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Construction of Optimal Mixed-Level Supersaturated Designs

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SUMMARY

This article describes some methods of construction of mixed level f_{NOD}-optimal supersaturated designs. The methods of construction exploit the layout and the property of Uniform designs and Hadamard matrices. Mathematical expression for $E(f_{NOD})$ and $E(\chi^2)$ criteria have been obtained for many designs constructed in this paper. Some examples are given to illustrate the methods of construction. A catalogue of 67 optimal mixed level supersaturated designs with at most 60 runs and 60 factors is prepared. Some other important features of the designs are also given in the catalogue. All designs are f_{NOD}- optimal while some designs are χ^2- optimal too.

Key words : Uniform design, Hadamard matrices, Efficiency criteria.

1. INTRODUCTION

Supersaturated Design (SSD) is essentially a fractional factorial design in which the number of runs is not enough to estimate the main effects of all the factors in the experiment. SSDs are mainly used in experimental situations where a large number of factors are to be tested but only few of the factors are active and the experimentation is expensive and also time consuming.

A common application of SSDs is the screening experiment. In screening experiment there are usually a large number of factors to be investigated, but it is believed that only a few of them will be active, and the few active factors have significant influence on the response. Identifying these few active factors correctly and economically is the main purpose of screening experiments. This phenomenon is commonly recognized as effect sparsity.

The construction of SSDs started with the use of random balance experiments by Satterthwaite (1959). Booth and Cox (1962) proposed an algorithm to construct systematic SSDs. A two-level SSD is balanced when the number of times each level appears in a column is same. Many methods have been proposed for constructing SSDs [see e.g., Nguyen (1996), Gupta and Chatterjee (1998), Butler *et al*. (2001), Xu and Wu (2003), Bulutoglu and Cheng (2004), Bulutoglu (2007), Liu *et al*. (2007a), Ryan and Bulutoglu (2007), Das *et al*. (2008), Gupta *et al*. (2008b) and Nguyen and Cheng (2008)].

Two-level SSDs have been studied extensively because of their application and ease of generation. But multi-level SSDs are often required in industrial and scientific experimentation for exploring nonlinear effects of the factors. It is never advised to reduce the factor levels to two if it would result in severe loss in information. Some references on multi-level factor designs include Yamada and Lin (1999), Fang *et al.* (2000), Lu and Sun (2001) and Liu *et al*. (2007b).

Mixed-level SSDs are also requested frequently in experimentation. In an experimental situation when several factors have same number of levels and one or two factors have different number of levels than the rest

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of the factors, mixed level SSDs become useful. The few factors with different levels may be called as factors of asymmetry and are important factors. Some recent references on mixed level SSDs are Fang *et al*. (2003), Li *et al*. (2004), Koukouvinos and Mantas (2005), Ai *et al*. (2007), Tang *et al*. (2007), Gupta *et al*. (2008a, 2009) and Chen and Liu (2008).

Most of the research on SSDs has been restricted to balanced designs. A mixed level SSD is said to be balanced if all the levels of all the factors (in every column of the design) appear a constant number of times in the design runs. In this paper also, we restrict ourselves to balanced SSDs only. This paper introduces some methods of constructing balanced, mixed level SSDs, derived essentially from the juxtaposition of Uniform designs and Hadamard matrices. The methods of construction are illustrated with the help of examples. $E(f_{NOD})$ and $E(\chi^2)$ criteria have been used to investigate the efficiency of the designs constructed. Mathematical expressions of $E(f_{NOD})$ and $E(\chi^2)$ have been obtained for the constructed SSDs. All designs obtained are f_{NOD}-optimal while some designs are also χ^2- optimal. A catalogue of 67 $E(f_{NOD})$-optimal designs is prepared and the $E(\chi^2)$-efficiency of the designs is also given.

We begin with some preliminaries in Section 2. Different non-orthogonality criteria for evaluation of SSDs are also defined in Section 2 for the sake of completeness. Proposed construction methods, mathematical expression of $E(f_{NOD})$ and $E(\chi^2)$ and some examples are given in Section 3. Some concluding remarks and a catalogue of optimal designs are given in Section 4.

Hadamard matrices used for generation of the designs have been taken from "http://www.iasri.res.in/WebHadamard/WebHadamard.htm" available at Design Resources Server, IASRI, New Delhi and Uniform designs used have been taken from http://www.math.hkbu.edu.hk/UniformDesign/ maintained by Chang-Xing Ma. The Uniform designs used are those with centered L_2-discrepancy. The 67 SSDs obtained are available at http://iasri.res.in/design Supersaturated_Design/Supersaturated.html.

2. PRELIMINARIES

This section is devoted towards giving some useful definitions for the sake of completeness.

