$-optimal) over the global class of competing plans under a model that includes the mean and all main effects, all 2-factor and higher order interactions being assumed negligible. We also need the notion of *projectivity* of a fractional factorial plan, defined below.

**Definition:** A fractional factorial plan involving $n$ factors is said to have projectivity $p$ if in every subset of $p(< n)$ factors, a complete factorial with possibly some repeated runs is produced.

It is well known that a regular fractional factorial plan of resolution $R$ (which is an orthogonal array of strength $R - 1$) has projectivity $R - 1$ but cannot have projectivity greater than $R - 1$. However, some non-regular fractions represented by orthogonal arrays of strength $g(\geq 2)$ can have projectivity greater than $g$. As an example, consider an $OA(12, 11, 2, 2)$ shown in Table 1. It can be verified that under *any* 3 columns, a complete $2^3$ factorial plus some repeated runs is produced and thus, the array in Table 1 has projectivity 3 even though the strength is 2. For more on projectivity of fractions represented by orthogonal arrays, a reference may be made to Box and Tyssedal (1996), Cheng (1995, 1998), Bulutoglu and Cheng (2003) and Dey (2005).

**Table 1**. An $OA(12, 11, 2, 2)$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Consider now a fractional factorial plan for a $2^n$ experiment represented by an $OA(N, n, 2, 2)$. Suppose a certain combination of levels of $k \leq n$ factors is a debarred combination. An orthogonal array of strength two which does not include the infeasible treatment combination(s) can always be constructed if the projectivity of the array is less than $k$. To see this, let $A \equiv OA(N, n, 2, 2)$ and let $A$ have projectivity less than $k$. It then follows from the definition of projectivity that there exist some $k$ columns of $A$ under which at least one of the relevant $2^k$ combinations is missing. If the set of such missing combination(s) includes the debarred combination, then we simply assign these $k$ columns to the corresponding $k$ factors involved in the debarred combination.

Now suppose the debarred combination, say $a_1 \ldots a_k$, appears under the $k$ columns mentioned above. Under these $k$ columns, at least one of the possible $2^k$ combinations is however missing as, the projectivity of the array is strictly less than $k$. Let $b_1 \ldots b_k$ be one such combination. Clearly, $a_1 \ldots a_k \neq b_1 \ldots b_k$, which implies that $a_i \neq b_i$ for at least one $i$, $1 \leq i \leq k$. In all the columns where $a_i \neq b_i$, interchange the two symbols. If we consider the resulting orthogonal array (after the suggested interchange of symbols) and continue to look at the same $k$ columns as before, the combination $a_1 \ldots a_k$ will be missing, as desired. Assign these $k$ columns to the relevant $k$ factors. We thus have the following result.

**Theorem 1.** In a 2-symbol orthogonal array of strength two, suppose a certain combination of levels of $k \leq n$ factors is a debarred combination. If the orthogonal array has projectivity less than $k$, then it is possible to obtain an orthogonal array which does not contain the debarred combination.

**Example.** Suppose one wants to conduct a $2^{11}$ experiment in 12 runs and chooses the $OA(12, 11, 2, 2)$ shown in Table 1 as the plan. As noted earlier, this array has projectivity 3. Suppose a combination of levels of $k$, $3 < k \leq 11$ is a debarred combination. First let $k = 11$ and suppose the combination with each factor at its low level (i.e., the combination (0, 0, . . ., 0)) is debarred. This is one of the combinations in the array in Table 1. Then, by interchanging the symbols 0 and 1 in each of the columns of the array, one gets an orthogonal array $OA(12, 11, 2, 2)$ which does not include the debarred combination. The same technique works if any other combination included in the array of Table 1 is a debarred combination. If the debarred combination is not one of those in Table 1, then the array in Table 1 itself can be taken as the array which does not include the debarred combination.

Next, consider the more interesting case when $k < 11$. Suppose $k = 4$ and suppose the combination (1, 1, 1, 0) involving the levels of the first four factors is a debarred combination. Since this combination appears in one of the rows of the array in Table 1, one needs to find another $OA(12, 11, 2, 2)$ which does not have this combination. Under the columns 7, 8, 10 and 11 of the array in Table 1, we find that the debarred combination does not appear. Therefore, we assign these columns to the first four factors to get an array which does not include the debarred combination. The original array is as in Table 1 and the transformed one is shown in Table 2.

Cheng (1995) showed that a fractional factorial plan represented by an orthogonal array $OA(N, n, 2, 2)$ with $N \neq 0 \pmod 8$ and $n \geq 4$ has projectivity three. Using this result and Theorem 1 therefore, one can obtain an $OA(N, n, 2, 2)$ which does not include the debarred combination for all $k > 3$.

