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Tables for Optimum Covariate Designs in PBIBD Set-ups

Ganesh Dutta[1], Premadhis Das[2] and Nripes K. Mandal[3*]
[1]*Basanti Devi College, Rashbehari Avenue, Kolkata*
[2]*University of Kalyani, Kalyani*
[3]*University of Calcutta, Kolkata*

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SUMMARY

In Dutta *et al.* (2009b) methods of construction of optimum covariate designs (OCDs) in partially balanced incomplete block design (PBIBD) set-ups have been proposed. Applying these methods actual designs can be obtained. Considering the suitable PBIBDs from Clatworthy's table (1973) lists of OCDs, categorised for different classes of designs, have been prepared here. The constructional methods have also been indicated for ready reference. Also some new results and constructional methods in the set-up of balanced factorial designs with orthogonal factorial structure (OFS) with the list of OCDs has been given.

Keywords: Block designs, Covariates, Optimal designs, Orthogonal arrays, Hadamard matrices, Khatri-Rao product, Kronecker product.

1. INTRODUCTION

Consider the non-stochastic controllable covariates model in a block design set-up

$$(\mathbf{Y}, \mu\mathbf{1}_N + \mathbf{X}_1\beta + \mathbf{X}_2\tau + \mathbf{Z}\gamma, \sigma^2\mathbf{I}) \tag{1.1}$$

where μ is the intercept term, $\beta^{b\times 1}$, $\tau^{v\times 1}$ and $\gamma^{c\times 1}$ correspond respectively to the vectors of block effects, treatment effects and covariate effects and σ^2 is the common variance of the observations. $\mathbf{Y}$ is the uncorrelated observation vector of order $N \times 1$, $\mathbf{X}_1$, $\mathbf{X}_2$ respectively are the incidence matrices of block effects, treatment effects, $\mathbf{Z}$ is the design matrix corresponding to the covariate effects and $\mathbf{1}_N$ is a vector of order N with all elements unity. For the covariates, it is assumed without loss of generality, the (location-scale)-transformed version i.e. $|z_{ij}| \le 1$. It is evident that for orthogonal estimation of treatment and block effect contrasts on one hand and covariate effects on the other, it is necessary and sufficient that

$$\mathbf{Z}'\mathbf{X}_1 = 0, \quad \mathbf{Z}'\mathbf{X}_2 = 0 \tag{1.2}$$

and for most efficient estimation of each of the regression parameters the following condition must also be satisfied (cf. Pukelsheim 1993)

$$\mathbf{Z}'\mathbf{Z} = N\mathbf{I}_c \tag{1.3}$$

A covariate design $\mathbf{Z}$ is said to be optimum, if it satisfies (1.2) and (1.3).

It is evident from (1.2) and (1.3) that (i) the columns of $\mathbf{Z}$ should be orthogonal to the columns of $\mathbf{X}_1$ and $\mathbf{X}_2$ and (ii) the columns of $\mathbf{Z}$ must be mutually orthogonal among themselves and the elements of $\mathbf{Z}$ must be ± 1.

The problem of choice of covariates in an experimental design set-up was earlier considered by Lopes Troya (1982a, 1982b), Liski *et al.* (2002), Das *et al.* (2003), Rao *et al.* (2003), Dutta (2004) and Dutta *et al.* (2007, 2009a, 2009b, 2010). Lopes Troya (1982a, 1982b) first considered the problem of choice of the $\mathbf{Z}$-matrix in a completely randomised design (CRD) model. Das *et al.* (2003) extended it to the set-up of

Corresponding author : Nripes K. Mandal
E-mail address : mandalnk2001@yahoo.co.in

randomised block designs (RBDs) and to some series of balanced incomplete block designs (BIBDs). Rao *et al.* (2003) re-visited the problem in CRD and RBD set-ups and identified the solutions as a mixed orthogonal array, thereby providing further insights and some new solutions. As mentioned earlier, the choice of optimum covariate designs (OCDs) depends heavily on the block design set-up as is evident from (1.2). In the case of incomplete block (IB) designs, the allocation of treatments to the plots of the blocks dictates the method of construction of the optimum covariate designs (OCDs). So in every case, as it will be seen subsequently that the construction of OCDs is suitably adopted for the constructional method of the block design itself. Das *et al.* (2003) considered some BIBDs with repeated blocks and some symmetric balanced incomplete block designs (SBIBDs) with parameters $b = v$, $r = k$, λ, constructed through Bose's difference method. Dutta (2004) also considered some series of BIBDs obtained through Bose's difference technique and some arbitrary BIBDs. Thereafter, Dutta *et al.* (2007) considered a series of SBIBDs which was obtained through projective geometry. Again Dutta *et al.* (2009b) extended the problem of OCDs to a large class of partially balanced incomplete block design (PBIBD) set-ups.

However, as is well known, there are different methods of construction leading to different IB designs. It is very difficult to construct maximum number of OCD for arbitrary IB design, as the choice of the **Z**-matrix depends on the construction of IB design itself. Dutta *et al.* (2010) considered this problem for any binary proper equi-replicate block design (BPEBD) $d(v, b, r, k)$ with $b = mv$, m being, a positive integer, ≥ 1 and prepared a list of OCDs in commonly used BPEBDs.

In all the investigations considered above, an uncorrelated homoscedastic model was assumed. For the correlated model, the issue of finding OCDs was considered in Dutta *et al.* (2009a). They considered this problem in standard split-plot and strip-plot design set-ups where variance-covariance matrices had special structures which were conveniently exploited to find the OCDs.

PBIBDs are popular among the experimenters and covariates are often used in them to reduce the error. It is necessary to allocate the covariate values in such a way that the regression coefficients are estimated in an optimum way. As in the BPEBDs, a list of OCDs in PBIBD set-ups seems to be useful for the users. The two-associate class PBIBDs from the tables of Clatworthy (1973) have been considered and lists of OCDs have been prepared with indication of methods for actual constructions proposed mainly in Dutta *et al.* (2009b). Henceforth, we will refer to Dutta *et al*. (2009b) as D(2009b). Also the problem of construction of OCDs in factorial designs, viz. in balanced factorial designs with orthogonal factorial structure (OFS) (cf. Gupta and Mukerjee 1989) has been considered and a list has been provided with in the two factors case.

Following Das *et al.* (2003), each column of the **Z**-matrix can be recast to a **W**-matrix where the element in the i^{th} row and j^{th} column of $\mathbf{W}^{(s)}$ is $z_{ij}^{(s)}$, $z_{ij}^{(s)}$ being the element of **Z** corresponding to the j^{th} treatment in the i^{th} block of the design for the s^{th} covariate. Corresponding to the block-treatment classification, optimality conditions (1.2) and (1.3) in terms of **W**-matrices reduce to:

($\mathbf{C}_1$) each **W**-matrix has all column-sums equal to zero ;

($\mathbf{C}_2$) each **W**-matrix has all row-sums equal to zero ;

($\mathbf{C}_3$) the grand total of all the entries in the Hadamard product (vide C.R. Rao 1973) of any two distinct **W**-matrices reduces to zero.

Henceforth, the conditions $\mathbf{C}_1$-$\mathbf{C}_3$ together are referred to as the single condition **C**.

It is to be noted that a covariate design **Z** for c covariates is equivalent to c **W**-matrices which are convenient to work with.

Definition 1.1. With respect to the model (1.1), the c **W**-matrices corresponding to the c covariates are said to be optimum if they satisfy the condition **C**.

Remark 1.1. It may be mentioned that if $c = 1$, only conditions $\mathbf{C}_1$ and $\mathbf{C}_2$ are to be satisfied by the **W**-matrix to be optimum.

In an incomplete binary design set-up, the basic principle of constructing optimum **W**-matrices i.e. the OCD, is to convert the incidence matrix **N** of the incomplete binary design to a **W**-matrix by placing judiciously ±1s in the non-zero cells of the incidence matrix so that the condition **C** mentioned above is satisfied.

The paper is organised as follows: In Section 2, main results for the construction of OCDs are first cited and then lists of OCDs in group divisible (GD), L_i-type and triangular PBIBDs have been given. In Section 3, some new results for the construction of OCDs in balanced factorial design set-up which forms a particular class of 3-associate PBIBD design based on rectangular association scheme are given and then a list of OCDs has been prepared in factorial design set-up.

2. OPTIMUM COVARIATE DESIGNS IN PBIBD SET-UPS

In this section, following D(2009b), statements of methods of construction of **W**-matrices satisfying the condition **C**, for different series of PBIBDs, are given. Here Hadamard matrices play the key role for constructing OCDs. **H** * denotes Hadamard matrix of order *.

