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Optimum Designs for Stress Strength Reliability

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SUMMARY

In a stress-strength model, the reliability of a system is measured by the probability of the strength of the system exceeding the environmental stress. As the strength of the system depends on a number of controllable factors, the accuracy in estimating the reliability can be enhanced by a proper choice of these factors. In this paper, we have assumed stress and strength to be independent exponential variables, with mean strength being a function of the controllable factors. The optimum design for the estimation of the system reliability has been proposed using suitable optimality criterion.

Key words : Stress-strength reliability, Exponential distribution, Optimum design, Central composite design.

1. INTRODUCTION

According to the simple stress-strength model of failure, a system fails when and only when the applied stress exceeds its strength. Since the stress is a function of the environment to which the system is subjected, it can be regarded as a random variable. Similarly, the strength of a system that is mass-produced depends on a number of factors, some of which are controllable, like material properties, manufacturing procedures etc., while the others are uncontrollable, and hence can also be treated as random. In such a stochastic environment the designer is interested in the reliability of the system subject to a stress. However, since the distributions of stress and strength are generally unknown, either fully or partially, estimation of reliability of the system becomes important. In estimating reliability, one may take into account the factors influencing stress and strength. To date, considerably little work has been done along this line, see Guttman *et al*. (1988), Weerahandi and Johnson (1990), Guttman and Papandonatos (1997). However, the existing literature does not include any study on choice of the controllable factors affecting the strength so as to improve the accuracy in estimation.

In this paper, we attempt to find an optimum continuous design for the controllable factors $\mathbf{z} = (z_1, z_2, \ldots, z_p)'$ that should be used in the experiment to achieve maximum accuracy in estimation of system reliability. We assume stress and strength to have independent exponential distributions, with the expected strength being a function of **z**. In Section 2, we derive the asymptotic variance of the maximum likelihood estimate (MLE) of the system reliability as a function of the design; in Section 3 the optimum design is derived using suitable criteria, and a discussion, including applications, is given in Section 4.

2. MLE OF THE SYSTEM RELIABILITY AND ITS ASYMPTOTIC VARIANCE

Let X and Y denote the stress and strength, respectively, which are assumed to be independently distributed with density functions $f(x)$ and $g(y)$, and means μ_X and μ_Y. We assume $\mu_Y = \eta(\mathbf{z}; \beta)$, a function of the controllable factors $\mathbf{z} = (z_1, z_2, \ldots, z_p)'$, where β denotes the vector of regression coefficients associated with **z**.

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Then, the reliability of the system is given by

$$R(\mathbf{z};\, \mu_X,\, \beta) = P\,(X < Y \mid z) \tag{2.1}$$

For μ_X unknown, we may find its MLE $\hat{\mu}_X$, based on a random sample of stresses $(X_1, X_2, \ldots, X_m)$. Similarly, to obtain an estimate of μ_Y, we consider a random sample of strengths $(Y_1, Y_2, \ldots, Y_n)$, associated with a n-point continuous design

$$\xi = \begin{Bmatrix} z_1, & z_2, & \ldots, & z_n \\ w_1, & w_2, & \ldots, & w_n \end{Bmatrix}$$

in the domain Ξ of $\mathbf{z}$, where $z_1, z_2, \ldots, z_n$ are the design points with weights $w_1, w_2, \ldots, w_n$, respectively,

$$w_i \geq 0,\ i = 1(1)n,\ \sum_{i=1}^{n} w_i = 1.$$

Based on this, we obtain the MLE $\hat{\beta}$ of β. The MLE of reliability is, therefore, $\hat{R}(\mathbf{z}) = R(\mathbf{z}; \hat{\mu}_X, \hat{\beta})$.

The asymptotic variance of $\hat{R}(\mathbf{z})$ is given by

$$\tilde{V}\,(\hat{R}(\mathbf{z})) = A'(\mu_X, \beta, \mathbf{z})\mathbf{I}^{-1}(\mu_X, \beta; \xi)\, A(\mu_X, \beta, \mathbf{z})$$

where

$$A'(\mu_X, \beta, \mathbf{z}) = \left(\frac{\partial}{\partial \mu_X} R(\mathbf{z}; \mu_X, \beta), \frac{\partial}{\partial \beta'} R(\mathbf{z}; \mu_X, \beta) \right)$$

and

$$\mathbf{I}(\mu_X, \beta, \xi) = \begin{pmatrix} \dfrac{1}{\operatorname{var}(\hat{\mu}_X)} & 0 \\ 0 & M(\xi) \end{pmatrix}$$

is the information matrix of (μ_X, β), with $\mathbf{M}(\xi)$ being the information matrix of β, given ξ, where information matrix is the variance of the score function, which is the gradient of the log-likelihood function.