2.1 Uniform Design

Uniform design is an efficient fractional factorial design originally proposed by Fang and Yuan (1980). A Uniform design, denoted as $\mathbf{U}_n(n^s)$, is an $n \times s$ array in n symbols with each column having n symbols appearing once, where n is the number of runs and also the number of levels of each factor and s is the number of factors in the design. The uniform designs used in this paper for construction of SSDs are those with centered L_2-discrepancy.

2.2 Hadamard Matrix

A square matrix $\mathbf{H}_n$ of order n and with entries +1 and –1 is said to be a Hadamard matrix of order n if and only if $\mathbf{H}_n\mathbf{H}'_n = \mathbf{H}'_n\mathbf{H}_n = n\mathbf{I}_n$. A necessary condition for the existence of a Hadamard matrix is that n must be an integer and n, $n/12$ or $n/20$ must be a power of 2. A Hadamard matrix is invariant with respect to permutation of rows and / or columns and also with respect to any scalar multiplication with –1 of any row or column. We shall write a Hadamard matrix of order n in semi normalized form as $\mathbf{H}_n = [\mathbf{1}_n\ \mathbf{L}_n]$, where $\mathbf{1}$ is a vector of all ones and $\mathbf{L}_n$ is an $n \times (n-1)$ matrix of the remaining columns of $\mathbf{H}_n$. For maintaining uniformity of notations, we shall recode the symbols –1 and +1 in $\mathbf{L}_n$ as 1 and 2, respectively, wherever required.

2.3 Evaluation Criteria of Mixed-level SSDs

Consider a mixed-level SSD represented as SSD-(n; $q_1 \times q_2 \times ... \times q_m$) with m factors, the j^{th} factor being experimented with q_j levels, $j = 1, 2, ..., m$ and number of design points or runs as n. Define $v = \dfrac{\sum_{j=1}^{m}(q_j - 1)}{n-1}$.

For an SSD, $v > 1$. We denote an SSD as an $n \times m$ matrix $\mathbf{X}$ with $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^j, ..., \mathbf{x}^m$ as its m columns, the j^{th} column $\mathbf{x}^j$ containing q_j symbols, $j = 1, 2, ..., m$. Fang *et al*. (2003) defined the following $E(f_{NOD})$ criterion for measuring non-orthogonality of the mixed-level SSDs:

$$f_{NOD}^{ij} = \sum_{u=1}^{q_i}\sum_{v=1}^{q_j}\left[n_{uv}^{(ij)} - \frac{n}{q_i q_j}\right]^2 \qquad (2.1)$$

Here the subscript *NOD* stands for non-orthogonality of the design; $n_{uv}^{(ij)}$ is the number of (u, v)-pairs in $(\mathbf{x}^i, \mathbf{x}^j)$ and n/q_iq_j stands for the average frequency of level-combination in each pair of columns $\mathbf{x}^i$ and $\mathbf{x}^j$, $i \neq j = 1, 2, ..., m$.

A criterion $E(f_{NOD})$ is defined as minimizing

$$E(f_{NOD}) = \sum_{1 \le i < j \le m} f_{NOD}^{ij} \Big/ \binom{m}{2} \qquad (2.2)$$

Fang *et al*. (2004) obtained a lower bound to $E(f_{NOD})$ which is sharper than the one obtained earlier by Fang *et al*. (2003) and is given in Theorem 2.1.

Theorem 2.1 (Fang *et al*. 2004)**.** For any balanced SSD-$(n; q_1 \times q_2 \times ... \times q_m)$

$$E(f_{NOD}) \ge \frac{n(n-1)}{m(m-1)} [(\gamma + 1 - \psi)(\psi - \gamma) + \psi^2] + C(n, q_1, ..., q_m) = L[E(f_{NOD})] \qquad (2.3)$$

where

$$C(n; q_1, q_2, ..., q_m) = \frac{nm}{m-1} - \frac{1}{m(m-1)}\left(\sum_{i=1}^{m} \frac{n^2}{q_i} + \sum_{i,j=1, j \neq i}^{m} \frac{n^2}{q_iq_j}\right)$$

depends on $\mathbf{X}$ only through $n, q_1, q_2, ..., q_m$.

Here $\psi = \dfrac{\sum_{i=1}^{m} n/q_i - m}{(n-1)}$, $\gamma = [\psi]$ and

$[x]$ denotes the integer part of x.