2.1 Singular Group Divisible (SGD) Design Set-up

Singular group divisible (SGD) designs considered here have the parameters

$$v = nv^*, b = b^*, r = r^*, k = nk^*, \quad \lambda_1 = r^*, \lambda_2 = \lambda^*, m = v^*, n = n \tag{2.1}$$

and are obtained by replacing each treatment of a BIBD($v^*, b^*, r^*, k^*, \lambda^*$) by group of n treatments (Bose *et al.* 1953). We state below the results obtained in D(2009b) in this connection.

Theorem 2.1. A set of t optimum **W**-matrices can be constructed for the SGD design with parameters in (2.1), where

(i) $t = c$, if c optimum **W**-matrices exist for an RBD with n treatments and r blocks;

(ii) $t = v^*(n - 1)(r - 1)$, if $\mathbf{H}_{v^*}$, $\mathbf{H}_n$ and $\mathbf{H}_r$ exist;

(iii) $t = v^*((n - 1)(r - 1) - (n - 2))$, if $n \equiv 2 \pmod 4$, $(n - 1)$ is a prime or a prime power and $\mathbf{H}_{v^*}$, $\mathbf{H}_r$ exist;

(iv) $t = v^*((n - 1)(r - 1) - (n - 2))$, if $\mathbf{H}_{v^*}$, $\mathbf{H}_{2n}$ and $\mathbf{H}_{\frac{r}{2}}$ exist;

(v) $t = v^*$, if n = even, r = even and $\mathbf{H}_{v^*}$ exists.

The theorem follows by noting that the incidence matrix for the SGD design in (2.1) can be written as

$$\mathbf{N} = (\mathbf{N}_1\ \mathbf{N}_2 \ldots \mathbf{N}_i \ldots \mathbf{N}_{v^*}) \tag{2.2}$$

where, $\mathbf{N}_i^{b\times n}$, corresponds to the n treatments ($\theta_{1i}, \theta_{2i}, \ldots, \theta_{ni}$) used to replace the i^{th} treatment of the BIBD, $i = 1, 2, ..., v^*$. Among the b rows of $\mathbf{N}_i$, r rows contain exclusively 1s and other elements are 0s for each i. For the time being, let it be assumed that c optimum **W**-matrices $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_c$ for an RBD with parameters n, r exist. Putting the elements of $\mathbf{W}_j$ of RBD (n, r) in the non-zero positions of each $\mathbf{N}_i$, a matrix $\mathbf{W}_j^*$ is obtained where

$$\mathbf{W}_j^* = (\mathbf{W}_{1j}^*\ \mathbf{W}_{2j}^* \ldots \mathbf{W}_{ij}^* \ldots \mathbf{W}_{v^*j}^*) \tag{2.3}$$

It is easy to verify that $\mathbf{W}_1^*, \mathbf{W}_2^*, \ldots, \mathbf{W}_c^*$ give optimum **W**-matrices for the SGD design given in (2.1) and thus (i) of the theorem follows.

Again if a Hadamard matrix of order v^* exists then the number of optimum **W**-matrices can be increased. For $l = 1, 2, \ldots, v^*$, a matrix $\mathbf{W}_{lj}^*$ is constructed as

$$\mathbf{W}_{lj}^* = \mathbf{h}_l \odot \mathbf{W}_j^* \tag{2.4}$$

where $\mathbf{h}_l$ is the l^{th} row of $\mathbf{H}_{v^*}$ and $\odot$ denotes the Khatri-Rao product. Thus varying l and j, v^*c optimum **W**-matrices are obtained.

It is proved in Das *et al.* (2003) and Rao *et al.* (2003) that the values of c are (a_1) $(n - 1)(r - 1)$, if $\mathbf{H}_n$ and $\mathbf{H}_r$ exist; (a_2) $(n - 1)(r - 1) - (n - 2)$, if $n \equiv 2 \pmod 4$ and $\mathbf{H}_r$ exists and (a_3) $(n - 1)(r - 1) - (n - 2)$, if $\mathbf{H}_{2n}$ and $\mathbf{H}_{\frac{r}{2}}$ exist. These values imply respectively (ii), (iii) and (iv) of the theorem when $\mathbf{H}_{v^*}$ exists.

Again if n and r are even, we can write an $r \times n$ matrix $\mathbf{W}_1$ as

$$\mathbf{W}_1 = \begin{pmatrix} \mathbf{J} & -\mathbf{J} \\ -\mathbf{J} & \mathbf{J} \end{pmatrix}$$

where **J** is a $\frac{r}{2} \times \frac{n}{2}$ matrix with all elements unity. It is easy to see that $\mathbf{W}_1$ gives an optimum **W**-matrix for an $r \times n$ RBD. As $\mathbf{H}_{v^*}$ exists, (v) of the theorem follows.

Remark 2.1. Exchanging the roles of r and n in (iii) and (iv) of Theorem 2.1, the following can be obtained

(i) If $r \equiv 2 \pmod 4$, $(r-1)$ is a prime or a prime power and $\mathbf{H}_{v^*}$ and $\mathbf{H}_n$ exist, then $t = v^*((n-1)(r-1)-(r-2))$ optimum **W**-matrices can be constructed for the SGD design with parameters in (2.1).

(ii) If $\mathbf{H}_{v^*}$, $\mathbf{H}_{2r}$ and $\mathbf{H}_{\frac{n}{2}}$ exist, then $t = v^*((n-1)(r-1)-(r-2))$ optimum **W**-matrices can be constructed for the SGD design with parameters in (2.1).

Remark 2.2. If v^* is an even integer but not a multiple of 4, then a set of t optimum **W**-matrices can be constructed for the SGD design with parameters in (2.1) by using $\mathbf{1}'_{v^*}$ and $\left(\mathbf{1}'_{\frac{v^*}{2}}, -\mathbf{1}'_{\frac{v^*}{2}}\right)$ to play the same role as that of the rows of $\mathbf{H}_{v^*}$. From (ii)-(iv) of Theorem 2.1 it follows that

(i) $t = 2(n-1)(r-1)$, if $\mathbf{H}_n$ and $\mathbf{H}_r$ exist;

(ii) $t = 2((n-1)(r-1)-(n-2))$, if $n \equiv 2 \pmod 4$, $(n-1)$ is a prime or a prime power and $\mathbf{H}_r$ exists;

(iii) $t = 2((n-1)(r-1)-(n-2))$, if $\mathbf{H}_{2n}$ and $\mathbf{H}_{\frac{r}{2}}$ exist.

Remark 2.3. It is easily seen that for the construction of optimum **W**-matrices for RBD(r, n), it is necessary that r and n must be even. If r, n and v^* are even but none of them are multiple of 4, then 2 optimum **W**-matrices can be constructed for the SGD design with parameters (2.1) by using two orthogonal rows as in Remark 2.2.

Remark 2.4. Suppose t_1 optimum **W**-matrices exist for the BIBD $(v^*, b^*, r^*, k^*, \lambda^*)$; then additional t_1 optimum **W**-matrices, orthogonal to the previous ones, can be constructed for a SGD design with parameters given in (2.1).

2.2 Semi-regular Group Divisible (SRGD) Design Set-up

Semi-regular group divisible (SRGD) design with parameters ($v = mn$, $b = n^2\lambda_2$, $r = n\lambda_2$, $k = m$, $\lambda_1 = 0$, λ_2, m, n) obtained from an orthogonal array, OA($n^2\lambda_2$, m, n, 2) (cf. Bose *et al.* 1953) are considered here. In this case, using the properties of orthogonal array (cf. Hedayat *et al.* 1999) OCDs can be constructed and the following theorem can be proved.

Theorem 2.2. (D(2009b)): Let $\mathbf{H}_n$, $\mathbf{H}_{m_1}$ and an OA($n^2\lambda_2$, m, n, 2) ($m_1 = k < m$) exist. Then $(n-1)(k-1)m_2$ optimum **W**-matrices can be constructed for an SRGD design with parameters $v = m_1 n$, $b = n^2\lambda_2$, $r = n\lambda_2$, $k = m_1$, $\lambda_1 = 0$, λ_2, m_1, n, where $m_1 + m_2 = m$ and $m_2 > 2$.