We shall assume X and Y to follow independent exponential distributions having densities

$$f(x) = \frac{1}{\mu_X} e^{-\frac{x}{\mu_X}},\ x \geq 0,\ \mu_X > 0$$

$$g(y) = \frac{1}{\mu_Y} e^{-\frac{y}{\mu_Y}},\quad y \geq 0,\ \mu_Y > 0,$$

$$\mu_Y = \eta\,(\mathbf{z}, \beta)$$

Then,

$$R(\mathbf{z};\, \mu_X,\, \beta) = \frac{1}{1 + \dfrac{\mu_X}{\eta(\mathbf{z};\beta)}} \tag{2.2}$$

and

$$\hat{R}(z) = \frac{1}{1 + \dfrac{\hat{\mu}_X}{\eta(\mathbf{z};\hat{\beta})}} \tag{2.3}$$

Let us assume $\eta\,(\mathbf{z};\, \beta)$ to be a polynomial of certain degree in the elements of $\mathbf{z}$ and write $\eta\,(\mathbf{z};\, \beta) = h'(\mathbf{z})\beta$, where $h(\mathbf{z})$ is a column vector involving powers of z_i's and their products.

Then,

$$\tilde{V}\,(\hat{R}(\mathbf{z})) = \frac{\mu_X^2}{[\,\eta(\mathbf{z};\beta)]^4 \left(1 + \dfrac{\mu_X}{\eta(\mathbf{z};\beta)}\right)^4} \left[\frac{\{\eta(\mathbf{z};\beta)\}^2}{m} + h'(\mathbf{z})\mathbf{M}^{-1}(\xi)h(\mathbf{z}) \right]$$

$$= a(\mathbf{z}) + c(\mathbf{z})\; b(\mathbf{z},\, \mathbf{M}) \tag{2.4}$$

where

$$\mathbf{M}(\xi) = \sum_{i=1}^{n} w_i h(z_i) h'(z_i) \tag{2.5}$$

$$a(\mathbf{z}) = \frac{\mu_X^2}{m\{\eta(\mathbf{z};\, \beta)\}^2 \left(1 + \dfrac{\mu_X}{\eta(\mathbf{z};\, \beta)}\right)^4}$$

$$b(z, I) = h'(z)\mathbf{M}^{-1}\,(\xi)\; h(\mathbf{z})$$

$$c(\mathbf{z}) = \frac{\mu_X^2}{\{\eta(\mathbf{z};\, \beta)\}^4 \left(1 + \dfrac{\mu_X}{\eta(\mathbf{z};\, \beta)}\right)^4} \tag{2.6}$$

As expected, the asymptotic variance of $\hat{R}(\mathbf{z})$ is a function of the unknown parameters and **z**. However, $\tilde{V}\ (\hat{R}(\mathbf{z}))$ depends on the design ξ only through $b(\mathbf{z}, \mathbf{M})$. To tackle this problem, we have used a simplified criterion of minimizing $b(\mathbf{z}, \mathbf{M})$ by a proper choice of design.

3. OPTIMUM DESIGNS

In this section, we attempt to find the optimum design for the estimation of $R(\mathbf{z}; \mu_X, \beta)$ by minimizing $b(\mathbf{z}, \mathbf{M})$, for the cases where $\eta(\mathbf{z}; \beta)$ is linear and quadratic in **z**, respectively.

We shall assume the domain of **z** to be $\mathbf{Z} = \{\mathbf{z}: \mathbf{z}'\mathbf{z} \le p\}$. With this assumption, the vertices of the symmetrized unit cube, $[-1; 1]^p$ come to lie on the boundary of the sphere $\mathbf{z}'\mathbf{z} \le p$. This is the appropriate generalization of the experimental domain [–1; 1] for the case of a single factor (cf. Pukelsheim 1993).