For a balanced mixed level SSD-$(n; q_1, q_2, ..., q_m)$-SSD, Yamada and Lin (1999) defined the following $E(\chi^2)$ criterion. For every pair of columns $(\mathbf{x}^i, \mathbf{x}^j)$, $\mathbf{i} \neq \mathbf{j} = 1, 2, ..., m$

$$\chi^2(\mathbf{x}^i, \mathbf{x}^j) = \frac{q_iq_j}{n} \sum_{u=1}^{q_i} \sum_{v=1}^{q_j} \left(n_{uv}^{(ij)} - \frac{n}{q_iq_j}\right)^2 \qquad (2.4)$$

Obviously, the value of $\chi^2(\mathbf{x}^i, \mathbf{x}^j)$ measures the non-orthogonality between two columns $\mathbf{x}^i$ and $\mathbf{x}^j$. Then the $E(\chi^2)$ value can be used to evaluate the overall non-orthogonality between the columns of $\mathbf{X}$, where $E(\chi^2)$ is defined as

$$E(\chi^2) = \frac{2}{m(m-1)} \sum_{1 \le i < j \le m} \chi^2(\mathbf{x}^i, \mathbf{x}^j) \qquad (2.5)$$

For a balanced design, Ai *et al*. (2007) obtained lower bound to the value of $E(\chi^2)$ as given in Theorem 2.2.

Theorem 2.2 (Ai, Fang and He 2007). For any balanced SSD-$(n; q_1 \times q_2 \times ... \times q_m)$,

$$E(\chi^2) \ge \frac{1}{m(m-1)(n-1)}\left(nm - \sum_{k=1}^{m} q_k\right)^2 + C_1(n; q_1, q_2, ..., q_m) = L[E(\chi^2)] \qquad (2.6)$$

where $C_1(n; q_1, q_2, ..., q_m)$

$$= \frac{1}{m(m-1)}\left[\left(\sum_{k=1}^{m} q_k\right)^2 - n\sum_{k=1}^{m} q_k\right] - n$$

In this paper we have used lower bounds $L[E(f_{NOD})]$, and $L[E(\chi^2)]$, given in Theorem 2.1 and Theorem 2.2, respectively.

For any design $d \in$ SSD-$(n; q_1 \times q_2 \times ... \times q_m)$, we define f_{NOD}-efficiency and χ^2-efficiency as

$$f_{NOD} - \text{efficiency} = \frac{L\left[E(f_{NOD})\right]}{E_d(f_{NOD})}$$

$$\chi^2 - \text{efficiency} = \frac{L\left[E(\chi^2)\right]}{E_d(\chi^2)} \qquad (2.7)$$

where $E_d(f_{NOD})$ and $E_d(\chi^2)$ are the values of $E(f_{NOD})$ and $E(\chi^2)$, respectively, for the design d. A design with high efficiency is acceptable and a design with f_{NOD}–efficiency [χ^2 – efficiency] one is f_{NOD} [χ^2]-optimal.

3. CONSTRUCTION OF F_{NOD}-OPTIMAL MIXED-LEVEL SUPERSATURATED DESIGNS

This Section gives a method of constructing f_{NOD}-optimal mixed level SSDs by juxtaposing Uniform designs with Hadamard matrices.

Theorem 3.1. Consider a Hadamard matrix $\mathbf{H}_n$ of order $n \ge 4$, written in semi-normal form as $\mathbf{H}_n = [\mathbf{1}_n \mid \mathbf{L}_n]$, and a Uniform design with n runs and s factors each at n levels as $\mathbf{U}_n(n^s) = [\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_s]$. Select any $p \le s$ columns of $\mathbf{U}_n(\mathbf{n}^s)$ and write them as an $n \times p$ array $\mathbf{M}_{n\times p} = [\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_p]$ The mixed level SSD- $(n;\ n^p \times 2^{n-1})$ given by $\mathbf{A} = [\mathbf{M}_{n\times p} \,\vdots\, \mathbf{L}_n]$, has $E(f_{NOD})$ and $E(\chi^2)$ as follows:

$$\mathrm{E}(f_{NOD}) = \frac{p[nm - n - p + 1]}{m(m-1)}$$

$$\mathrm{E}(\chi^2) = \frac{np[(p-1)(m-1) + 2(n-1)]}{m(m-1)}$$

where $m = p + n - 1$.