The above method is illustrated as follows:

Let the orthogonal array OA($n^2\lambda_2$, m, n, 2) denoted by the matrix **A** with $n^2\lambda_2$ rows and m columns and it be partitioned into two sub-matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ i.e. $\mathbf{A} = (\mathbf{A}_1 \mid \mathbf{A}_2)$ where $\mathbf{A}_1$ corresponds to first m_1 columns and $\mathbf{A}_2$ corresponds to last m_2 ($m_2 = m - m_1$). Using $\mathbf{A}_1$, an SRGD design with parameters $v = m_1 n$, $b = n^2\lambda_2$, $r = n\lambda_2$, $k = m_1$, m_1, n, $\lambda_1 = 0$, λ_2; $m_1 + m_2 = m$ and $m_2 > 2$ is constructed, where the $n^2\lambda_2$ rows of $\mathbf{A}_1$ give the blocks of the SRGD design. Let $\mathbf{H}_n$ be written as

$$\mathbf{H}_n = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-1}, \mathbf{1}] \qquad (2.5)$$

and the n symbols in each column of $\mathbf{A}_2$ be replaced by the n elements of $\mathbf{h}_j$, $j = 1, 2, \ldots, (n-1)$ and the new array $\mathbf{A}_2^*(j) = \left(\mathbf{a}_1^*(j), \mathbf{a}_2^*(j), \ldots, \mathbf{a}_{m_2}^*(j)\right)$ thus obtained is still an orthogonal array of strength 2, but with the two symbols +1 and –1 in each column. Let the incidence matrix of the SRGD design corresponding to the orthogonal array $\mathbf{A}_1$ be denoted as $\mathbf{N}^{b\times v}$ with the i^{th} row as, $\mathbf{n}_i = (n_{i1}, n_{i2}, \ldots, n_{iv})$, $n_{ij} = 0$ or 1; $1 \le i \le b$, $1 \le j \le v$. The $k = m_1$ non-zero elements of each row of **N** are replaced by the k elements (± 1) of $\mathbf{h}_u^*$, the u^{th} column of the Hadamard matrix $\mathbf{H}_k = \left[\mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_{k-1}^*, \mathbf{1}\right]$ in that order and thus a matrix $\mathbf{N}_u^*$ is obtained with the i^{th} row as $\mathbf{n}_i^*(u) = \left(n_{i1}^*(u), n_{i2}^*(u), \ldots, n_{iv}^*(u)\right)$ $u = 1, 2, \ldots, k-1$; $i = 1, 2, \ldots b$. Next construct $m_2(k-1)(n-1)$ optimum **W**-matrices $\mathbf{W}(j, u, q)$ by taking the Khatri-Rao product of $\mathbf{a}_q^*(j)$ and $\mathbf{N}_u^*$ i.e.

$$\mathbf{W}(j,u,q) = \mathbf{a}_q^*(j) \odot \mathbf{N}_u^* = \begin{pmatrix} a_{1q}^*(j) \\ \vdots \\ a_{bq}^*(j) \end{pmatrix} \odot \begin{pmatrix} \mathbf{n}_1^*(u) \\ \vdots \\ \mathbf{n}_b^*(u) \end{pmatrix} \qquad (2.6)$$

$q = 1, 2, ..., m; q = 1, 2, ..., m_2;$
$u = 1, 2, ..., k-1; j = 1, 2, ..., n.$

Remark 2.5. It follows from Theorem 2.2 that the maximum number of **W**-matrices that can be constructed depends on the maximum value of $m_2(m_1 - 1)(n-1)$ where $m_1 > 0$, $m_2 > 0$, $m_1 + m_2 = m$ and each of m_1, n is such that $\mathbf{H}_{m1}$ and $\mathbf{H}_n$ exist.

Remark 2.6.

(a) If $n = 4t + 2$, $t\ (\geq 0)$ an integer, then it is possible to construct $(k-1)m_2$ optimum **W**-matrices for the SRGD with the above parameters as the elements of $\left(\mathbf{1}'_{\frac{n}{2}}, -\mathbf{1}'_{\frac{n}{2}}\right)$ are used to replace the n symbols in the columns of $\mathbf{A}_2$.

(b) Similarly, if $k = 4t + 2$, $t\ (\geq 0)$ an integer, then it is possible to construct $(n-1)m_2$ optimum **W**-matrices for the SRGD with the above parameters.

(c) Again, if $n = 4t + 2$, $k = 4t^* + 2$, $t, t^* (\geq 0)$ an integer, then in the same way, it is possible to construct m_2 optimum **W**-matrices for the SRGD with the above parameters.

2.3 Regular Group Divisible (RGD) Design Set-up

It is known that if from a BIBD with parameters v^*, b^*, r^*, k^*, $\lambda^* = 1$, all the r^* blocks in which a particular treatment occurs are deleted, then an RGD design with parameters $v = v^* - 1$, $b = b^* - r^*$, $r = r - 1$, $k = k^*$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = r^*$, $n = k^* - 1$ can be obtained (Bose *et al.* 1953). It is difficult to construct a covariate design optimally for such a GD design obtained from such an arbitrary BIBD. However, for some series of BIBDs, it is possible to provide OCDs.

Let the series of BIBDs with parameters:

$$v^* = 4(3t+1),\ b^* = (4t+1)(3t+1),$$
$$r^* = 4t+1,\ k^* = 4,\ \lambda^* = 1 \tag{2.7}$$

be considered with the initial blocks:

$$\left(x_1^{2i'}, x_1^{2t+2i'}, x_2^{\alpha+2i'}, x_2^{\alpha+2t+2i'}\right);$$
$$\left(x_2^{2i'}, x_2^{2t+2i'}, x_3^{\alpha+2i'}, x_3^{\alpha+2t+2i'}\right);$$
$$\left(x_3^{2i'}, x_3^{2t+2i'}, x_1^{\alpha+2i'}, x_1^{\alpha+2t+2i'}\right);$$
$$(0_1, 0_2, 0_3, \infty);\ i' = 0, 1, \ldots, t-1 \tag{2.8}$$

where x is a primitive root of GF($4t + 1$); 1, 2, 3 are the three symbols attached to x, α is an odd integer and ∞ is the invariant treatment symbol (cf. Bose 1939, pp. 384). If the initial block containing the treatment symbol ∞ in (2.8) is deleted and others are developed, then an RGD design with parameters

$$v = 3(4t+1),\ b = 3t(4t+1),\ r = 4t,\ k = 4,$$
$$\lambda_1 = 0,\ \lambda_2 = 1,\ m = 4t+1,\ n = 3 \tag{2.9}$$

is obtained. The $(4t + 1)$ groups obtained by developing $(0_1, 0_2, 0_3)$ over GF($4t + 1$) give the association scheme for the above RGD design.

The following theorem provides OCDs for the series with parameters given in (2.9).

Theorem 2.3. (D(2009b)): If $\mathbf{H}_t$ exists, then $3t$ optimum **W**-matrices can be constructed for the RGD design with parameters given in (2.9).

Method can be described as follows:

Let the $3t$ initial blocks other than $(0_1, 0_2, 0_3, \infty)$ of (2.9) be divided into t sets of 3 blocks each, the i^{th} set being

$$\mathrm{S}_i = \Big\{\left(x_1^{2i'}, x_1^{2t+2i'}, x_2^{\alpha+2i'}, x_2^{\alpha+2t+2i'}\right),$$
$$\left(x_2^{2i'}, x_2^{2t+2i'}, x_3^{\alpha+2i'}, x_3^{\alpha+2t+2i'}\right),$$
$$\left(x_3^{2i'}, x_3^{2t+2i'}, x_1^{\alpha+2i'}, x_1^{\alpha+2t+2i'}\right)\Big\}$$
$$i = i' + 1 = 1, 2, \ldots, t.$$

Also let each of the initial blocks of S_i be displayed in the form of row vectors of the incidence matrix. Development of the initial blocks of S_i will give rise to the sub-incidence matrix $\mathbf{N}_i$ of order $3(4t + 1) \times 3(4t + 1)$ where

$$\mathbf{N}_i = \begin{pmatrix} \mathbf{N}_1^{(i)} & \mathbf{N}_2^{(i)} & 0 \\ 0 & \mathbf{N}_1^{(i)} & \mathbf{N}_2^{(i)} \\ \mathbf{N}_2^{(i)} & 0 & \mathbf{N}_1^{(i)} \end{pmatrix}$$

It is easy to see that $\mathbf{N}_1^{(i)}$ and $\mathbf{N}_2^{(i)}$ matrices corresponding to two portions of the initial blocks of S_i, are obtained by cyclically permuting row vectors of each of the matrices. For $j = 1, 2$, the two non-zero positions of the first row of $\mathbf{N}_j^{(i)}$ is replaced by 1 and