Case 1: Linear Regression

Let us assume

$$\eta(\mathbf{z}; \beta) = \beta_0 + \sum_{i=1}^{p} \beta_i z_i \tag{3.1}$$

Here, $h'(\mathbf{z}) = (1, \mathbf{z}')$
so that $h'(\mathbf{z})\, h(\mathbf{z}) = 1 + \mathbf{z}'\mathbf{z}$

Now, under linear regression, for a design x the moment matrix is given by

$$\mathbf{M}(\xi) = \begin{pmatrix} 1 & [1] & [2] & \dots & [p] \\ & [11] & [12] & \dots & [1p] \\ \dots & \dots & \dots & \dots & \dots \\ & & & & [pp] \end{pmatrix}$$

with

$$[i] = \sum_{k=1}^{n} w_k z_{ik}\ ,\ [i\,j] = \sum_{k=1}^{n} w_k z_{ik} z_{jk}$$

where z_{ik} is the i-th element of the k-th design point z_k.

Since $b(\mathbf{z}, \mathbf{M})$ depends on **z**, to find the optimum design, one approach would be to minimize $\max_{\mathbf{z}\in \mathrm{Z}} b(\mathbf{z}, \mathbf{M})$ with respect to the design. But, by the Equivalence Theorem (cf. Kiefer and Wolfowitz 1960), this amounts to maximizing the determinant of the information matrix for the linear model (3.1) above. Now, the problem of maximizing the determinant of $\mathbf{M}(\xi)$ is invariant with respect to permutations and sign changes of the factors. Hence, a D-optimum design will be invariant, and we may therefore confine our attention to the class of invariant designs (cf. Pal and Mandal 2008). Further, we may make use of the fact that for the factor space $Z = \{\mathbf{z}: \mathbf{z}'\mathbf{z} \le p\}$, a design ξ whose moment matrix has the form

$$\mathbf{M}(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{I}_p \end{bmatrix}$$

is Kiefer optimal (cf. Pukelsheim 1993) and hence is necessarily D-optimal. Such a design can always be constructed with the help of a Hadamard matrix. Similar dominance results are also available in Liski *et al.* (2002).

Example. Let $p = 3$. Consider a Hadamard matrix of order 4 in the standard form:

$$\mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = (1\ D)$$

Since, $\mathbf{H}_4' \mathbf{H}_4 = 4\, \mathbf{I}_4$, the moment matrix of the optimum design, which puts equal weights ¼ at the four design points (1, 1, 1)′, (–1, 1, –1)′, (1, –1, –1)′ and (–1, –1, 1)′, is given by $\mathbf{M} = \mathbf{I}_4$ and hence is optimum.

Case II: Quadratic Regression

Let

$$\eta(\mathbf{z}; \beta) = \beta_0 + \sum_{i=1}^{p} \beta_i z_i + \sum_{i=1}^{p}\sum_{j=1}^{p} \beta_{ij} z_i z_j$$
$$\beta_{ij} = \beta_{ji},\ \forall i \ne j \tag{3.2}$$

This is a non-singular model.

Here, let us write

$$\beta = (\beta_0, \beta_{11}, \beta_{22}, \dots, \beta_{12}, \beta_{13}, \dots, \beta_{p-1,\,p}, \beta_1, \beta_2, \dots, \beta_p)'$$

$$h(z) = (1,\ z_1^2, z_2^2, \dots, z_p^2, z_1 z_2,\ z_1 z_3, \dots, z_{p-1} z_p,\ z_1, z_2, \dots, z_p)'$$

To find the optimum design using the minimax criterion considered in Case I, we again take help of the celebrated *Equivalence Theorem* (cf. Kiefer and Wolfowitz 1960) which states that the problem of minimizing $\max_{\mathbf{z}\in Z} h'(\mathbf{z})\mathbf{M}(\xi)^{-1} h(\mathbf{z})$ w.r.t. the design ξ is equivalent to maximizing $|\mathbf{M}(\xi)|$ w.r.t. ξ, where $|\mathbf{M}(\xi)|$ is the determinant of the moment matrix $\mathbf{M}(\xi)$.

It may be noted that the problem of maximizing $|\mathbf{M}(\xi)|$ remains invariant under

(a) permutation of the co-ordinates of $\mathbf{z}$;

(b) sign change of co-ordinates of $\mathbf{z}$.

Let $\mathfrak{I}$ denote the class of all permutation matrices and sign change matrices of order $p \times p$.