Proof. The SSD in **A** can be written as

$$\mathbf{A} = \left[\underbrace{\mathbf{u}_1\ \ \mathbf{u}_2\ \ \cdots\ \ \mathbf{u}_p}_{\text{Group 1}} \vdots\ \underbrace{\mathbf{L}_n}_{\text{Group 2}}\right]$$

Case I. $E(f_{NOD})$ for **A**

For the design **A**, we can rewrite (2.2) as

$$E(f_{NOD}) = \frac{1}{\binom{m}{2}}\left[\sum_{1\le i<j\le p} f_{NOD}^{ij} + \sum_{p+1\le i<j\le m} f_{NOD}^{ij} + \sum_{i=1}^{p}\sum_{j=1}^{n-1} f_{NOD}^{ij}\right] \quad (3.1)$$

Since Group 1 consists of p columns of $\mathbf{U}_n(n^s)$,

$$\sum_{1\le i<j\le p} f_{NOD}^{ij} = \frac{p(p-1)(n-1)}{2} \quad (3.2)$$

Similarly, for Group 2 consisting of n–1columns of $\mathbf{H}_n$

$$\sum_{p+1\le i<j\le m} f_{NOD}^{ij} = 0 \quad (3.3)$$

The third component of (3.1) is $\sum_{i=1}^{p}\sum_{j=1}^{n-1} f_{NOD}^{ij}$, which has contribution from p columns of $\mathbf{U}_n(n^s)$ and n–1columns of $\mathbf{H}_n$. There will be $2n$ pairs of symbols but only n pairs appear together once. It is not required to know which n pairs appear together once. The remaining n pairs do not appear together.

Therefore,

$$\sum_{i=1}^{p}\sum_{j=1}^{n-1} f_{NOD}^{ij} = \frac{pn(n-1)}{2} \quad (3.4)$$

Using (3.2), (3.3) and (3.4) in (3.1) gives

$$E(f_{NOD}) = \frac{p[nm - n - p + 1]}{m(m-1)} \quad (3.5)$$

Case II. $E(\chi^2)$ for **A**

For the design **A**, we can write (2.5) as

$$E(\chi)^2 = \frac{1}{\binom{m}{2}}\left[\sum_{1\le i<j\le p} \chi^2(\mathbf{x}^i, \mathbf{x}^j) + \sum_{p+1\le i<j\le m} \chi^2(\mathbf{x}^i, \mathbf{x}^j) + \sum_{i=1}^{p}\sum_{j=1}^{n-1} \chi^2(\mathbf{x}^i, \mathbf{x}^j)\right] \quad (3.6)$$

Following on the lines of Case I above, we have

$$\sum_{1\le i<j\le p} \chi^2(\mathbf{x}^i, \mathbf{x}^j) = \frac{p(p-1)(n-1)n}{2} \quad (3.7)$$

$$\sum_{p+1\le i<j\le m} \chi^2(\mathbf{x}^i, \mathbf{x}^j) = 0 \quad (3.8)$$

and

$$\sum_{i=1}^{p}\sum_{j=1}^{n-1} \chi^2(\mathbf{x}^i, \mathbf{x}^j) = pqn \quad (3.9)$$

Using (3.7), (3.8) and (3.9) in (3.6) gives

$$E(\chi^2) = \frac{np[(p-1)(n-1) + 2(n-1)]}{m(m-1)} \quad (3.10)$$

The proof is thus complete.

Theorem 3.2. For any balanced mixed level SSD-$(n;\ n^p \times 2^{n-1})$, denoted as an $n \times m$ array, $m = p + n - 1$ the lower bounds to $E(f_{NOD})$ and $\mathrm{E}(\chi^2)$ are the following:

$$L[E(f_{NOD})] = \frac{p[nm - n - p + 1]}{m(m-1)}$$

$$L[E(\chi^2)] = \frac{np[(p-1)(m-1) + 2(n-1)]}{m(m-1)}$$

Proof. We first obtain the lower bound to $E(f_{NOD})$.

Case I. $L[E(f_{NOD})]$

$L[E(f_{NOD})]$ for the designs constructed by Method 3.1, can be obtained by using Theorem 2.1 and equation (2.3).

For the designs under consideration, $q_1 = q_2 = \ldots = q_p = n$ and $q_{p+1} = q_{p+2} = \ldots = q_m = 2$ and number of runs is also n. Now we calculate the following:

$$\psi = \frac{\sum_{i=1}^{m} n/q_i - m}{(n-1)} = \frac{n}{2} - 1 \qquad (3.11)$$

$$\gamma = [\psi] = \frac{n}{2} - 1,$$ because n is a Hadamard number and, therefore, $\frac{n}{2} - 1$ is an integer. (3.12)

Now

$$\sum_{i=1}^{m} \frac{n^2}{q_i} + \sum_{i,j=1,j\neq i}^{m} \frac{n^2}{q_i q_j}$$

$$= \sum_{i=1}^{p} \frac{n^2}{q_i} + \sum_{i=p+1}^{m} \frac{n^2}{q_i} + \sum_{i=1}^{p} \sum_{j=1,\, i\neq j}^{p} \frac{n^2}{q_i q_j}$$

$$+ \sum_{i=p+1}^{m} \sum_{j=p+1,\, i\neq j}^{m} \frac{n^2}{q_i q_j} + 2\sum_{i=1}^{p} \sum_{j=p+1}^{m} \frac{n^2}{q_i q_j}$$

$$= n^2 p + (n-1)\frac{n^3}{4} + p(p-1)$$

Therefore

$$C(n;\, n^p,\, 2^{n-1}) = \frac{nm}{(m-1)} - \frac{1}{m(m-1)}\left[n^2 p + (n-1)\frac{n^3}{4} + p(p-1)\right] \qquad (3.13)$$