–1 successively and then this row is permuted cyclically in the same way as $\mathbf{N}_j^{(i)}$ was obtained. The resultant matrix is denoted by $\mathbf{W}_j^{(i)}$. Replacing $\mathbf{N}_j^{(i)}$ s of $\mathbf{N}_i$ by $\mathbf{W}_j^{(i)}$ s, one would get a matrix $\mathbf{W}_{i1}$ of order $3(4t + 1) \times 3(4t + 1)$ which can be displayed as

$$\mathbf{W}_{i1} = \begin{pmatrix} \mathbf{W}_1^{(i)} & \mathbf{W}_2^{(i)} & 0 \\ 0 & \mathbf{W}_1^{(i)} & \mathbf{W}_2^{(i)} \\ \mathbf{W}_2^{(i)} & 0 & \mathbf{W}_1^{(i)} \end{pmatrix}$$

Then two other matrices viz. $\mathbf{W}_{i2}$ and $\mathbf{W}_{i3}$ are constructed from $\mathbf{W}_{i1}$ and $\mathbf{N}_i$ respectively, where

$$\mathbf{W}_{i2} = \begin{pmatrix} \mathbf{W}_1^{(i)} & -\mathbf{W}_2^{(i)} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_1^{(i)} & -\mathbf{W}_2^{(i)} \\ -\mathbf{W}_2^{(i)} & \mathbf{0} & \mathbf{W}_1^{(i)} \end{pmatrix}$$

$$\mathbf{W}_{i3} = \begin{pmatrix} \mathbf{N}_1^{(i)} & -\mathbf{N}_2^{(i)} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_1^{(i)} & -\mathbf{N}_2^{(i)} \\ -\mathbf{N}_2^{(i)} & \mathbf{0} & \mathbf{N}_1^{(i)} \end{pmatrix}$$

It can be easily checked that these three matrices satisfy the condition **C** for each i, $i = 1, 2,\ldots,t$. If Hadamard matrix $\mathbf{H}_t = (h_{ml})$ exists, the number of **W**-matrices can be increased t times and $3t$ optimum **W**-matrices viz. $\mathbf{W}_j(m)$ can be constructed through the Khatri-Rao product, where

$$\mathbf{W}_j(m) = \begin{pmatrix} h_{m1} \\ \vdots \\ h_{mt} \end{pmatrix} \odot \begin{pmatrix} \mathbf{W}_{1j} \\ \vdots \\ \mathbf{W}_{tj} \end{pmatrix} \qquad (2.10)$$

$\forall\, m = 1, 2, \ldots t$ and $j = 1, 2, 3$

Remark 2.7. Here as $b = tv$ and $r = tk$, $3t$ **W**-matrices can be constructed alternatively from Theorem 2.2 of Dutta *et al.* (2010) on BPEBDs, since $\mathbf{H}_4$ and $\mathbf{H}_t$ exist.

Remark 2.8. If t is even but not multiple of 4, then 6 optimum **W**-matrices can be constructed for the RGD design by finding two orthogonal rows of size t.

2.4 L_i Design Set-up

It is known that, for $2 \le i < s + 1$, L_i-type PBIBD with parameters

$$v = s^2,\ b = si,\ r = i,\ k = s,\ \lambda_1 = 1,\ \lambda_2 = 0 \qquad (2.11)$$

exists if $(i - 2)$ mutually orthogonal latin squares exist (cf. Raghavarao 1971, pp. 171). Optimum **W**-matrices for the series (2.11) can be constructed through the following theorem.

Theorem 2.4. (D(2009b)): If $\mathbf{H}_s$ and $\mathbf{H}_{i-2}$ exist, then $(i-3)(s-1)$ optimal **W**-matrices can be constructed for the L_i-type PBIBD with parameters given in (2.11).

The above theorem is described as follows:

Suppose s^2 treatments of the L_i design are arranged in the following $s \times s$ natural square:

$$\begin{matrix} 1 & 2 & 3 & \ldots & s \\ s+1 & s+2 & s+3 & \ldots & 2s \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (s-1)s+1 & (s-1)s+2 & (s-1)s+3 & \ldots & s^2 \end{matrix}$$

The L_i design is constructed in the usual way (cf. Raghavrao 1971) where the blocks are partitioned into different sets as:

$\mathbf{G}_1$ = the set of s blocks corresponding to the s rows of the natural square,

$\mathbf{G}_2$ = the set of s blocks corresponding to the s columns of the natural square,

$\mathbf{G}_{j+2}$ = the set of s blocks containing s symbols each. These s blocks correspond to the s symbols of L_j, the j^{th} orthogonal latin square, when it is superimposed on the natural square, $\forall\, j = 1, 2,\ldots,(i - 2)$.

Each set is transformed into a sub-incidence matrix of order $s \times s^2$ with elements 0 and 1. The matrix obtained from pth set $(p = 1, 2,\ldots,i)$ is denoted by $\mathbf{U}_p$, represented as

$$\mathbf{U}_p = (\mathbf{U}_{p1}\ \mathbf{U}_{p2} \ldots \mathbf{U}_{ps}) \qquad (2.12)$$

where $\mathbf{U}_{pm}$ is the m^{th} sub-matrix of order $s \times s$ corresponding to the treatments in the m^{th} row of the natural square, $m = 1, 2, \ldots, s$.

For fixed α, α', the s non-zero elements of $\mathbf{U}_{1m}$ are replaced by those of the m^{th} row of $\mathbf{h}_\alpha \otimes h'_{\alpha'}$, $m = 1, 2,\ldots, s$, where $\mathbf{h}_\alpha$ is the α^{th} column of $\mathbf{H}_s = [\mathbf{h}_1,\ldots,\mathbf{h}_{s-1}, \mathbf{1}]$. The matrix $\mathbf{U}_1$ is thus changed to $\mathbf{U}_1^{(\alpha,\alpha')}$, whose row-sums vanish.

It is always possible to define a $s \times s^2$ matrix $\mathbf{U}_2^{(\alpha,\alpha')}$ such that all the row and column-sums of the $2s \times s^2$ matrix $\mathbf{V}^{(\alpha,\alpha')}$ vanish, where

$$\mathbf{V}^{(\alpha,\alpha')} = \begin{pmatrix} \mathbf{U}_1^{(\alpha,\alpha')} \\ U_2^{(\alpha\,\alpha')} \end{pmatrix} \tag{2.13}$$

Now varying α and α' over 1, 2,. . . , s – 1, $(s-1)^2$ such $\mathbf{V}^{(\alpha,\alpha')}$ matrices can be obtained and these matrices actually give $(s-1)^2$ optimum **W**-matrices for the design when $i = 2$.

When $i > 2$, for p = 3, 4,. . . , i, an $s \times s^2$ matrix $\mathbf{U}_p^{(\alpha)}$ is constructed from $\mathbf{U}_p$ as

$$\mathbf{U}_p^{(\alpha)} = \mathbf{h}'_\alpha \otimes \mathbf{U}_p$$

Again a matrix $\mathbf{V}^{*(\alpha,q)}$ of order $s(i-2) \times s^2$ is constructed through the following Khatri-Rao product as

$$\mathbf{V}^{*(\alpha,q)} = \begin{pmatrix} h_{1q}^* \\ \vdots \\ h_{(i-2)q}^* \end{pmatrix} \odot \begin{pmatrix} \mathbf{U}_3^{(\alpha)} \\ \vdots \\ \mathbf{U}_i^{(\alpha)} \end{pmatrix} \qquad 2.14)$$

$\forall\, \alpha = 1, 2,\ldots, (s-1);\ q = 1, 2, \ldots, (i-3)$

where $\mathbf{h}_q^* \left(h_{1q}^*, h_{2q}^*, \ldots, h_{(i-2)q}^*\right)'$ $(\neq (1, 1,...,1)')$ is the q^{th} column of $\mathbf{H}_{i-2}$. Varying α and q, $(s-1)(i-3)$ such $\mathbf{V}^{*(\alpha,q)}$ matrices can be obtained. Again, $(s-1)(i-3)$ matrices $\mathbf{W}^{(\alpha,q)}$ giving OCDs can be constructed by pairing $(s-1)(i-3)$ matrices $\mathbf{V}^{(\alpha,q)}$ defined in (2.14) with $(s-1)(i-3)$ matrices defined in (2.13), i.e.

$$\mathbf{W}^{(\alpha,q)} = \begin{pmatrix} \mathbf{V}^{(\alpha,q)} \\ \mathbf{V}^{*(\alpha,q)} \end{pmatrix} \begin{matrix} \forall \alpha = 1, 2, ..., (s-1); \\ q = 1, 2, ..., (i-3) \end{matrix} \tag{2.15}$$

where $\mathbf{V}^{(\alpha,q)}$ corresponds to the q^{th} matrix (q = 1, 2,. . ., $(i-3)$, $(i-3) < s-1$) from $(s-1)$ matrices $\mathbf{V}^{(\alpha,\alpha')}$, α' = 1, 2,. . .,$(s-1)$ for fixed α.