It is also possible to construct an $OA(N, n, 2, 2)$ where $N = 0 \pmod 8$, such that it does *not* have projectivity three. This construction is based on Hadamard matrices. A Hadamard matrix $\mathbf{H}_N$ of order $N$ is an $N \times N$ matrix with entries $\pm 1$ such that $\mathbf{H}'_N \mathbf{H}_N = N\mathbf{I}_N$, where $\mathbf{I}_N$ is an identity matrix of order $N$ and a prime over a matrix denotes its transpose. A positive integer $N$ is called a *Hadamard number* if $\mathbf{H}_N$, a Hadamard matrix of order N exists.

**Table 2.** An $OA(12, 11, 2, 2)$ not containing the debarred combination

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The following facts about Hadamard matrices are well known :

(i) One can always write a Hadamard matrix of order $N$ with its first column consisting of only +1's, i.e., one can write $\mathbf{H}_N = [\mathbf{1}_N \ \mathbf{B}]$, where for a positive integer $s$, $\mathbf{1}_s$ is an $s \times 1$ vector of all ones.

(ii) The existence of $\mathbf{H}_N$, $N \geq 4$ is equivalent to that of an $OA(N, N - 1, 2, 2)$.

A set of 3 distinct columns of $\mathbf{H}_N (N \geq 4)$ is said to have the *Hadamard property* if the Hadamard product of any two columns in the set equals the third. Recall that the Hadamard product of two vectors $\mathbf{a} = (a_1, \ldots, a_n)'$ and $\mathbf{b} = (b_1, \ldots, b_n)'$ is defined as $\mathbf{a} * \mathbf{b} = (a_1b_1, \ldots, a_nb_n)'$. We now have the following result.

**Theorem 2.** Let $t \geq 4$ be a Hadamard number and let $\mathbf{H}_t$ be written as $\mathbf{H}_t = [\mathbf{1}_t\ \mathbf{B}]$, where $\mathbf{B}$ is a $t \times (t-1)$ matrix with entries $\pm 1$. Then the orthogonal array $\mathbf{C} = OA(2t, 2t-1, 2, 2)$ given by

$$\mathbf{C} = \begin{bmatrix} \mathbf{B} & \mathbf{H}_t \\ \mathbf{B} & -\mathbf{H}_t \end{bmatrix}$$

does not have projectivity three.

**Proof.** It is easy to verify that $\mathbf{C}$ is an $OA(2t, 2t-1, 2, 2)$. Let $\mathbf{a}_i$ be the $i$th column of $\mathbf{B}$. Consider the three columns of $\mathbf{C}$ given by

$$\begin{bmatrix} \mathbf{a}_i & \mathbf{1}_t & \mathbf{a}_i \\ \mathbf{a}_i & -\mathbf{1}_t & -\mathbf{a}_i \end{bmatrix}.$$

It is not hard to see that these three columns of $\mathbf{C}$ have the Hadamard property. This implies that under these three columns, the combinations $(-1, -1, 1)$, $(-1, 1, -1)$, $(1, -1, -1)$ and $(1, 1, 1)$ occur equally often as a row of $C$ while the other 4 combinations, viz., $(-1, -1, -1)$, $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$ do not appear at all. It follows then that $C$ does not have projectivity three, thus completing the proof.

For each Hadamard number $t \geq 4$, Theorem 2 gives an $OA(2t, 2t-1, 2, 2)$ which does not have projectivity three. Combining this fact with the one in Theorem 1, one can obtain orthogonal arrays not including a debarred combination for all $k \geq 3$.

Furthermore, numerical investigations show that for $N \neq 0 \pmod 8$, there exist 2-symbol orthogonal arrays of strength two derived from $\mathbf{H}_N$ and not having projectivity *four* for each of the following values of $N \leq 100$ :

$$N = 12, 20, 28, 36, 44, 52, 60, 68, 76, 84, 92, 100.$$

Hence, using such orthogonal arrays, one can derive orthogonal arrays not including the debarred combination for all $k \geq 4$.

For regular fractions, one can always construct the desired array with $k \geq 3$. To see this, consider a regular fraction of a $2^n$ factorial with resolution III or more. Recall that such a regular fraction is characterized by a set of defining contrasts with the property that the resolution of the fraction is $R$ if the smallest interaction in the set of defining contrasts has $R$ factors. The interactions in the defining contrast set are called defining words. If the fraction under consideration has resolution III, then the projectivity of the corresponding orthogonal array is 2 (and never more than 2) and the technique described earlier works with $k \geq 3$. Next, suppose the resolution $R$ is IV or more. Choose a word of length $R$ in the defining contrast subgroup and consider $R-3$ letters, say $l_1, \ldots, l_{R-3}$ appearing in the chosen defining word. Delete each of $l_1, \ldots, l_{R-3}$ from every defining word, wherever they appear. It is then not hard to see that the fraction defined by the new defining words is a regular fraction of resolution III. Thus, again we have a fraction represented by an orthogonal array of strength two and projectivity 2 and we can proceed as earlier with $k \geq 3$.

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