Now using (3.11) and (3.12) in $[(\gamma + 1 - \psi)(\psi - \gamma^2) + \psi^2]$ we get

$$[(\gamma + 1 - \psi)(\psi - \gamma) + \psi^2] = \left(\frac{n}{2} - 1\right)^2 \qquad (3.14)$$

Now using (3.13) and (3.14) in (2.3) we get the following:

$$L[E(f_{NOD})] = \frac{n(n-1)}{m(m-1)}\left(\frac{n}{2}-1\right)^2 + \frac{nm}{(m-1)} - \frac{1}{m(m-1)}\left[n^2 p + (n-1)\frac{n^3}{4} + p(p-1)\right]$$

$$= \frac{p}{m(m-1)}\,[nm - n - p + 1] \qquad (3.15)$$

Case II. $L[E(\chi^2)]$

$L[E(\chi^2)]$ for the designs constructed by Method 3.1, can be obtained by using Theorem 2.2 and equation (2.6).

For the designs under consideration, $q_1 = q_2 = \ldots = q_p = n$ and $q_{p+1} = q_{p+2} = \ldots = q_m = 2$ and number of runs is also n. Now we calculate the following:

$$C_1(n;\, n^p \times 2^{n-1}) = \frac{1}{m(m-1)}\left[\left(\sum_{k=1}^{m} q_k\right)^2 - n\sum_{k=1}^{m} q_k\right] - n$$

$$= \frac{[(pn + 2n - 2)(pn + n - 2)]}{m(m-1)} - n \qquad (3.16)$$

Using (3.16) in (2.6) gives

$$L[E(\chi^2)] = \frac{1}{m(m-1)(n-1)}\left(nm - \sum_{k=1}^{m} q_k\right)^2 + C_1(n;\, np \times 2^{n-1})$$

$$= \frac{1}{m(m-1)(n-1)}\,[nm - np - (n-1)\,2]^2 + \frac{[(pn + 2n - 2)(pn + n - 2)]}{m(m-1)} - n$$

$$= \frac{np[(p-1)(n-1) + 2(n-1)]}{m(m-1)} \qquad (3.17)$$

The proof is thus complete.

Corollary 3.1. Let $\mathbf{H}_n = [\mathbf{1}_n \mid \mathbf{L}_n]$ and $\mathbf{U}_n(n^s) = [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_s]$ be as in Theorem 3.1. The mixed level SSD-$(n;\, n \times 2^{n-1})$ given by $\mathbf{A} = [\mathbf{u}_1 \vdots \mathbf{L}_n]$, has $E(f_{NOD}) = 1$ and $E(\chi^2) = 2$.

Proof. Follows by taking $p = 1$ in Theorem 3.1.

Result 3.1. Applying Theorem 3.1, Corollary 3.1 and Theorem 3.2 in (2.7) gives that the mixed level SSD-$(n;\, n^p \times 2^{n-1})$ and SSD-$(n;\, n \times 2^{n-1})$, are both f_{NOD}-optimal and χ^2-optimal.

Example 3.1. An f_{NOD}- optimal mixed level SSD $(8;\, 8^3 \times 2^7)$ obtained from $\mathbf{U}_8(8^7)$ and $\mathbf{H}_8$. Consider the following Hadamard matrix $\mathbf{H}_8$ in semi-normalized form, obtained from http://www.iasri.res.in/WebHadamard/WebHadamard.htm:

2	2	2	2	2	2	2	2
2	1	2	1	2	1	2	1
2	2	1	1	2	2	1	1
2	1	1	2	2	1	1	2
2	2	2	2	1	1	1	1
2	1	2	1	1	2	1	2
2	2	1	1	1	1	2	2
2	1	1	2	1	2	2	1

Further, obtain the following Uniform design $\mathbf{U}_8(8^7)$ from http://www.math.hkbu.edu.hk/UniformDesign/:

7	7	1	4	2	5	2
8	5	6	5	5	8	8
3	4	3	8	8	7	3
5	2	8	3	6	4	1
4	6	2	2	7	2	7
2	3	5	1	1	6	5
1	8	7	6	4	3	4
6	1	4	7	3	1	6

Juxtaposing the first three columns of $\mathbf{U}_8(8^7)$ and the last seven columns of $\mathbf{H}_8$ above gives the following desired mixed level SSD (8; $8^3 \times 2^7$):