Remark 2.9. In the case of L_4 design with $v = 16$ constructed from 2 mutually orthogonal latin squares, three more optimum **W**s can be constructed. Actually

$$\begin{pmatrix} \mathbf{V}^{(1,1)} \\ -\mathbf{V}^{*(1,1)} \end{pmatrix}, \begin{pmatrix} \mathbf{V}^{(2,1)} \\ -\mathbf{V}^{*(2,1)} \end{pmatrix}, \text{ and } \begin{pmatrix} \mathbf{V}^{(3,1)} \\ -\mathbf{V}^{*(3,1)} \end{pmatrix}$$

give additional 3 OCDs together with the 3 given in (2.15).

Remark 2.10. If $\mathbf{H}_s$ and/or $\mathbf{H}_{i-2}$ do not exist but s and $i-2$ are even then also a number of optimum **W**-matrices can be constructed for the L_i design for suitable values of i and s. Actually vectors $\left(\mathbf{1}'_{\frac{s}{2}}, -\mathbf{1}'_{\frac{s}{2}}\right)$ and $\left(\mathbf{1}'_{\frac{i-2}{2}}, -\mathbf{1}'_{\frac{i-2}{2}}\right)$ are used for one column of $\mathbf{H}_s$ and/or of $\mathbf{H}_{i-2}$ respectively.

(a) If $i \equiv 0 \pmod 4$ and $s \equiv 0 \pmod 4$, then it is possible to construct $(s-1)$ optimum **W**-matrices for the L_i design.

(b) If $i \equiv 2 \pmod 4$ and $s \equiv 2 \pmod 4$, then it is possible to construct $(i-3)$ optimum **W**-matrices for the L_i design.

(c) If $i = 2$ and $s \equiv 2 \pmod 4$, then it is possible to construct one optimum **W**-matrix for the L_2 design with the above parameters.

(d) If $i = 0 \pmod 4$, $s \equiv 2 \pmod 4$, then it is possible to construct one optimum **W**-matrix for the L_i.

It is known that an L_2 design with the parameters

$$v = s^2,\ b = s(s-1),\ r = 2(s-1),$$
$$k = 2s,\ \lambda_1 = s,\ \lambda_2 = 2 \tag{2.16}$$

exists (cf. Clatworthy 1967), where the blocks are formed by combining all possible pairs of rows and all possible pairs of columns of the $s \times s$ natural square. In this case, the covariate values can be chosen optimally and the following theorem can be proved.

Theorem 2.5. (D(2009b)): If $\mathbf{H}_s$ exists, then $(s-1)^2$ optimum **W**-matrices can be constructed for the L_2 design.

The method is explained through an example when $s = 4$. The incidence matrix **N** (partitioned into 8 submatrices $\mathbf{N}_{pq}$, p = 1, 2, q = 1, 2, 3, 4) of the design looks like

Blocks ↓ Treatments →

Blocks	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
Row	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
pairs	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
↑	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
↓	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
Column	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
pairs	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
↑	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1

Writing $\mathbf{H}_4$ as $\mathbf{H}_4 = (\mathbf{1}, \mathbf{h}_2, \mathbf{h}_3, \mathbf{h}_4)$, $\mathbf{h}_j = (h_{j1}, h_{j2}, h_{j3}, h_{j4})'$, the 3 unit vectors of $\mathbf{N}_{1q}$, $q = 1, 2, 3, 4$ are replaced by $h_{iq}\mathbf{h}'_j$. The non-null elements in the l^{th} column of each row of each of $\mathbf{N}_{21}$, $\mathbf{N}_{22}$, $\mathbf{N}_{23}$ and $\mathbf{N}_{24}$ are replaced by $-h_{iq}\mathbf{h}'_j$, $l = 1, 2, 3, 4$ so that the column-sums of $\mathbf{N}$ vainsh. The resultant materices give a $\mathbf{W}$-matrix. Varying i and j give q $(= (s-1)^2)$ OCDs.

Remark 2.11. If $s \equiv 2 \pmod 4$, then one optimum $\mathbf{W}$-matrix can be constructed for the L_2 design.

Let us now consider a series of L_2 designs which can be obtained from a BIBD D^* where

$$D^* = \text{BIBD}(v^* = s, b^*, r^*, k^*, \lambda^*) \quad (2.17)$$

The i^{th} treatment in D^* is replaced by the treatments in the i^{th} row of the $s \times s$ natural square. This procedure leads to a design with b^* blocks, each of size sk^*. Adjoin to these b^* blocks, another b^* blocks, obtained by replacing the i^{th} treatment in D^* by the treatments in the i^{th} column of the $s \times s$ natural square, $i = 1, 2, \ldots, s$. Then these $2b^*$ blocks give an L_2 design D^{**} with parameters (cf. Raghavarao 1971, pp. 159)

$$v = s^2,\ b = 2b^*,\ r = 2r^*,\ k = sk^*,\ \lambda_1 = r^* + \lambda^*,\ \lambda_2 = 2\lambda^* \quad (2.18)$$

Theorem 2.6. (D(2009b)): A set of q optimum $\mathbf{W}$-matrices can be constructed for the L_2 design with parameters given in (2.18) where

(i) $q = 2s(s-1)(r^*-1)$ if $\mathbf{H}_s$ and $\mathbf{H}_{r^*}$ exist;

(ii) $q = 4((s-1)(r^*-1)-(s-2))$ if $s \equiv 2 \pmod 4$, $(s-1)$ is a prime or a prime power and $\mathbf{H}_{r^*}$ exists;

(iii) $q = 4((s-1)(r^*-1)-(s-2))$ if $\mathbf{H}_{2s}$ and $\mathbf{H}_{\frac{r^*}{2}}$ exist;

(iv) $q = 4$ if s and r^* are both even but not a multiple of 4.

As in the case of SGD in Theorem 2.1, the incidence matrix $\mathbf{N}$ of the BIBD transforms to

$$\mathbf{N}_1^{b^* \times s^2} = \left(\mathbf{N}_1^{(R)}, \ldots, \mathbf{N}_s^{(R)}\right)$$

where the s elements of the BIBD are replaced by those of the s rows of the square and the non-null elements of $\mathbf{N}_j^{(R)}$ form the incidence matrix of a $r^* \times s$ RBD. In the same way $\mathbf{N}$ transforms to $\mathbf{N}_2^{b^* \times s^2} = \left(\mathbf{N}_1^{(C)}, \ldots, \mathbf{N}_s^{(C)}\right)$ when the elements in the columns of the natural square are used for replacement. In $\mathbf{N}_2^{b^* \times s^2}$ the s^2 treatments are in the order of those in the columns. Here $\mathbf{N}_i^{(C)}$s have similar properties like those of $\mathbf{N}_i^{(R)}$s. If $\mathbf{W}_1, \ldots, \mathbf{W}_c$ are c optimum $\mathbf{W}$-matrices for $r^* \times s$ RBD, then by replacing the non-null elements of $\mathbf{N}_1$ and $\mathbf{N}_2$ by these of $\mathbf{W}_j$, $2c$ matrices defined as

$$\mathbf{W}_j^{(R)} = \left(\mathbf{W}_{1j}^{(R)}, \ldots, \mathbf{W}_{sj}^{(R)}\right)$$

$$\mathbf{W}_j^{(C)} = \left(\mathbf{W}_{1j}^{(C)}, \ldots, \mathbf{W}_{sj}^{(C)}\right)$$

can be constructed. Again as $\mathbf{H}_s$ exists, matrices $\mathbf{W}_{lj}^{(R)*}$ and $\mathbf{W}_{lj}^{(C)*}$ can be constructed as

$$\mathbf{W}_{lj}^{(R)*} = \mathbf{h}_l \odot \mathbf{W}_j^{(R)},\ \mathbf{W}_{lj}^{(C)*} = h_l \odot \mathbf{W}_j^{(C)}$$

where $\mathbf{h}_l$ is the column of $\mathbf{H}_s$, $l = 1, \ldots, s$. Next we permute the columns of $\mathbf{W}_{lj}^{(C)*}$ so that they correspond to the treatments in the rows of the square viz. $1, 2, \ldots, s^2$. Let the changed matrix be denoted by $\mathbf{W}_{lj}^{(C)**}$. Then

$$\mathbf{W}_{lj}^{(1)} = \begin{pmatrix} \mathbf{W}_{lj}^{(R)} \\ \mathbf{W}_{lj}^{(C)**} \end{pmatrix},\ \mathbf{W}_{lj}^{(2)} = \begin{pmatrix} \mathbf{W}_{lj}^{(R)} \\ -\mathbf{W}_{lj}^{(C)**} \end{pmatrix}$$

$$j = 1, \ldots, c;\ l = 1, \ldots, s$$

gives $2sc$ OCDs for the L_2 design in (2.18).