Let ξ^T be the design obtained from ξ by virtue of the transformation $\mathbf{z} \to T\mathbf{z}$, $T \varepsilon \mathfrak{I}$, and let,

$$\bar{\mathbf{M}} = \frac{1}{t}\sum_{T\in\mathfrak{I}} \mathbf{M}(\xi^T)$$

where t denotes the cardinality of $\mathfrak{I}$.

Then, if $\phi(\mathbf{M}) = \log|\mathbf{M}|$, which is convex in $\mathbf{M}$, we have

$$\begin{aligned}\phi(\bar{\mathbf{M}}) &= \phi(\mathbf{M}(\bar{\xi}))\\ &\le \frac{1}{t}\sum_{T\in\mathfrak{I}} \phi(\mathbf{M}(\xi^T))\\ &= \phi(\mathbf{M}(\xi))\end{aligned}$$

where $\bar{\xi}$ denotes a design with information matrix $\bar{\mathbf{M}}$ (cf. Pal and Mandal 2008).

Thus, we may restrict our search for optimum design within the subclass of symmetric designs. Consider the following designs:

$$\xi_0 = \{\mathbf{z} \mid \mathbf{z}'\mathbf{z} = 0\} \tag{3.3}$$

$$\tilde{\xi}_{\sqrt{p}} = \frac{n_c\xi_c + n_s\xi_s}{n} \tag{3.4}$$

$$n_c = p^2,\ n_s = 2^{p-k},\ n = n_c + n_s$$

$\xi_c = \dfrac{1}{2^k}$ fraction of a 2^p factorial experiment with levels ± 1

ξ_s = set of star points of the form $(\pm\sqrt{p},0,0,...,0)$, $(0,\pm\sqrt{p},0,...,0)$,..., $(0, 0,..., \pm\sqrt{p})$

Let, $\mathbf{M}_c$ and $\mathbf{M}_s$ denote the moment matrices corresponding to the designs ξ_c and ξ_s, respectively. Then, it can be easily seen that

$$\mathbf{M}_c = \begin{pmatrix} \mathbf{1} & \mathbf{1}'_p & 0 & 0\\ & \mathbf{J}_p & 0 & 0\\ & & \mathbf{I}_{\binom{p}{2}} & 0\\ & & & \mathbf{I}_p \end{pmatrix}$$

$$\mathbf{M}_s = \begin{pmatrix} \mathbf{1} & \mathbf{1}'_p & 0 & 0\\ & p\mathbf{I}_p & 0 & 0\\ & & 0 & 0\\ & & & \mathbf{I}_p \end{pmatrix} \tag{3.5}$$

where

$$\mathbf{J}_p = \mathbf{1}_p\mathbf{1}'_p$$

Hence, the moment matrix of the design $\tilde{\xi}_{\sqrt{p}}$, given by (3.4), is

$$\mathbf{M}_P = \frac{n_c\mathbf{M}_c + n_s\mathbf{M}_s}{n} \tag{3.6}$$

Now, we may use the following result for the full model

$$\eta(\mathbf{z};\beta) = \beta_0 + \sum_{i=1}^{p}\beta_i z_i + \sum_{i=1}^{p}\sum_{j=1}^{p}\beta_{ij}z_i z_j \tag{3.7}$$

which is singular (see Pukelsheim 1993):

Kiefer Optimality : Given a symmetric design ξ, there exists a design $\xi^* = (1-\alpha)\xi_0 + \alpha\tilde{\xi}_{\sqrt{p}}$, concentrated at $\mathbf{z}'\mathbf{z} = 0$ and $\mathbf{z}'\mathbf{z} = p$, such that $\xi^* \succ \xi$ in the sense of Loewner Order Dominance.

Such a design ξ^* is called a central composite design (CCD). Note that the design ξ^* is completely characterized by α.

Since the Loewner Order Dominance of ξ^* for the full model (3.7) implies the same for ξ^* in the non singular set-up (3.2), we may utilize the above result to reduce the class of symmetric designs substantially.