1	7	1	2	2	2	2	2	2	2
8	5	6	1	2	1	2	1	2	1
3	4	3	2	1	1	2	2	1	1
5	2	8	1	1	2	2	1	1	2
4	6	2	2	2	2	1	1	1	1
2	3	5	1	2	1	1	2	1	2
1	8	7	2	1	1	1	1	2	2
6	1	4	1	1	2	1	2	2	1

Here we can take any of the three columns of $\boldsymbol{U}_8(8^7)$ instead of taking the first three columns. For this design $E(f_{NOD}) = 2.33$ and $E(\chi^2) = 7.47$ and the design is both f_{NOD}-optimal and χ^2-optimal.

Example 3.2. An f_{NOD}-optimal mixed level SSD (8; 8×2^7) obtained from $\mathbf{U}_8(8^7)$ and $\mathbf{H}_8$. Consider the Hadamard matrix $\mathbf{H}_8$ in semi-normalized form and the Uniform design $\mathbf{U}_8(8^7)$ as in Example 3.1.

Juxtaposing the first column of $\mathbf{U}_8(8^7)$ with the last seven columns of $\mathbf{H}_8$ gives the desired mixed level SSD (8; 8×2^7) as follows:

7	2	2	2	2	2	2	2
8	1	2	1	2	1	2	1
3	2	1	1	2	2	1	1
5	1	1	2	2	1	1	2
4	2	2	2	1	1	1	1
2	1	2	1	1	2	1	2
1	2	1	1	1	1	2	2
6	1	1	2	1	2	2	1

Here we can take any of the seven columns of $\mathbf{U}_8(8^7)$ instead of taking the first column. For this design, $E(f_{NOD}) = 1.00$ and $E(\chi^2) = 2.00$, and the design is both f_{NOD}-optimal and χ^2-optimal.

Method 3.1. Consider again a Hadamard matrix $\mathbf{H}_n$ of order $n \geq 4$, written in semi-normal form as $\mathbf{H}_n = [\mathbf{1}_n \mid \mathbf{L}_n]$. Consider a Uniform Design $\mathbf{U}_t(t^s)$, with s factors each at t levels in t runs, where $t = \dfrac{n}{2}$. The $t \times s$ array, $\mathbf{U}_t(t^s)$, may be written as $\mathbf{U}_t(t^s) = [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_s]$, where $\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_s$ are the columns of $\mathbf{U}_t(t^s)$. Select first two columns of $\mathbf{U}_t(t^s)$ and write them as an $n \times 1$ array $\mathbf{D}_{n\times 1} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}$. Then, $\mathbf{A} = [\mathbf{D}_{n\times 1} \vdots \mathbf{L}_n]$ is an f_{NOD}- optimal mixed level SSD-(n; $t \times 2^{n-1}$) with one factor having t (=n/2) levels and ($n - 1$) factors having two levels and n runs.

Remark 3.1. For the mixed level SSD-(n; $t \times 2^{n-1}$) given by $\mathbf{A} = [\mathbf{D}_{n\times 1} \vdots \mathbf{L}_n]$, obtained by Method 3.1, it is not possible to obtain mathematical expressions for $E(f_{NOD})$ and $E(\chi)^2$, similar to the ones obtained for the designs generated in Theorem 3.1 and Corollary 3.1. The reason is that in both the equations (3.1) and (3.6), it is not possible to evaluate the third term. The values of $n_{uv}^{(ij)}$ will be 0, 1 or 2, but it is not possible to know their frequencies, although the total of their frequencies is n. But we have used this method of construction using $\mathbf{H}_n$, $n \leq 60$ and $\mathbf{U}_t(t^s)$, $t \leq 30$ and obtained f_{NOD}-optimal mixed level SSDs-(n; $t \times 2^{n-1}$). These designs are given in Table 4.2. The χ^2–efficiency of these designs is also given in the Table.

Example 3.3. An f_{NOD}-optimal Mixed level SSD (8; 4×2^7) obtained from $\mathbf{U}_4(4^3)$ and $\mathbf{H}_8$. A uniform design $\mathbf{U}_4(4^3)$ obtained from http://www.math.hkbu.edu.hk/UniformDesign/ is the following:

3	4	3
2	1	4
1	3	1
4	2	2

Using the first two columns of $\mathbf{U}_4(4^3)$ and juxtaposing the new column with the last seven columns of $\mathbf{H}_8$ from Example 3.1 gives the desired mixed level SSD (8; 4×2^7) as follows.