Remark 2.12. If the assumptions on r^* and s are interchanged, then the number of optimum **W**-matrices in (ii) and (iii) of Theorem 2.6 changes to

(i) $q = 2s((s-1)(r^*-1)-(r^*-2))$ if $r^* \equiv 2 \pmod 4$, (r^*-1) is a prime or a prime power and $\mathbf{H}_s$ exists; and

(ii) $q = 2s((s-1)(r^*-1)-(r^*-2))$ if $\mathbf{H}_{2r^*}$ and $\mathbf{H}_{\frac{s}{2}}$ exist; respectively.

Remark 2.13. L_2 design with parameters $v = s^2$, $b = s(s-1)$, $r = 2(s-1)$, $k = 2s$, $\lambda_1 = s$, $\lambda_2 = 2$ was obtained by Clatworthy (1967) by constructing blocks with every possible pairs of rows and columns of the natural square of side s. The same design may also obtained from the method used to obtain L_2 design with parameters given in (2.18) by considering irreducible BIBD($v^* = s$, $b^* = \binom{s}{2}$, $r^* = s-1$, $k^* = 2$, $\lambda^* = 1$) where every possible pair of treatments forms blocks.

So, using the same procedure described in Theorem 2.5, we can construct $(s-1)^2$ optimum **W**-matrices for the L_2 design with parameters in (2.18) when $\mathbf{H}_s$ exists.

Remark 2.14. If s is even but not a multiple of 4 and r^* is an odd positive integer then one optimal **W**-matrix can be constructed for the L_2 design with parameters in (2.18).

Remark 2.15. If p optimum **W**-matrices exist for the BIBD D* with parameters given in (2.17), then an improvement in the number of covariates can be made. Actually $2p + q$ optimum **W**-matrices can be constructed for such L_2 designs where q is the number of OCDs given in Theorem 2.6.

It is known that the existence of the BIBD ($v^* = s$, b^*, r^*, k^*, λ^*) mentioned in (2.17) implies the existence of another L_2 design with parameters

$$v = s^2,\ b = 2sb^*,\ r = 2r^*,\ k = k^*,\ \lambda_1 = \lambda^*,\ \lambda_2 = 0 \tag{2.19}$$

(cf. Raghavarao 1971, pp. 160). The above design can be obtained by writing down the BIBD with the symbols occurring in the rows and in the columns of the natural square. The following theorem gives a method of constructing **W**-matrices for this design set-up with parameters given in (2.19).

Theorem 2.7. (D(2009b)): If t optimum **W**-matrices exist for the BIBD with parameters given in (2.17) and $\mathbf{H}_{2s}$ exists, then $2st$ optimum **W**-matrices for the L_2 design with parameters given in (2.19) can be constructed.

Proof. By assumption, for fixed j, $j = 1, 2,...,2s$, t optimum **W**-matrices $\mathbf{W}_1^*(j)$, $\mathbf{W}_2^*(j)$,..., $\mathbf{W}_t^*(j)$ for the j^{th} set of blocks forming a BIBD exist. Again, by assumption $\mathbf{H}_{2s}$ exists and is written as $\mathbf{H}_{2s} = (h_{\alpha j})$. It can be seen that the following $2st$ matrices

$$\mathbf{W}_i(\alpha) = \begin{pmatrix} h_{\alpha 1} \\ \vdots \\ h_{\alpha 2s} \end{pmatrix} \odot \begin{pmatrix} \mathbf{W}_i^*(1) \\ \vdots \\ \mathbf{W}_i^*(2s) \end{pmatrix}$$

$\forall\ \alpha = 1, 2, ..., 2s;\ i = 1, 2,..., t$

give optimum **W**-matrices.

Remark 2.16. If s is an odd integer, then $2t$ optimal **W**-matrices for the L_2 design with parameters given in (2.19) can be constructed.

2.5 Triangular Design Set-up

Since the irreducible BIBD with the parameters $v^* = b^* = n-1$, $r^* = k^* = n-2$, $\lambda^* = n-3$ exists for all integral $n \geq 4$, the triangular design with parameters

$$v = \frac{n(n-1)}{2},\ b = n(n-1),\ r = 2(n-2),$$
$$k = n-2,\ \lambda_1 = n-3,\ \lambda_2 = 0 \tag{2.20}$$

exists for all $n \geq 4$ (cf. Raghavarao 1971, pp. 154). This design can be obtained by writing the solution of the BIBD with the $(n-1)$ treatments in the $1^{st}, 2^{nd}, \ldots, n^{th}$ rows of the triangular association scheme. Optimum **W**-matrices for the series (2.20) can be constructed with the help of the following theorem.

Theorem 2.8. (D(2009b)): If $n = p^h + 1$, where p is a prime number, h is a positive integer such that $p^h \equiv 1 \pmod 4$ and $n > 4$, then $(n-1)(n-4) + (n-2)$ optimum **W**-matrices for the triangular design with parameters given in (2.20) can be constructed.

The proof of the theorem is very much involved and OCDs are constructed for one such design. Interested reader is referred to D(2009b).

Table 1. (OCDs in SGD designs)

(v, b, r, k, λ_1, λ_2, m, n) denote the parameters of the PBIBD and (v^* b^* r^* k^* λ) denote the parameters of the BIBD used to construct the PBIBD. t denotes the number of optimum **W**-matrices and t_1 denotes the additional number of **W**-matrices mentioned in Remark 2.4.

Sl. no.	Design no.	v	b	r	k	λ_1	λ_2	m	n	v^*	b^*	r^*	k^*	λ	t	$t + t_1$	Method of construction
1	S1	6	3	2	4	2	1	3	2	3	3	2	2	1	1	2	T 2.1(i), R 2.4
2	S2	6	6	4	4	4	2	3	2	3	6	4	2	2	2	5	T 2.1(i), R 2.4
3	S3	6	9	6	4	6	3	3	2	3	9	6	2	3	1	2	T 2.1(i), R 2.4
4	S4	6	12	8	4	8	4	3	2	3	12	8	2	4	4	11	T 2.1(i), R 2.4
5	S5	6	15	10	4	10	5	3	2	3	15	10	2	5	1	2	T 2.1(i), R 2.4
6	S7	8	12	6	4	6	2	4	2	4	12	6	2	2	1	5	T 2.1(v), R 2.4
7	S9	10	10	4	4	4	1	5	2	5	10	4	2	1	2	5	T 2.1(i), R 2.4
8	S10	10	20	8	4	8	2	5	2	5	20	8	2	2	4	11	T 2.1(i), R 2.4
9	S12	12	30	10	4	10	2	6	2	6	30	10	2	2	1	3	R 2.3, R 2.4
10	S13	14	21	6	4	6	1	7	2	7	21	6	2	1	1	2	T 2.1(i), R 2.4
11	S15	18	36	8	4	8	1	9	2	9	36	8	2	1	4	11	T 2.1(i), R 2.4
12	S17	22	55	10	4	10	1	11	2	11	55	10	2	1	1	2	T 2.1(i), R 2.4
13	S19	8	8	6	6	6	4	4	2	4	8	6	3	4	0	4	T 2.1(v), R 2.4
14	S21	9	3	2	6	2	1	3	3	3	3	2	2	1	1	1	R 2.4
15	S22	9	6	4	6	4	2	3	3	3	6	4	2	2	2	2	R 2.4
16	S23	9	9	6	6	6	3	3	3	3	9	6	2	3	1	1	R 2.4
17	S24	9	12	8	6	8	4	3	3	3	12	8	2	4	4	4	R 2.4
18	S25	9	15	10	6	10	5	3	3	3	15	10	2	5	1	1	R 2.4
19	S26	10	10	6	6	6	3	5	2	5	10	6	3	3	0	1	T 2.1(i)
20	S29	12	12	6	6	6	2	4	3	4	12	6	2	2	1	1	R 2.4
21	S31	12	20	10	6	10	4	6	2	6	20	10	3	4	0	2	R 2.3
22	S33	14	14	6	6	6	2	7	2	7	14	6	3	2	0	1	T 2.1(i)
23	S35	15	10	4	6	4	1	5	3	5	10	4	2	1	2	2	R 2.4
24	S36	15	20	8	6	8	2	5	3	5	20	8	2	2	4	4	R 2.4
25	S37	18	12	4	6	4	1	9	2	9	12	4	3	1	0	3	T 2.1(i)
26	S39	18	24	8	6	8	2	9	2	9	24	8	3	2	0	7	T 2.1(i)
27	S40	18	30	10	6	10	2	6	3	6	30	10	2	2	1	1	R 2.4
28	S42	21	21	6	6	6	1	7	3	7	21	6	2	1	1	1	R 2.4
29	S44	26	26	6	6	6	1	13	2	13	26	6	3	1	0	1	T 2.1(i)
30	S45	27	36	8	6	8	1	9	3	9	36	8	2	1	4	4	R 2.4
31	S48	33	55	10	6	10	1	11	3	11	55	10	2	1	1	1	R 2.4
32	S50	42	70	10	6	10	1	21	2	21	70	10	3	1	0	1	T 2.1(i)
33	S51	10	5	4	8	4	3	5	2	5	5	4	4	3	3	6	T 2.1(i), R 2.4
34	S52	10	10	8	8	8	6	5	2	5	10	8	4	6	6	13	T 2.1(i), R 2.4
35	S53	12	3	2	8	2	1	3	4	3	3	2	2	1	1	4	T 2.1(i), R 2.4