The moment matrix of the CCD $\xi^* = (1-\alpha)\xi_0 + \alpha\tilde{\xi}_{\sqrt{p}}$, where ξ_0 and $\tilde{\xi}_{\sqrt{p}}$ are given by (3.3) and (3.4) respectively, comes out to be

$$\mathbf{M}(\xi^*) = (1-\alpha)\mathbf{M}(\xi_0) + \alpha\mathbf{M}(\tilde{\xi}_{\sqrt{p}})$$

$$= \begin{bmatrix} \mathbf{1} & \alpha\mathbf{1}'_p & 0 & 0 \\ & \alpha\{b\mathbf{J}_p + (1\text{-}b)p\mathbf{I}_p\} & 0 & 0 \\ & & \alpha b\mathbf{I}_{\binom{p}{2}} & 0 \\ & & & \alpha\mathbf{I}_p \end{bmatrix} \qquad (3.8)$$

where

$$b = \frac{n_c}{n}$$

By virtue of Kiefer Optimality, our problem thus reduces to finding α, $0 \le \alpha \le 1$, such that $|\mathbf{M}(\xi^*)|$ is maximized.

Clearly,

$$|\mathbf{M}(\xi^*)| = \alpha^{2p+\binom{p}{2}} b^{\binom{p}{2}} |p(1-b)\mathbf{I}_p + (b-\alpha)\mathbf{J}_p|$$

$$= \alpha^{2p+\binom{p}{2}} b^{\binom{p}{2}} p[p(1-b)]^{p-1}(1-\alpha)$$

which is maximum at

$$\alpha = \alpha_0 = \frac{2p+\binom{p}{2}}{2p+\binom{p}{2}+1} \qquad (3.9)$$

Thus, we have the following theorem:

Theorem 3.1. The optimum design for estimating the stress-strength reliability in a model with exponentially distributed stress and strength, and expected strength defined by the quadratic regression model (3.2) in the domain $Z = \{\mathbf{z}: \mathbf{z}'\mathbf{z} \le p\}$, is given by the central composite design $\xi^* = (1-\alpha_0)\xi_0 + \alpha_0\tilde{\xi}_{\sqrt{p}}$, where ξ_0 and $\tilde{\xi}_{\sqrt{p}}$ are as stated in (3.3) and (3.4), respectively,

and $\alpha_0 = \dfrac{2p+\binom{p}{2}}{2p+\binom{p}{2}+1}$

Example. Let $p = 4$. Then $\tilde{\xi}_{\sqrt{p}}$ is given by

$$\tilde{\xi}_{\sqrt{p}} = \frac{n_c\xi_c + n_s\xi_s}{n}$$

where

$$n_c = p^2,\ n_s = 2^{p-k},\ n = n_c + n_s$$

For $k = 1$, ξ_c = ½ fraction of 2^4 factorial experiment with levels ±1, corresponding to the identifying equation I = ABCD, where A, B, C and D represent the four factors:

A	B	C	D
–1	–1	–1	–1
1	1	–1	–1
1	–1	1	–1
1	–1	–1	1
–1	1	1	–1
–1	1	–1	1
–1	–1	1	1
1	1	1	1

ξ_s corresponds to the eight star points given by

(± 2,0,0,0) (0, ± 2,0,0,0), (0,0, ± 2,0) and (0, 0, 0, ± 2)

Since n =16 and n = 8,

$\tilde{\xi}_{\sqrt{4}} = \dfrac{16\xi_c + 8\xi_s}{24}$, and the optimum design is given by $\xi^* = (1-\alpha_0)\xi_0 + \alpha_0\tilde{\xi}_{\sqrt{4}}$, with $\alpha_0 = 14/15$.

4. DISCUSSION

In this investigation, we have considered the problem of determining the optimum design for estimating the system reliability in a stress-strength model, when the strength of the system is known to depend on a number of controllable factors, which are the covariates. A suitable criterion has been used to derive the optimum design when the mean strength of the system is taken to be a function of the covariates.

Two functional forms have been considered, viz. linear and quadratic.

Applications of the models can be found in agriculture, as well as in industry. In agriculture, for example, the yield of a crop depends, among others, on the soil quality. The soil needs to have the elastic behaviour for good yield. The strength of the soil is its capacity to retain the property of elasticity, and this depends on a number of factors like soil composition, (grain size, shape of particles, mineralogy etc.), state (loose, dense, over-consolidated, stiff, soft etc.), structure (arrangement of particles within soil mass, the manner in which the particles are packed or distributed) and loading conditions. Some of these factors are controllable. The stress on the soil is the prevailing shear stress. In industry, the strength of a manufactured product depends on its design, which in turn is dependent on a number of controllable parameters. These parameters are, therefore, the covariates affecting the strength of the product. For example, the strength of alkali activated slag concrete depends on the type of alkaline activator used, type and fineness of the slag, etc. The product breaks down when the environmental stress actng on it exceeds its strength.

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