3	2	2	2	2	2	2	2
2	1	2	1	2	1	2	1
1	2	1	1	2	2	1	1
4	1	1	2	2	1	1	2
4	2	2	2	1	1	1	1
1	1	2	1	1	2	1	2
3	2	1	1	1	1	2	2
2	1	1	2	1	2	2	1

Instead of using the first two columns of $\mathbf{U}_4(4^3)$, one could use any two columns of $\mathbf{U}_4(4^3)$. For this design, $E(f_{NOD}) = 0.86$ and the design is f_{NOD}-optimal. Further, the design has $E(\chi^2) = 0.86$ with χ^2 – efficiency = 0.71.

4. DISCUSSION

This article introduces construction of mixed level SSDs by juxtaposition of Uniform designs and the Hadamard matrices. All the designs generated through this method are f_{NOD}-optimal. Some series of designs are both f_{NOD} and χ^2-optimal. Such designs are catalogued in Table 4.1. It is not possible to establish algebraically the optimality of designs obtained in Method 3.1. However, using this method f_{NOD}-optimal SSDs have been obtained and are given in Table 4.2. The χ^2–efficiency of these designs is also given. All the designs given in the Tables 4.1 and 4.2 are available at

Table 4.1. f_{NOD} and χ^2-optimal mixed level SSDs obtained using Theorem 3.1 and Corollary 3.1

Sl. No.	Design	$E(f_{NOD})$	$E(\chi^2)$	Sl. No.	Design	$E(f_{NOD})$	$E(\chi^2)$
1	$(4;4.2^3)$	1.00	2.00	28	$(16;16^8.2^{15})$	5.45	34.15
2	$(4;4^2.2^3)$	1.50	3.60	29	$(16;16^9.2^{15})$	5.87	39.13
3	$(8;8.2^7)$	1.00	2.00	30	$(16;16^{10}.2^{15})$	6.25	44.00
4	$(8;8^2.2^7)$	1.75	4.67	31	$(16;16^{11}.2^{15})$	6.60	48.74
5	$(8;8^3.2^7)$	2.33	7.47	32	$(16;16^{12}.2^{15})$	6.92	53.33
6	$(8;8^4.2^7)$	2.80	10.18	33	$(16;16^{13}.2^{15})$	7.22	57.78
7	$(8;8^5.2^7)$	3.18	12.73	34	$(16;16^{14}.2^{15})$	7.50	62.07
8	$(8;8^6.2^7)$	3.50	15.08	35	$(16;16^{15}.2^{15})$	7.76	66.21
9	$(8;8^7.2^7)$	3.77	17.23	36	$(20;20.2^{19})$	1.00	2.00
10	$(12;12.2^{11})$	1.00	2.00	37	$(20;20^2.2^{19})$	1.90	5.43
11	$(12;12^2.2^{11})$	1.83	5.08	38	$(20;20^3.2^{19})$	2.71	9.87
12	$(12;12^3.2^{11})$	2.54	8.70	39	$(20;20^4.2^{19})$	3.45	15.02
13	$(12;12^4.2^{11})$	3.14	12.57	40	$(20;20^5.2^{19})$	4.13	20.65
14	$(12;12^5.2^{11})$	3.67	16.50	41	$(20;20^6.2^{19})$	4.75	26.60
15	$(12;12^6.2^{11})$	4.13	20.38	42	$(20;20^7.2^{19})$	5.32	32.74
16	$(12;12^7.2^{11})$	4.53	24.16	43	$(20;20^8.2^{19})$	5.85	38.97
17	$(12;12^8.2^{11})$	4.89	27.79	44	$(20;20^9.2^{19})$	6.33	45.23
18	$(12;12^9.2^{11})$	5.21	31.26	45	$(20;20^{10}.2^{19})$	6.79	51.47
19	$(12;12^{10}.2^{11})$	5.50	34.57	46	$(20;20^{11}.2^{19})$	7.21	57.66
20	$(12;12^{11}.2^{11})$	5.76	37.72	47	$(20;20^{12}.2^{19})$	7.60	63.75
21	$(16;16.2^{15})$	1.00	2.00	48	$(20;20^{13}.2^{19})$	7.97	69.73
22	$(16;16^2.2^{15})$	1.88	5.29	49	$(20;20^{14}.2^{19})$	8.31	75.59
23	$(16;16^3.2^{15})$	2.65	9.41	50	$(20;20^{15}.2^{19})$	8.64	81.31
24	$(16;16^4.2^{15})$	3.33	14.04	51	$(20;20^{16}.2^{19})$	8.94	86.89
25	$(16;16^5.2^{15})$	3.95	18.95	52	$(20;20^{17}.2^{19})$	9.23	92.33
26	$(16;16^6.2^{15})$	4.50	24.00	53	$(20;20^{18}.2^{19})$	9.50	97.62
27	$(16;16^7.2^{15})$	5.00	29.09	54	$(20;20^{19}.2^{19})$	9.76	102.76