Sl. no.	Design no.	v	b	r	k	λ_1	λ_2	m	n	v^*	b^*	r^*	k^*	λ	t	$t + t_1$	Method of construction
36	S54	12	6	4	8	4	2	3	4	3	6	4	2	2	2	11	T 2.1(i), R 2.4
37	S55	12	9	6	8	6	3	3	4	3	9	6	2	3	1	12	T 2.1(i), R 2.4
38	S56	12	12	8	8	8	4	3	4	3	12	8	2	4	4	25	T 2.1(i), R 2.4
39	S57	12	15	10	8	10	5	3	4	3	15	10	2	5	1	20	T 2.1(i), R 2.4
40	S58	12	15	10	8	10	6	6	2	6	15	10	4	6	1	3	R 2.3
41	S59	14	7	4	8	4	2	7	2	7	7	4	4	2	3	6	T 2.1(i), R 2.4
42	S60	14	14	8	8	8	4	7	2	7	14	8	4	4	6	13	T 2.1(i), R 2.4
43	S62	16	12	6	8	6	2	4	4	4	12	6	2	2	1	45	T 2.1(iii), R 2.4
44	S65	18	18	8	8	8	3	9	2	9	18	8	4	3	6	13	T 2.1(i), R 2.4
45	S66	20	10	4	8	4	1	5	4	5	10	4	2	1	2	11	T 2.1(i), R 2.4
46	S67	20	15	6	8	6	2	10	2	10	15	6	4	2	0	2	R 2.3
47	S68	20	20	8	8	8	2	5	4	5	20	8	2	2	4	25	T 2.1(i), R 2.4
48	S70	24	30	10	8	10	2	6	4	6	30	10	2	2	1	39	R 2.2(ii)
49	S71	26	13	4	8	4	1	13	2	13	13	4	4	1	3	6	T 2.1(i), R 2.4
50	S72	26	26	8	8	8	2	13	2	13	26	8	4	2	6	13	T 2.1(i), R 2.4
51	S73	28	21	6	8	6	1	7	4	7	21	6	2	1	1	12	T 2.1(i), R 2.4
52	S76	32	40	10	8	10	2	16	2	16	40	10	4	2	1	17	T 2.1(v), R 2.4
53	S77	36	36	8	8	8	1	9	4	9	36	8	2	1	4	25	T 2.1(i), R 2.4
54	S79	44	55	10	8	10	1	11	4	11	55	10	2	1	1	20	T 2.1(i), R 2.4
55	S80	50	50	8	8	8	1	25	2	25	50	8	4	1	6	13	T 2.1(i), R 2.4
56	S99	12	12	10	10	10	8	6	2	6	12	10	5	8	0	2	R 2.3
57	S100	15	3	2	10	2	1	3	5	3	3	2	2	1	1	1	R 2.4
58	S101	15	6	4	10	4	2	3	5	3	6	4	2	2	2	2	R 2.4
59	S102	15	9	6	10	6	3	3	5	3	9	6	2	3	1	1	R 2.4
60	S103	15	12	8	10	8	4	3	5	3	12	8	2	4	4	4	R 2.4
61	S104	15	15	10	10	10	5	3	5	3	15	10	2	5	1	1	R 2.4
62	S105	18	18	10	10	10	5	9	2	9	18	10	5	5	0	1	T 2.1(i)
63	S107	20	12	6	10	6	2	4	5	4	12	6	2	2	1	1	R 2.4
64	S111	22	22	10	10	10	4	11	2	11	22	10	5	4	0	1	T 2.1(i)
65	S112	25	10	4	10	4	1	5	5	5	10	4	2	1	2	2	R 2.4
66	S113	25	20	8	10	8	2	5	5	5	20	8	2	2	4	4	R 2.4
67	S115	30	30	10	10	10	2	6	5	6	30	10	2	2	1	1	R 2.4
68	S116	35	21	6	10	6	1	7	5	7	21	6	2	1	1	1	R 2.4
69	S119	42	42	10	10	10	2	21	2	21	42	10	5	2	0	1	T 2.1(i)
70	S120	45	36	8	10	8	1	9	5	9	36	8	2	1	4	4	R 2.4
71	S121	50	30	6	10	6	1	25	2	25	30	6	5	1	0	1	T 2.1(i)
72	S123	55	55	10	10	10	1	11	5	11	55	10	2	1	1	1	R 2.4
73	S124	82	82	10	10	10	1	41	2	41	82	10	5	1	0	1	T 2.1(i)

Table 2. (OCDs in SRGD designs)

		v	b	r	k	λ_1	λ_2	m	n	OA $(n^2\lambda_2, m, n, 2)$	m_2	$c = (n-1)(k-1)m_2$ (cf. T 2.2)
74	SR1	4	4	2	2	0	1	2	2	OA(4, 3, 2, 2)	1	1
75	SR2	4	8	4	2	0	2	2	2	OA(8, 7, 2, 2)	5	5
76	SR3	4	12	6	2	0	3	2	2	OA(12, 11, 2, 2)	9	9
77	SR4	4	16	8	2	0	4	2	2	OA(16, 15, 2, 2)	13	13
78	SR5	4	20	10	2	0	5	2	2	OA(20, 19, 2, 2)	17	17
79	SR9	8	16	4	2	0	1	2	4	OA(16, 5, 4, 2)	1	3
80	SR10	8	32	8	2	0	2	2	4	OA(32, 10, 4, 2)	8	24
81	SR13	12	36	6	2	0	1	2	6	OA(36, 7, 6, 2)	5	5
82	SR15	16	64	8	2	0	1	2	8	OA(64, 9, 8, 2)	6	42
83	SR17	20	100	10	2	0	1	2	10	OA(100, 11, 10, 2)	9	9
84	SR36	8	8	4	4	0	2	4	2	OA(8, 7, 2, 2)	3	9
85	SR37	8	12	6	4	0	3	4	2	OA(12, 11, 2, 2)	7	21
86	SR39	8	16	8	4	0	4	4	2	OA(16, 15, 2, 2)	11	33
87	SR40	8	20	10	4	0	5	4	2	OA(20, 19, 2, 2)	15	45
88	SR44	16	16	4	4	0	1	4	4	OA(16, 5, 4, 2)	1	9
89	SR45	16	32	8	4	0	2	4	4	OA(32, 10, 4, 2)	6	54
90	SR49	32	64	8	4	0	1	4	8	OA(64, 9, 8, 2)	5	105
91	SR51	40	100	10	4	0	1	4	10	OA(100, 11, 10, 2)	7	21
92	SR66	12	8	4	6	0	2	6	2	OA(8, 7, 2, 2)	1	1
93	SR67	12	12	6	6	0	3	6	2	OA(12, 11, 2, 2)	5	5
94	SR69	12	16	8	6	0	4	6	2	OA(16, 15, 2, 2)	9	9
95	SR70	12	20	10	6	0	5	6	2	OA(20, 19, 2, 2)	13	13
96	SR74	24	32	8	6	0	2	6	4	OA(32, 10, 4, 2)	4	12
97	SR78	48	64	8	6	0	1	6	8	OA(64, 9, 8, 2)	3	21
98	SR91	16	12	6	8	0	3	8	2	OA(12, 11, 2, 2)	3	21
99	SR92	16	16	8	8	0	4	8	2	OA(16, 15, 2, 2)	7	49
100	SR93	16	20	10	8	0	5	8	2	OA(20, 19, 2, 2)	11	77
101	SR95	32	32	8	8	0	2	8	4	OA(32, 10, 4, 2)	2	42
102	SR97	64	64	8	8	0	1	8	8	OA(64, 9, 8, 2)	1	49
103	SR106	20	12	6	10	0	3	10	2	OA(12, 11, 2, 2)	1	1
104	SR107	20	16	8	10	0	4	10	2	OA(16, 15, 2, 2)	5	5
105	SR108	20	20	10	10	0	5	10	2	OA(20, 19, 2, 2)	9	9

Table 3. (OCDs in RGD designs)

		v	b	r	k	λ_1	λ_2	m	n	t	$3t$	Method of construction
106	R114	15	15	4	4	0	1	5	3	1	3	D(2009b)
107	R129	27	54	8	4	0	1	9	3	2	6	T 2.3

Table 4. (OCDs in L_i and Tringular designs)

Sl. no.	Design no.	v	b	r	k	λ_1	λ_2	n_1	n_2	$v^* = s$	b^*	r^*	k^*	λ^*	c^*	c	Method of construction
1	LS1	9	18	4	2	1	0	4	4	3	3	2	2	1	1	2	R 2.16
2	LS2	9	36	8	2	2	0	4	4	3	6	4	2	2	2	4	R 2.16
3	LS5	25	100	8	2	1	0	8	16	5	10	4	2	1	2	4	R 2.16
4	LS28	16	8	2	4	1	0	6	9							9	T 2.4
5	LS45	25	50	8	4	3	0	8	16	5	5	4	4	2	3	6	R 2.16
6	LS46	49	98	8	4	2	0	12	36	7	7	4	4	2	3	6	R 2.16
7	LS72	9	6	4	6	3	2	4	4	3	3	2	2	1	1	1	R 2.15
8	LS73	9	12	8	6	6	4	4	4	3	6	4	2	2	2	2	R 2.15
9	LS74	36	12	2	6	1	0	10	25							1	R 2.10(c)
10	LS98	16	12	6	8	4	2	6	9							9	T 2.5
11	LS102	64	16	2	8	1	0	14	49							49	T 2.4
12	LS104	64	32	4	8	1	0	28	35							14	T 2.4, R 2.9
13	LS106	64	48	6	8	1	0	42	21							21	T 2.4
14	LS135	25	20	8	10	5	2	8	16	5	10	4	2	1	2	2	R 2.15
15	LS137	100	20	2	10	1	0	18	81							1	R 2.10(c)
16	LS139	100	40	4	10	1	0	36	63							1	R 2.10(d)
17	T38	15	30	8	4	3	0	8	6							14	T 2.8

Remark 2.17. For $n \equiv 0 \pmod 4$, then n optimum **W**-matrices can be constructed for the above triangular design.

A list of OCDs for suitable two-associate PBIB designs divided as Singular (S), Semi-regular (SR), Regular (R), Latin square types (LS), Triangular (T) design obtained from Clatworthy (1973, page 107-311) is given below. In the Constructional Method column, T stands for Theorem and R for Remark mentioned in Section 2.

3. BALANCED FACTORIAL DESIGNS WITH ORTHOGONAL FACTORIAL STRUCTURE (OFS)

It is known that the kronecker product of the two incidence matrices of two BIBDs gives the incidence matrix of a rectangular PBIBD (cf. Vartak 1954). Rectangular design gives a balanced factorial design (with two-factor) with OFS (cf. Gupta and Mukerjee 1989). Again it was proved in Dutta *et al.* (2009b) that if OCDs exist for two BIBDs then OCDs exist for the kronecker product design. Thus, choosing appropriate BIBDs, OCDs for a class of balanced two-factor factorial designs with OFS have been constructed by taking Kronecker product of the OCDs of the component BIBDs.

The following table gives the number of OCDs in the BIBD set-ups with $v \leq 15$ (cf. Raghavarao 1971, pp. 91-92). The methods of construction are given in Das *et al.* (2003), Dutta *et al.* (2010). Also these are stated here for ready reference. Any two BIBDs from Table 5 can be used to generate an OCD in the two-factor balanced factorial design set-up.

Now we state the following theorems and remarks.

Table 5. OCDs in BIBDs, any two of which will give OCDs for balanced factorial design

Sl. no.	Design no.	v	b	r	k	λ	c	Method of construction
1	3	5	10	4	2	1	2	R 3.1 or T 3.3
2	4	5	5	4	4	3	3	T 3.1 or T 3.2
3	9	6	15	10	4	6	2	Example 3.2 in Section 3 of Dutta *et al.* (2010)
4	11	7	7	4	4	2	3	T 3.1 or T 3.2
5	12	7	21	6	2	1	1	T 3.1 or T 3.2
6	13	7	7	6	6	5	1	R 3.2
7	18	9	36	8	2	1	4	R 3.1 or T 3.3
8	19	9	18	8	4	3	6	R 3.1 or T 3.3
9	21	9	9	8	8	7	7	T 3.1 or T 3.2
10	30	11	11	6	6	3	1	R 3.2
11	31	11	55	10	2	1	1	T 3.1 or T 3.2
12	32	11	11	10	10	9	1	R 3.2
13	37	13	13	4	4	1	3	T 3.1 or T 3.2
14	40	13	26	12	6	5	2	R 3.3
15	44	15	15	8	8	4	7	T 3.2
16	47	16	16	6	6	2	1	R 3.2
17	49	16	16	10	10	6	1	R 3.2
18	56	19	19	10	10	5	1	R 3.2
19	57	19	57	12	4	2	3	T 3.2
20	61	21	42	12	6	3	2	R 3.2
21	Dual of 64	23	23	12	12	6	11	T 3.2
22	66	25	50	8	4	1	6	T 3.2
23	Dual of 71	27	27	14	14	7	1	R 3.2
24	75	31	31	6	6	1	1	R 3.2
25	76	31	31	10	10	3	1	R 3.2
26	85	45	45	12	12	3	11	T 3.2
27	87	57	57	8	8	1	7	T 3.2
28	91	91	91	10	10	1	1	R 3.2

Theorem 3.1. (Das *et al.* (2003)): Suppose a symmetric balanced incomplete block design (SBIBD) with parameters $v = b$, $r = k$ and λ has been constructed by applying Bose's difference technique (Bose 1939), starting with an initial block of size k and developing the same. Moreover, suppose that $\mathbf{H}_k$ exists. Then $(k - 1)$ optimum **W**-matrices can be constructed.

Remark 3.1. (Das *et al.* (2003)): If a BIBD(mv, v, mk, k, λ) is formed by developing m initial blocks each of size k, then $m(k - 1)$ optimum **W**-matrices can be constructed whenever $\mathbf{H}_m$ and $\mathbf{H}_k$ exist.

Theorem 3.2. (Dutta *et al.* (2010)): For any Binary Proper Equi-replicate Block Design (BEPBD) d(v, b, r, k) with $b = mv$, m (≥1) a positive integer, $(k - 1)$ optimum **W**-matrices can be constructed provided $\mathbf{H}_k$ exists.

Remark 3.2. (Dutta *et al.* (2010)): It follows that in Theorem 3.3 if k is even, then at least one optimum **W**-matrix can always be constructed.

Theorem 3.3. (Dutta *et al.* (2010)): For a k-resolvable BEPBD with $b = mv$, it is possible to construct $m(k-1)$ optimum **W**-matrices, provided $\mathbf{H}_k$ and $\mathbf{H}_m$ exist.

Remark 3.3. (Dutta *et al.* (2010)): Let $\mathbf{H}_m$ exist, $\mathbf{H}_k$ do not exist and k be even. Then following the Theorem 3.3, m optimum **W**-matrices can be constructed for a resolvable BPEBD.

A list of OCDs for BIBDs (cf. Raghavarao 1971, pp. 91-94) is given in Table 5.

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