Table 4.2. f_{NOD}-optimal mixed level SSDs obtained using Method 3.1

Sl. No.	Design	$E(f_{NOD})$	$E(\chi^2)$	$L[E(\chi^2)]$	χ^2–efficiency
1	$(8;4.2^7)$	0.86	0.86	0.61	0.71
2	$(12;6.2^{11})$	0.91	0.91	0.66	0.73
3	$(16;8.2^{15})$	0.93	0.93	0.68	0.73
4	$(20;10.2^{19})$	0.95	0.95	0.70	0.74
5	$(24;12.2^{23})$	0.96	0.96	0.71	0.74
6	$(28;14.2^{27})$	0.96	0.96	0.71	0.74
7	$(32;16.2^{31})$	0.97	0.97	0.72	0.74
8	$(36;18.2^{35})$	0.97	0.97	0.72	0.74
9	$(40;20.2^{39})$	0.97	0.97	0.72	0.74
10	$(44;22.2^{43})$	0.98	0.98	0.73	0.74
11	$(52;26.2^{51})$	0.98	0.98	0.73	0.75
12	$(56;28.2^{55})$	0.98	0.98	0.73	0.75
13	$(60;30.2^{59})$	0.98	0.98	0.73	0.75

http://iasri.res.in/design/Supersaturated_Design/Supersaturated.html. The Hadamard matrix used to construct designs in the catalogue have been taken from the link http://www.iasri.res.in/WebHadamard/WebHadamard.htm maintained at Design Resources Server. The Uniform designs are taken from http://www.math.hkbu.edu.hk/UniformDesign/ maintained by Chang-Xing Ma.

We now describe an interesting problem. Consider again the mixed level SSD-$(n;\, n \times 2^{n-1})$, obtained as $\mathbf{A} = [\mathbf{u}_1 \vdots \mathbf{L}_n]$ in Corollary 3.1. Suppose that the n levels of $\mathbf{u}_1$ are such that $n = s_1 \times s_2 \times \ldots \times s_r$. Using the replacement technique one can get from SSD-$(n;\, n \times 2^{n-1})$ another SSD-$(n;\, s_1 \times s_2 \times \ldots \times s_r \times 2^{n-1})$ on replacing the n level factor with r factors at $s_1, s_2, \ldots, s_r$ levels respectively. The design has n runs. Intuitively, it appears that the new design will also be f_{NOD}-optimal and χ^2-optimal. But this may not be so. Consider the following example:

Example 4.1. Suppose we have an SSD $(12;\, 12 \times 2^{11})$ constructed using Theorem 3.1. Therefore by Corollary 3.1, the design has $E(f_{NOD}) = 1$ and $E(\chi^2) = 2$, and is both f_{NOD}-optimal and χ^2-optimal. The design is

5	1	1	1	1	1	1	1	1	1	1	1
4	2	1	2	1	1	1	2	2	2	1	2
3	2	2	1	2	1	1	1	2	2	2	1
7	1	2	2	1	2	1	1	1	2	2	2
1	2	1	2	2	1	2	1	1	1	2	2
6	2	2	1	2	2	1	2	1	1	1	2
11	2	2	2	1	2	2	1	2	1	1	1
10	1	2	2	2	1	2	2	1	2	1	1
9	1	1	2	2	2	1	2	2	1	2	1
8	1	1	1	2	2	2	1	2	2	1	2
2	2	1	1	1	2	2	2	1	2	2	1
12	1	2	1	1	1	2	2	2	1	2	2

Now using replacement technique we can construct an SSD $(12;\, 4 \times 3 \times 2^{11})$. The constructed design is as follows:

2	2	1	1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	1	1	2	2	2	1	2
1	3	2	2	1	2	1	1	1	2	2	2	1
3	1	1	2	2	1	2	1	1	1	2	2	2
1	1	2	1	2	2	1	2	1	1	1	2	2
2	3	2	2	1	2	2	1	2	1	1	1	2
4	2	2	2	2	1	2	2	1	2	1	1	1
4	1	1	2	2	2	1	2	2	1	2	1	1
3	3	1	1	2	2	2	1	2	2	1	2	1
3	2	1	1	1	2	2	2	1	2	2	1	2
1	2	2	1	1	1	2	2	2	1	2	2	1
4	3	1	2	1	1	1	2	2	2	1	2	2

This design has $E(f_{NOD}) = 1.31$ and $E(\chi^2) = 0.77$. The design is f_{NOD}-optimal but not χ^2-optimal. In fact, the χ^2–efficeincy of the design is 0.73. This problem needs to be investigated further and work in this direction is in progress.

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